BASICS OF CAVITIES AND RESONATORS

Cavity Quantum Optomechanics

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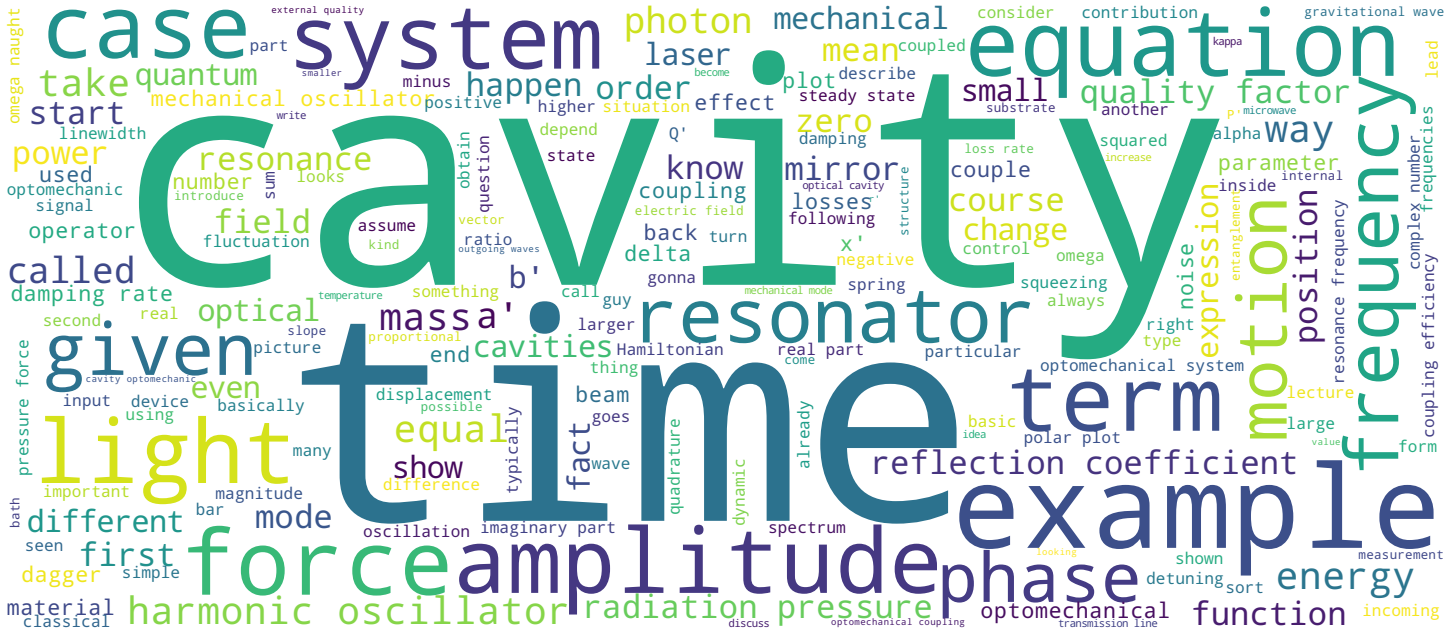
TU Delft



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WELCOME & OUTLINE

In this lesson, you will learn about the basics of cavities and resonators

The topics we will cover will include:

- **The equations of motion of a damped harmonic oscillator**
- **The frequency response function of a cavity**
- **Coupling a cavity to a transmission line**
- **S-Parameters**
- **Internal and external quality factors**

https://commons.wikimedia.org/wiki/File:Animated-mass-spring-faster.gif

Okay. Welcome. My name is Professor Gary Steele from the Delft University of Technology and I will be giving you a lecture today on the basics of cavities and resonators as part of the cavity optomechanics mooc. So my aim is to teach you about some basics of cavities and resonators and the topics we'll cover will include the equations of motion of a damped harmonic oscillator, the frequency response function of a cavity, coupling a cavity to a transmission line, the S-parameters that you see when you couple a cavity to transmission line and an application of those in particular, extracting the internal and external quality factors from S-parameter measurements.

Notes

Summary

0m 00s

THE DAMPED HARMONIC OSCILLATOR

$$F = ma = -kx - cv$$

Differential Equation:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

Undamped natural frequency

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Damping Rate

$$\gamma = \frac{c}{m}$$

https://en.wikipedia.org/wiki/Harmonic_oscillator

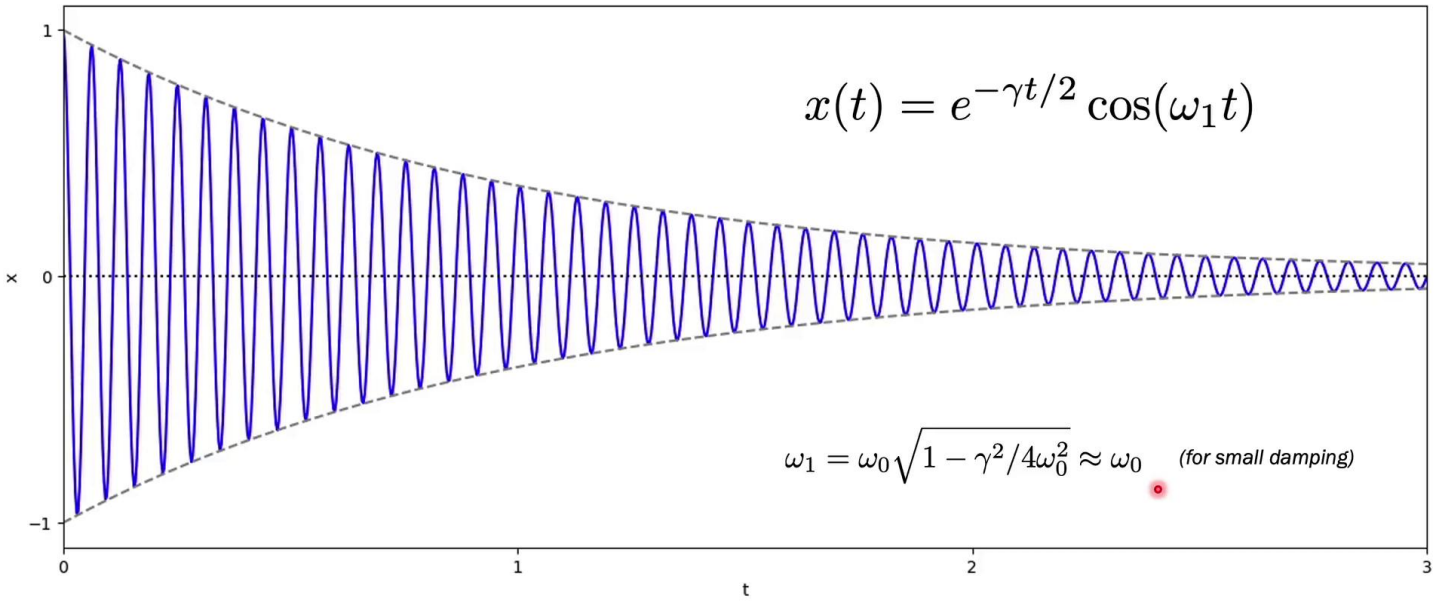
So I'll start with the basics. So we're going to study this guy over here, mass bouncing up and down the spring. The equation of motion for this mass on the spring is given by Newton's laws up here and force equals mass times acceleration. And in the case of this spring, there are two forces. There's a force, restoring force minus 'kx' and a damping force which is, which all right here is minus 'c', some coefficient, some damping coefficient times velocity. So I'm going to define two new variables for this to translate this into a differential equation. I'm going to find omega-naught which I'll call the undamped natural frequency and that's just given by the square root of 'k' over 'm'. And there's also, I'm also going to define gamma which will be a damping rate and this is given by the friction coefficient 'c' divided by the mass. If I do so, I will get this differential equation here and this is one common parameterization of the mass of the harmonic oscillator that we will see in many places. And our goal is to study well, I'll show in the coming slides, we'll take a look at some of the solutions of this. So in particular, what I'm gonna look at is what will happen if I pull this mass out with my finger, hold it steady and just let it go.

