BASICS OF ELASTICITY

Cavity Quantum Optomechanics

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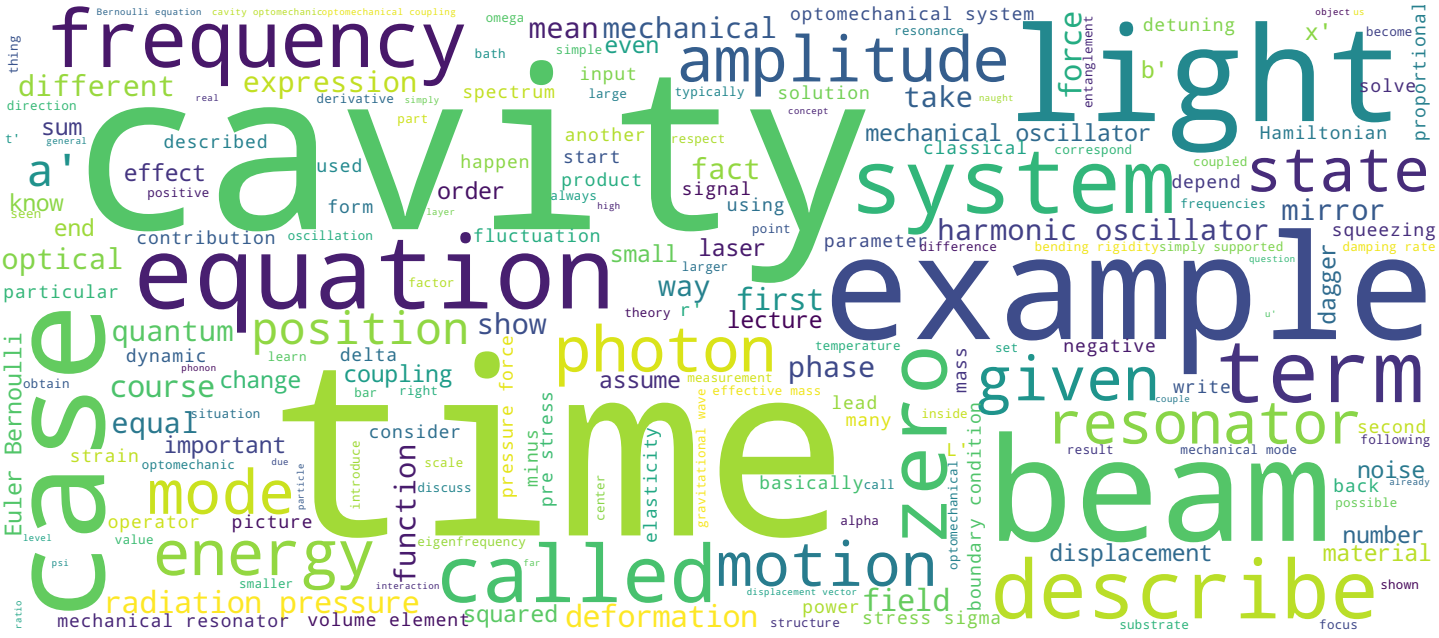

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OUTLINE

In this lecture, you will learn how to describe the vibrational modes of (nano)mechanical resonators using the theory of elasticity:

- **Deformation of a solid produces stresses and strains**
- **Bending and elongation dominate the vibration of beams**
- **The vibrational modes are characterized by their spatial profile and their eigenfrequency**
- **The dynamics of the mode maps on that of a simple harmonic oscillator**

Welcome. My name is Eva Weig and I'm a professor at the University of Konstanz in Germany. In the previous lecture, you were introduced to the model of the damped harmonic oscillator. In this lecture, we will take a closer look at mechanical resonators and you will learn about the basics of elasticity. This is lecture seven of the mooc on cavity optomechanics. Here is a brief outline. In this lecture, we will use the theory of elasticity to describe the vibrational modes of mechanical resonators. We will see how the deformation of a solid produces stresses and strains, that bending and elongation dominate the vibrations of beams and that the vibrational modes are characterized by their spatial profile and by their eigenfrequency. Finally, we will get back to the concept of the harmonic oscillator and see how the simple model introduced in the previous lecture works to accurately describe the resonator.

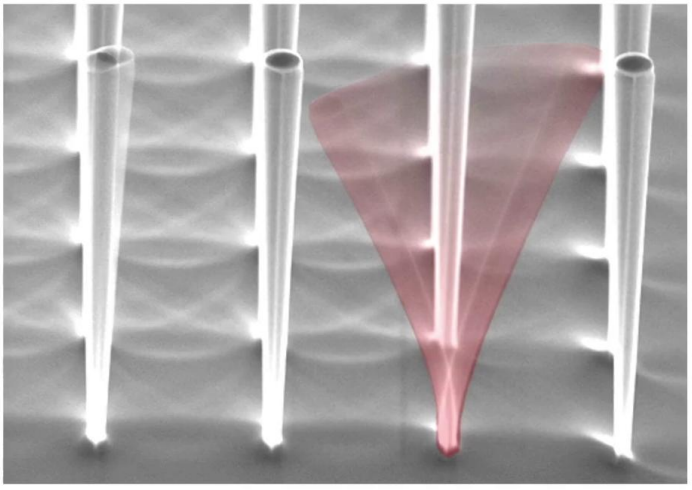
Notes

Summary

0m 00s

(NANO)MECHANICAL RESONATOR

- The mechanical elements in cavity optomechanical systems are freely suspended beams, strings, plates or membranes.
- This lecture will describe how to describe their vibrational motion.
- And we will make a connection to the simple mass-on-a-spring-model discussed in the previous lecture



This doesn't look like a mass on a spring…