Notes

Summary

0m 48s

FREE MOTION (PULL IT AND LET GO)



So what I will see is the following. So I pull it out to normalize position one and I bling and let it go. What will happen is it will oscillate up and down. This is what harmonic oscillators do. But in addition because there is some friction, there will be also a damping of this motion following this exponential envelope here and the decay constant of that envelope is given by the damping rate. And the oscillation frequency of this decaying oscillation is given by what I'll call omega one. Omega one is actually not quite the resonance frequency of the harmonic oscillator, of the undamped harmonic oscillator but is slightly modified by this term here. Now, in optomechanics we'll be mostly studying high-Q objects, cavities and resonators and that means the damping is small relative to the natural frequency and so to a very good approximation and this will just be equal to omega-naught what we will call the resonance frequency.

Notes

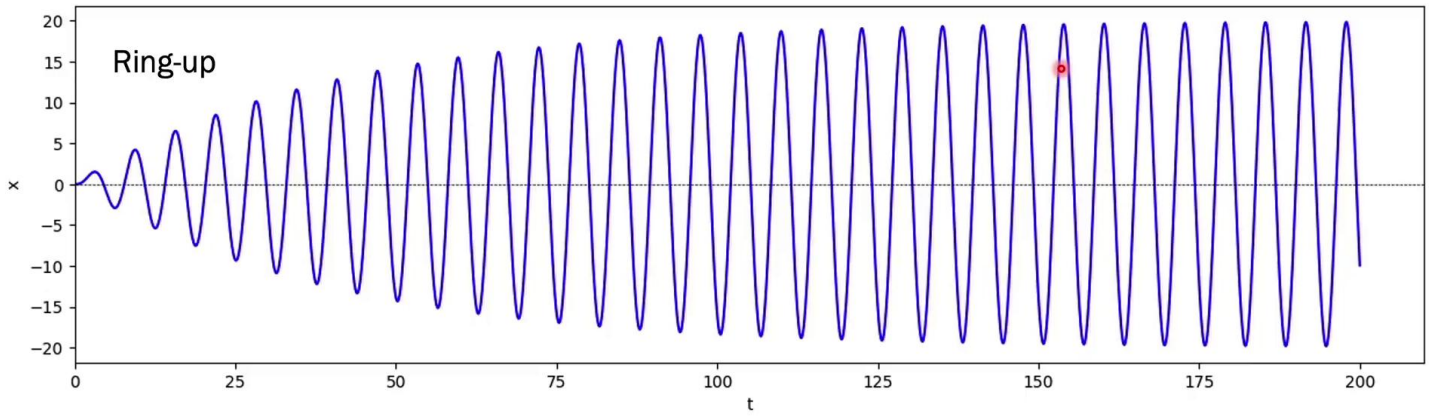
Summary

2m 19s

RESPONSE TO AN OSCILLATING FORCE

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = F_0\cos(\omega t)$$

Turn on force at t=0.
What does x do?



Now, let's take a look at a different thing. Let's imagine the force the force the mass is experiencing is not zero but say, oscillating in time, say, I'm pushing it only with my finger up and down somehow with this force here. And now the question is if I turn on this force, this oscillating force at 't' equals zero, what will the position do as a function of time? What we will see is the following. This is, you know, if particularly I'm, this is a case where this omega is oscillating at the resonance frequency. Then what will happen is that the the harmonic oscillator's amplitude will start to grow and but at some point due to the damping that growth will saturate and we will come to a steady state amplitude and this is something that we often call ring up because we're starting to ring. If I just let go of the mass that's what we call ring down and the opposite of that is what I'm showing here which is we're ringing up the mass up into a steady state oscillation amplitude.

Notes

Summary

3m 27s

(Intermezzo for a fun puzzle...)

Now, actually there's a funny little intermezzo. We're going to take for a little puzzle for those that are interested.

Notes

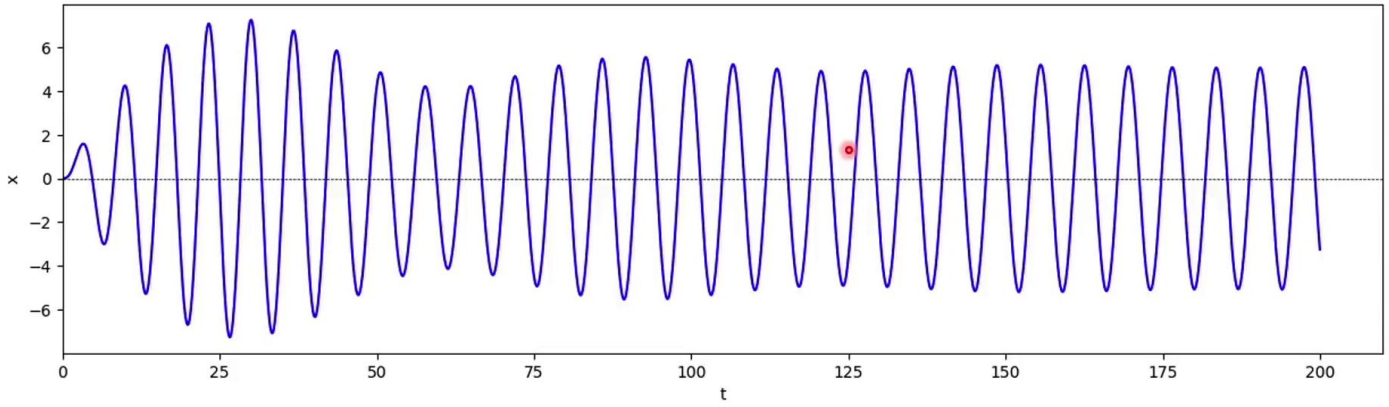
Summary

4m 39s

FUN PUZZLE

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = F_0\cos(\omega t)$$

Turn on force at t=0, non resonant...
What does x do?



Can you understand this? And can you estimate what frequency I've used?

Let's say I do this but pick a frequency which is not resonant with natural frequency of the mass. What does position 'x' do as a function of time? Now, this is the answer actually. It does something funky. It sort of starts to grow but then it starts to bounce up and down for a little while and eventually comes to a steady state and for the experts in the audience, you can think about why this is happening and a good question for homework if you're interested is, can you estimate what frequency I've used to make this plot?

Notes

Summary

4m 48s

(Back to main story)

So let's go back to our main story though.

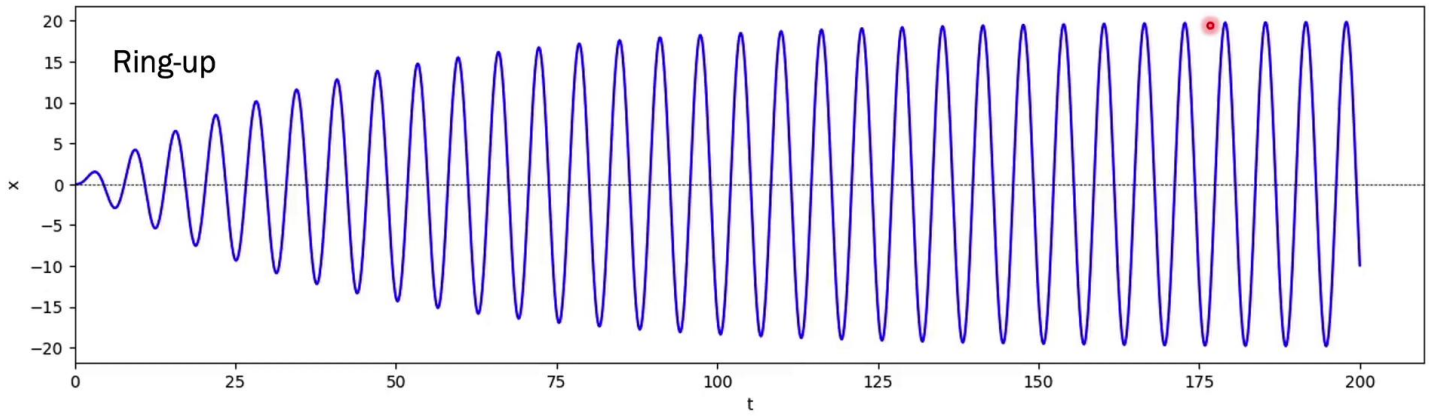
Notes

Summary

5m 30s

RESPONSE TO AN OSCILLATING FORCE

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = F_0\cos(\omega t)$$ What is steady state amplitude?



So this is where we were. We were ringing this guy up and eventually this when we are applying a constant oscillating force, we will reach some steady state amplitude. And now the question I want to ask next is what is the steady state amplitude that we reach depending on the frequency we choose.

Notes

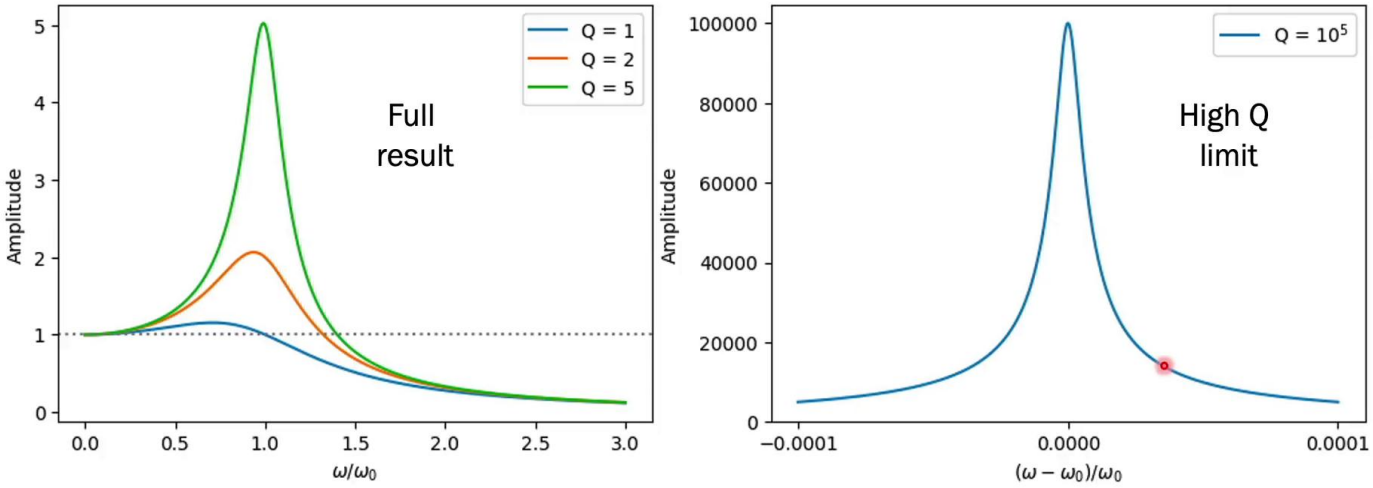
Summary

5m 33s

STEADY-STATE RESPONSE FUNCTION

$$A(\omega) = \frac{F}{m}\frac{1}{\sqrt{\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2}}$$

$$A(\omega) \approx \frac{F}{2m\omega_0}\frac{1}{\sqrt{(\omega - \omega_0)^2 + (\gamma/2)^2}}$$



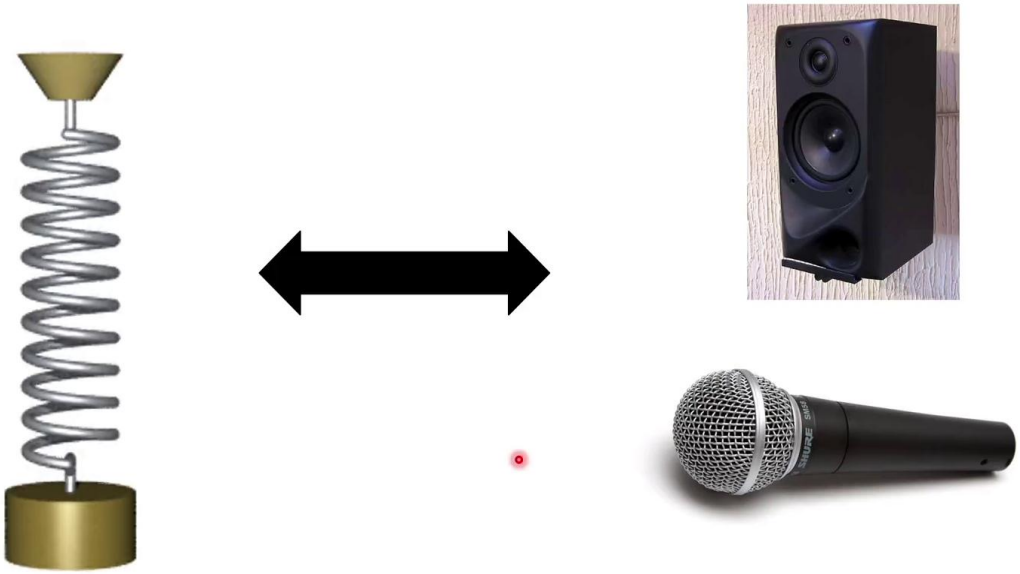
So that's looks like this so this is the full solution over here. So the amplitude we can just get this by solving the equations that were on the previous slide. The full amplitude looks is given by this function here and this is that function plotted for different values of 'Q' where 'Q' is just the ratio of omega-naught to the damping rate. So for low 'Q' resonators, it looks a little bit like this and as I start to increase the 'Q', the amplitude on resonance gets higher and higher and higher and it's peaked, it's peaked at the resonant frequency and it's also interesting to see that resonant amplitude is just equal to 'Q' times force over mass. If you have a sufficiently high Q's and this is small Q's of only order of one, two or five. If you have a sufficiently high 'Q', then you can make an approximation of this formula and this is a common approximation we pretty much always make in optomechanics. And that is to approximate this harmonic oscillator response function with a Lorentzian and you can see the Lorentzian is also peaked, also has the same characteristics that the amplitude on resonances just 'Q' times of the DC, the static response of the resonator but you can see the one thing it misses is that even in the high-Q response, there is some asymmetry on the left and right side of this peak but that is now gone in this Lorentzian approximation.

Notes

Summary

5m 54s

COUPLING RESONATORS TO SOURCES AND DETECTORS



https://commons.wikimedia.org/wiki/File:Loudspeaker.arp.500pix.jpg
https://commons.wikimedia.org/wiki/File:Micro_Shure_SM58.jpg

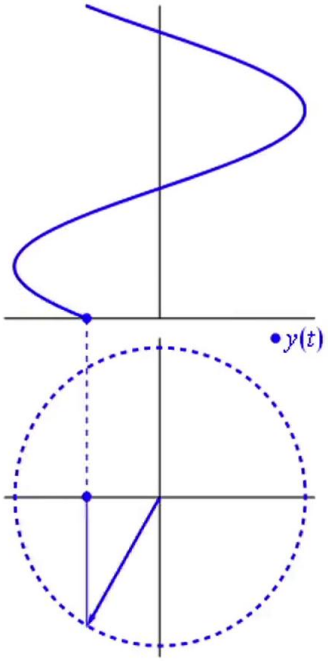
So now I've covered what I wanted to cover in the basics of resonators and harmonic oscillators and now I want to enter another topic which is, how to couple this resonator to sources and detectors. So far I've been talking about pushing it with my finger. Of course, in real life, I don't actually go and push my mechanical resonators with my finger in the lab and so how do we excite cavities or resonators? We do that by having a source which is somehow weakly coupled to this harmonic oscillator that can excite it and then we get signal back out of it leaking out and that's what we try to detect over here.

Notes

Summary

7m 32s

PHASOR NOTATION



Complex notation can be handy for representing oscillating signals

$$V_0 \cos(\omega t + \theta) = \mathrm{Re}(V_0 e^{i\theta}\ e^{i\omega t})$$

Amplitude and phase of an oscillating signal can be represented by complex-valued “phasor”:

$$\tilde{V} = V_0 e^{i\theta}$$

https://commons.wikimedia.org/wiki/File:Unfasor.gif

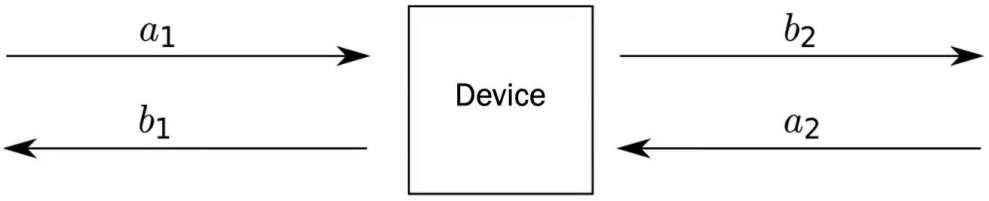
So to treat this really well and properly, I want to remind the you guys, hopefully remind you guys of something called phasor notation. Phasor notation is a handy way that we use in Physics and Engineering to represent oscillating signals. So, you know, in real life of course, a position of my of my mass is a real number. It's like a cosine omega 't' and also the electric field in my electromagnetic wave is also a real number and also the voltage in my circuit is also real number, but it's mathematically convenient actually to represent these by this complex number over here remembering, of course, that in the end we will take the real part in order to get the actually oscillating signal. So actually we can we can separate out this guy here and we can give him a little tilde on the top if we want but and now we can see that this number here is a vector in the complex plane and that represents the amplitude and the phase of the oscillating signal we're studying and that's something that is often called a phasor.

Notes

Summary

8m 13s

S-PARAMETERS



a,b = Phasors of incoming / outgoing waves

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

S_{xx} Complex-valued

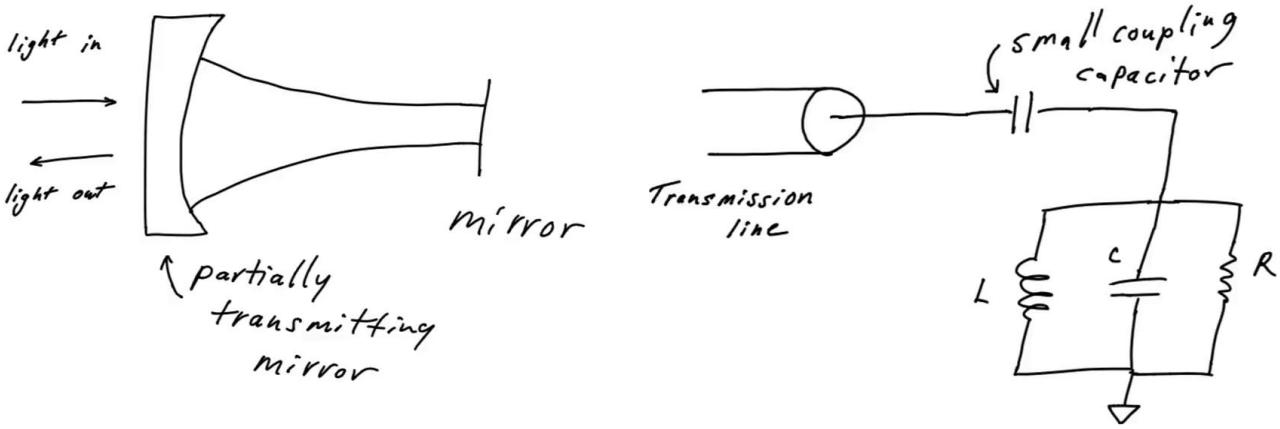
So now, the next topic I want to introduce very briefly is the idea of S-parameter. So now we're gonna have a device and we're gonna have waves coming in incident on my device and then those waves are gonna bounce off my device and go back out. This is actually even a more fancy topic that we want a more fancy device that we won't talk about today but where we even have, for example, waves that are coming in from the other side and also bouncing off the device. And 'a' and 'b' in these pictures are the phasors of the incoming and outgoing waves. The phasors we saw in the previous slide so the question is a very general question. How do I relate the incoming and outgoing waves of this device to each other. We do that we can parameterize that by something called the S-parameter matrix, scattering matrix and the incoming and outgoing waves are complex numbers and represented by these phasors and the incoming waves are complex numbers and in these parameters S11, S21, S12, S22, these are complex numbers that translate the incoming the combinations of incoming waves and the outgoing waves.

Notes

Summary

9m 34s

COUPLING A CAVITY TO A TRANSMISSION LINE



Voltage (field amplitude) reflection coefficient: S_{11}

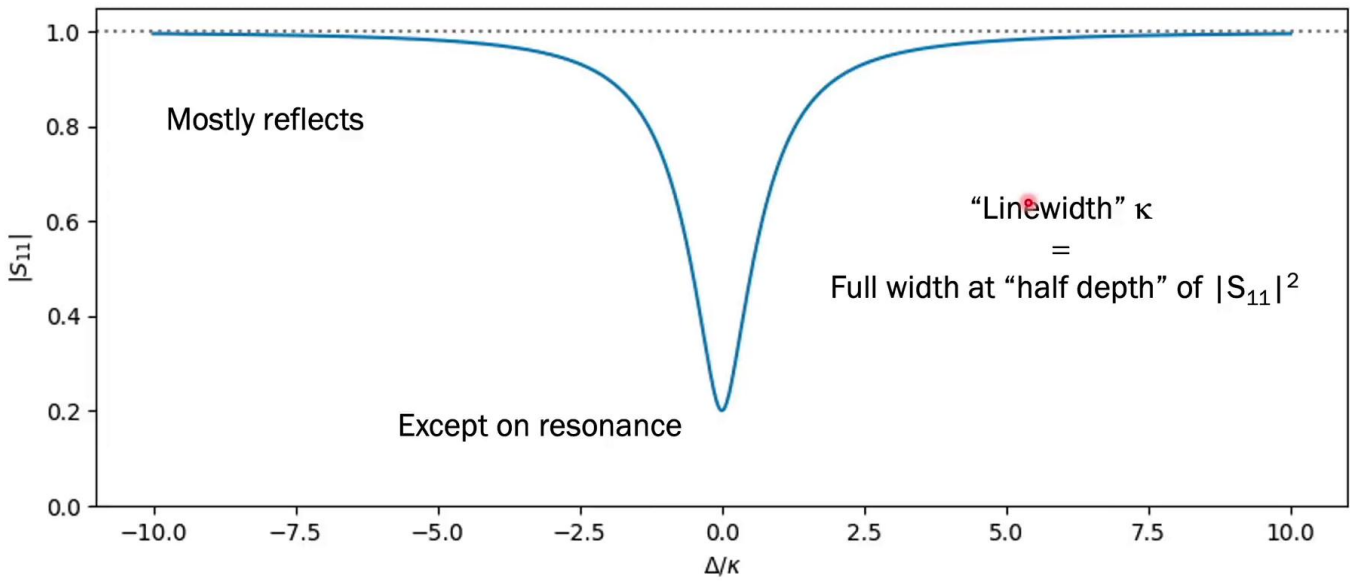
Now, how we do this in real life? This is depends on what type of optomechanical system you are studying. There you can make cavities out of light and then the cavity might consist of, for example, light bouncing off back and forth between two mirrors and the way we couple these cavities, we make one of these mirrors partially transmitting so that we can send light waves in and then measure light waves that come back out. In a microwave domain it's also very similar except that our cavity is typically either literally or can be represented by an LC circuits where there are oscillations of charge back and forth on this capacitor plate through this inductor. And then our waves are sent in and out on a transmission line, for example coaxial cable, and are equivalent of a partially transmitting mirror as this can be, for example, a small coupling capacitor. That's a small amount of this of the current from this wave coming in, reach our circuit and potentially excite it. But this small coupling capacitor is just like the mirror over here also keeps the currents that are oscillating this circuit from decaying too quickly back out into the transmission line. So these are both one-port cavities and so the S-matrix is actually just a number S11 which is also sometimes just called the reflection coefficient.

Notes

Summary

10m 54s

CAVITY REFLECTION COEFFICIENT



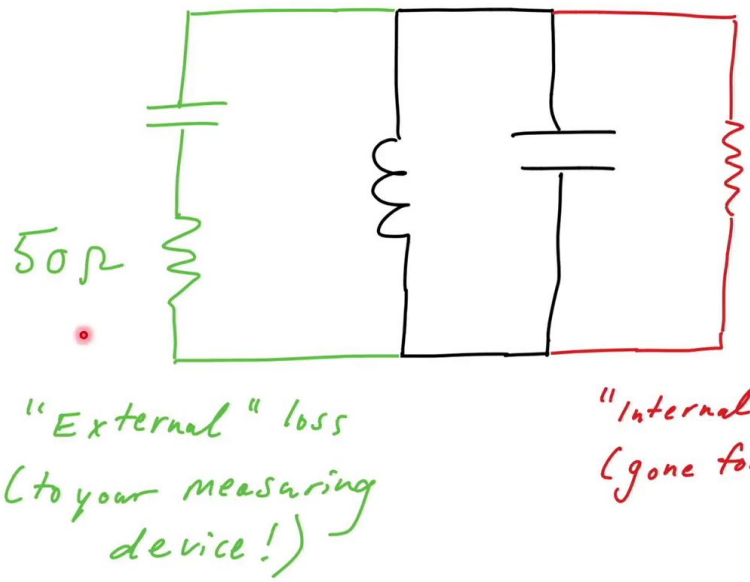
So let's take a look at what the reflection coefficient of my cavity will do as a function of detuning from the resonance. And so actually in these one-port cavity scenarios, I just sketched actually most of the time your cavity just reflects. So most of the time the light just bounces off the mirror or by microwaves off your capacitor and reflection coefficient is just one. The difference is that so as you approach the resonance, actually what you will see is a dip. This is because you're starting to get a lot of power to go inside of the cavity and then it can lose a bit of power and then you see the magnitude of the reflection coefficient go down. And now actually the linewidth, there's the linewidth, the sort of width full width at maximum not of this, but actually the S11-squared of this dip that is is equal to what we call the linewidth of this resonance.

Notes

Summary

12m 19s

INTERNAL AND EXTERNAL LOSSES



Coupling Efficiency:

$$\eta = \frac{\kappa_e}{\kappa}$$

Both contribute to your total linewidth!

$$\kappa = \kappa_i + \kappa_e$$

How can I tell them apart?

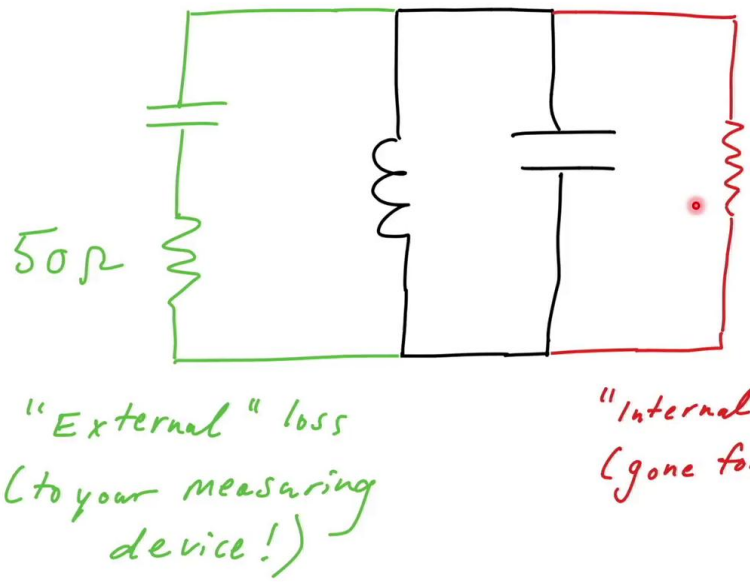
Now, where does there's an important concept when we talk about about losses in coupled cavities and, in particular, there can be two types of losses. So your LC circuit might be shunted by a resistor inside of it and if so, then some of its oscillations will be damped through, for example, losses in this resistor. This could be light scattering out of your cavity in an optical cavity or could be the loss of your dielectrics. But in addition, we've coupled our cavity also to the outside world on purpose to be able to both excite it but also more importantly, to be able to measure what's coming out if we don't, yeah, we actually want the signal from our cavity eventually to go to our detector otherwise we can tell nothing about what's going on. And and so actually these, there are two ways two types of losses in our cavity and so there is two quality factors. We typically define two quality factors. One for the internal quality factor, which is losses in this resistor, internal resistor and one that is a quality factor. Yeah, this one we do our best but sometimes we can't control them. The other one is the losses to our external circuit which is actually the coupling to our external circuit and this is something we usually have some sort of control over.

Notes

Summary

13m 25s

INTERNAL AND EXTERNAL LOSSES



Coupling Efficiency:

$$\eta = \frac{\kappa_e}{\kappa}$$

Both contribute to your total linewidth!

$$\kappa = \kappa_i + \kappa_e$$

How can I tell them apart?

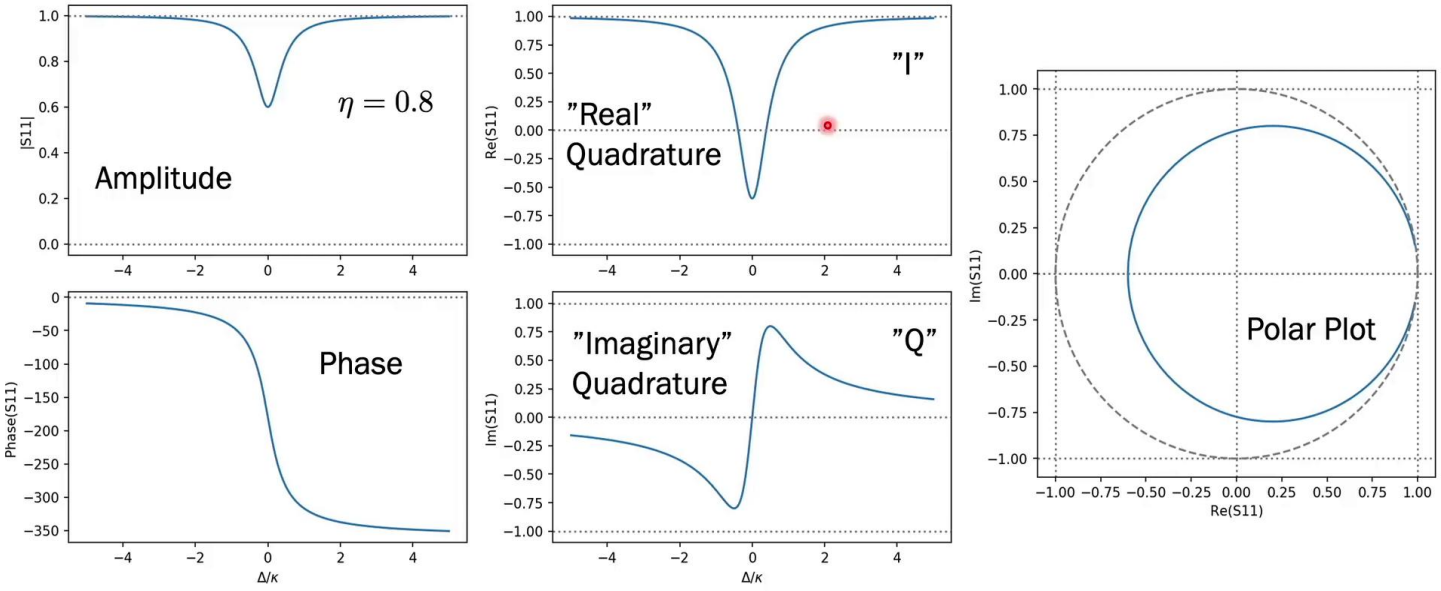
We can define actually the ratio of that external coupling to the total loss rate, external loss rate to the total loss rate is something called the coupling efficiency represented by a number eta. And actually these losses, these internal losses and external losses, they both contribute to my total linewidth. But now I can't go in and take a look at that little resistor and see how much power it's losing. My oscillations are losing to my internal resistor and the big question is, is there some sort of way that I can tell these two couplings, these two loss rates apart.

Notes

Summary

14m 48s

ANSWER: S_{11} IS COMPLEX VALUED



$$S_{11} = 1 - \frac{\eta\kappa}{i\Delta + \kappa/2} \qquad \mathrm{Re}[S_{11}(\omega_0)] = 1 - 2\eta$$

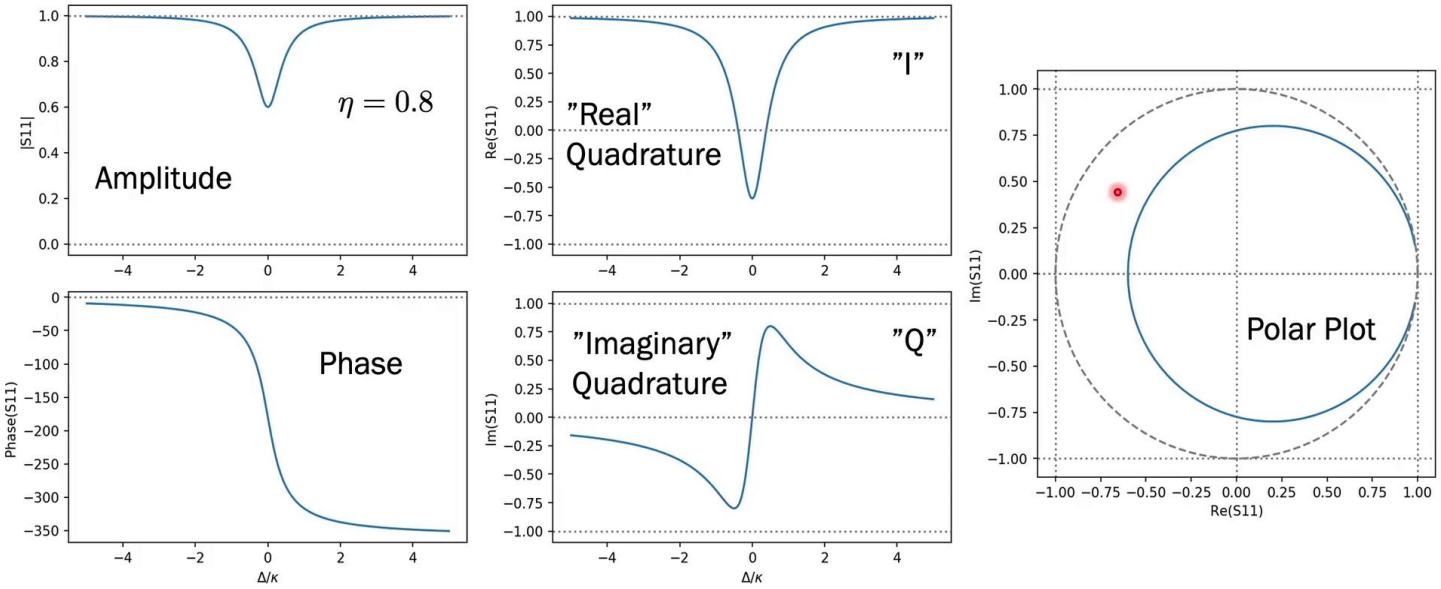
So it turns out you can and that is because, you know, what this kappa here that is the linewidth that's this width, this S11 here, but actually this S11 is not just a magnet, is not just a reflection coefficient of power but it's telling you about the power that's reflected but it actually contains information about the phase and that's because it's a complex number. So here I plot actually this, this is a formula for the reflection coefficient of of a one-port cavity and here I plot this in whole bunch of different ways. So this is probably the one people are most used to, the amplitudes. If you look at what happens to the phase actually, this is an over coupled copy and the phase goes through a two-pi phase shift. My personal preference for looking at at these at the reflection coefficient is to break it up into what are called quadratures and these are the two, the real and imaginary part of this this reflection coefficient and these are the plots of the quadratures for this cavity. And you can see that this one here looks has a dip. It looks a bit like the amplitude. It's sometimes called, sometimes people might call it the amplitude quadrature.

Notes

Summary

15m 29s

ANSWER: S_{11} IS COMPLEX VALUED



$$S_{11} = 1 - \frac{\eta\kappa}{i\Delta + \kappa/2} \qquad \mathrm{Re}[S_{11}(\omega_0)] = 1 - 2\eta$$

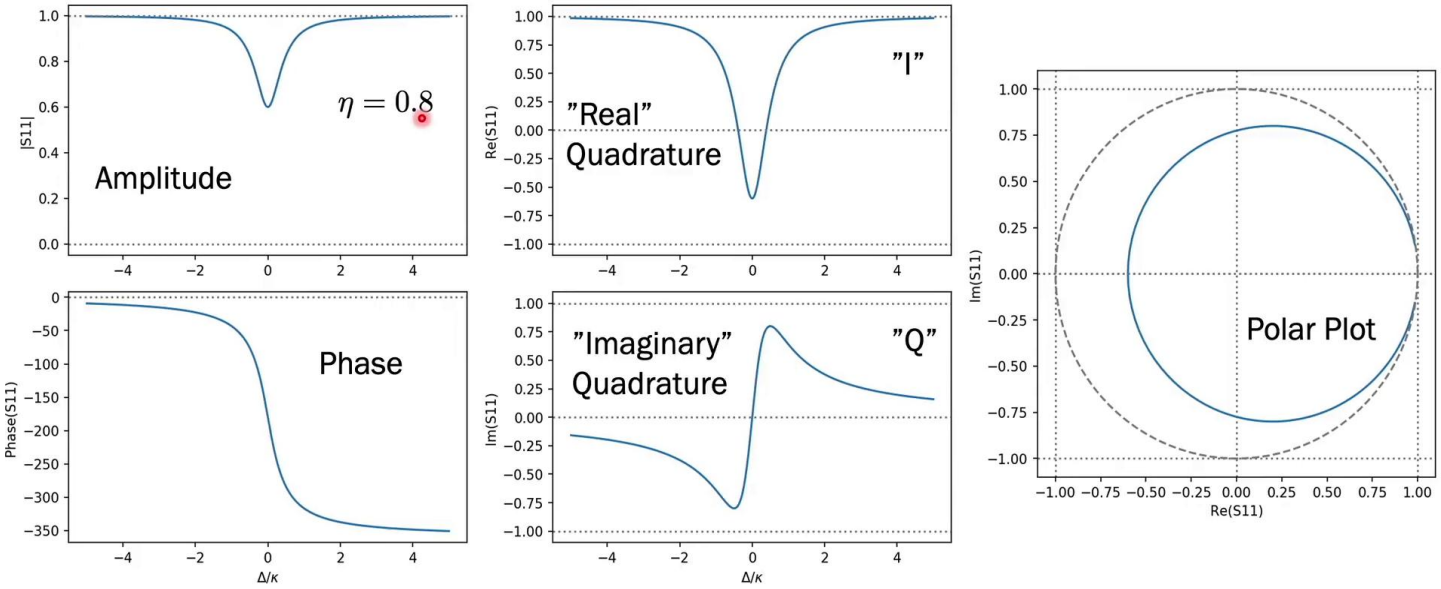
It's also sometimes called 'I' from 'I' and 'Q' or sometimes people call it 'P' from 'P' and 'Q'. And then down here there's another quadrature called the imaginary which is the just imaginary part of S11. Sometimes it's called the 'Q' quadrature. It's also sometimes called the phase quadrature because it has a slope at zero detuning, a lot like the phase has a slope at zero detuning whereas the amplitude has no slope at zero detuning. And finally, there's one more way you can look at this cavity responses by looking at plotting the real versus imaginary part and that makes polar plot as you see here and actually this resonance traces out of the circle in the polar plot.

Notes

Summary

16m 53s

ANSWER: S_{11} IS COMPLEX VALUED



$$S_{11} = 1 - \frac{\eta\kappa}{i\Delta + \kappa/2} \qquad \mathrm{Re}[S_{11}(\omega_0)] = 1 - 2\eta$$

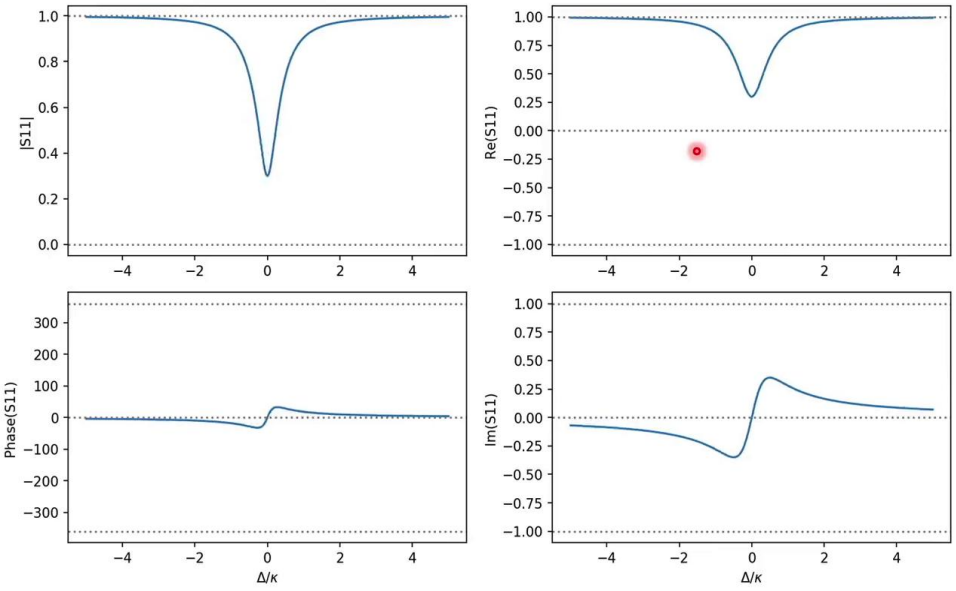
So this is what the polar plot here I've already indicated with my mouse in the previous slide but here it shows you what happens as you sweep the frequency. This little orange dot shows you where I am on this frequency sweep and in the polar plot, you can actually see what happens. We start on one side and we go downwards and then we get super fast. We zoom around and we come back to the other side slowly. So now what does this relate us? How does this relate to the internal and external quality factors?

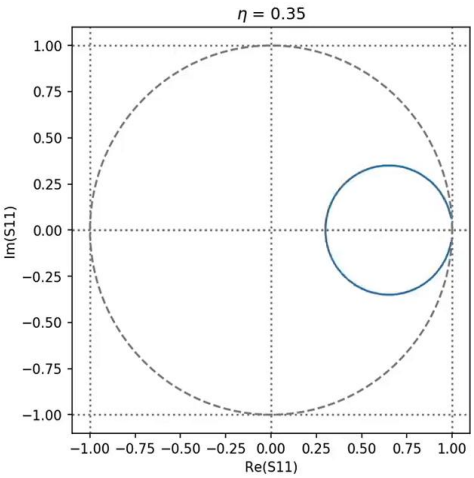
Notes

Summary

17m 35s

CHANGING COUPLING EFFICIENCY





$$S_{11} = 1 - \frac{\eta\kappa}{i\Delta + \kappa/2} \qquad \mathrm{Re}[S_{11}(\omega_0)] = 1 - 2\eta$$

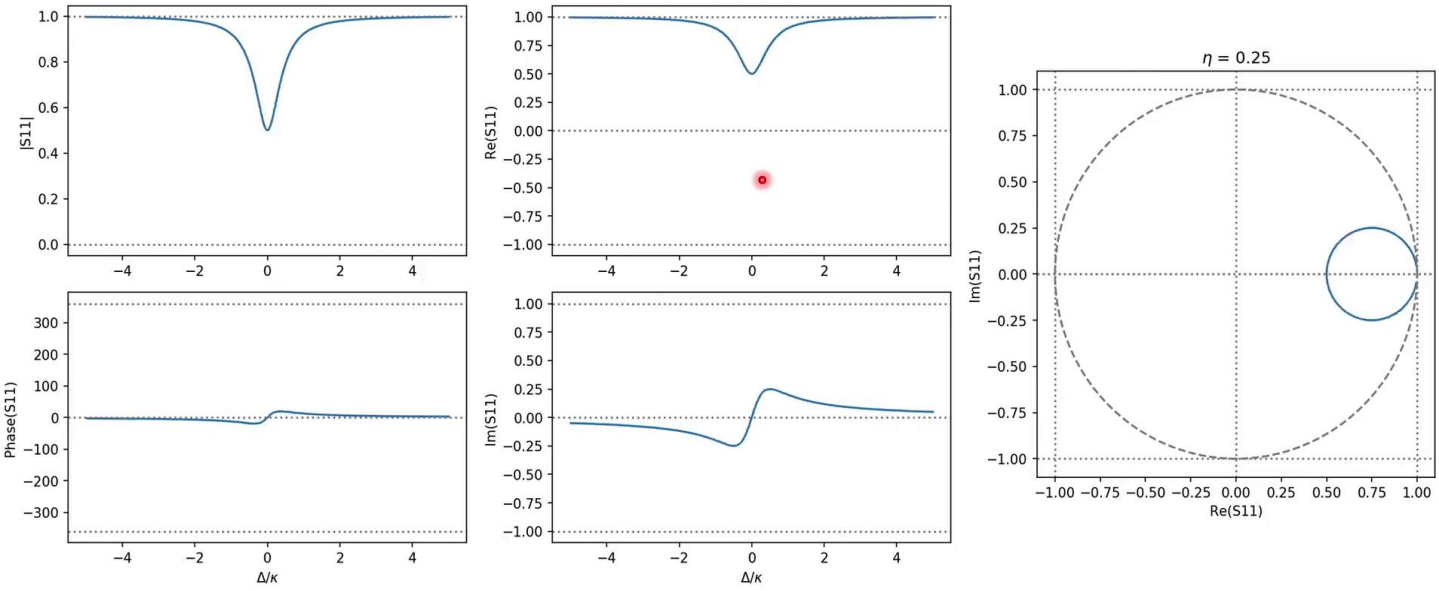
Well, actually if you pick eta is equal to 0.2, you can see there's actually no difference in the magnitude of the reflection coefficient but you can see already there's a big difference in the phase because now the phase is no longer even rapping two by two pi at all and has a different shape. It goes up down up instead of down all the way. So you can, even though you don't see any main difference in the magnitude of S11, you can actually see it very clearly in the quadratures actually. The real part has only a little dip and imaginary part also has only a little dip here. And you can also see it clearly, you know, what's going on in the polar plot. This circle actually shrinks and now I've got the smaller shrunken circle here. Now, if you want to actually numerically extract the quality factor, the internal external quality factors from this, how do you do it? And my personal favorite way is if you just look at the real part, the real part of the reflection coefficient at resonance is just equal to one minus two eta so I can read this off here. That's about 0.6 and therefore, I can know that eta is equal 0.2. So this is then just to hammer this all home. I now, I want to show you one last animation.

Notes

Summary

18m 08s

CHANGING COUPLING EFFICIENCY



$$S_{11} = 1 - \frac{\eta\kappa}{i\Delta + \kappa/2} \qquad \mathrm{Re}[S_{11}(\omega_0)] = 1 - 2\eta$$

This is changing the coupling efficiency of my cavity. You can see that as I change my coupling efficiency, all that's happening is the circle is growing until unity coupling efficiency it reaches the unit circle at here. The amplitude for this reflection our one-port cavity actually goes down and then back up again. That's kind of a funny thing. And that makes it a bit tricky always to get the internal and external quality factor that are only the magnitude of the reflection coefficient but if you look at the quadratures or the phase but the real quadrature is very clear. It just keeps dropping and hits minus one when we hit coupling efficiency of one.

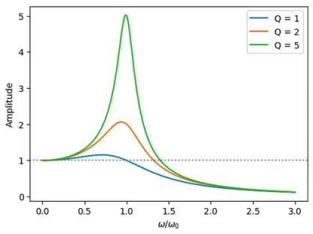
Notes

Summary

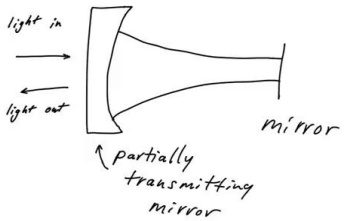
19m 28s

CONCLUSION / SUMMARY

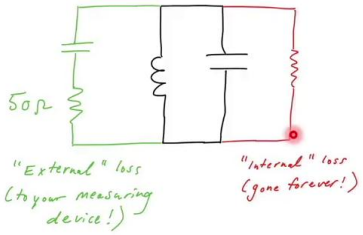
1. Resonance



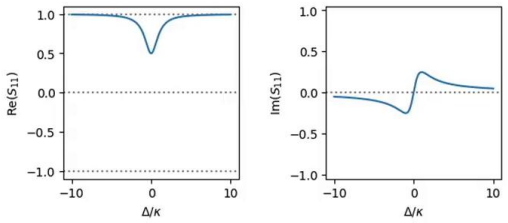
2. Exciting and listening



3. Internal and external loss



4. Information encoded in complex response



Code for plots and animations available at https://gitlab.tudelft.nl/gsteele/omt-mooc

Good, so that's what I wanted to tell you about to remind you what we've seen. I looked at the motion. We looked at the equations of motion of a mass on the spring to understand resonances and, you know, resonators and cavities. We looked at the resonant response of resonators and cavities. I talked a little bit about how to excite and how to listen to the to the oscillations of your cavity. We talked about the internal and external losses of cavity and we talked about how to interpret the real and imaginary parts of the reflection coefficient in order to separate out the internal and external loss.

Notes

Summary

20m 10s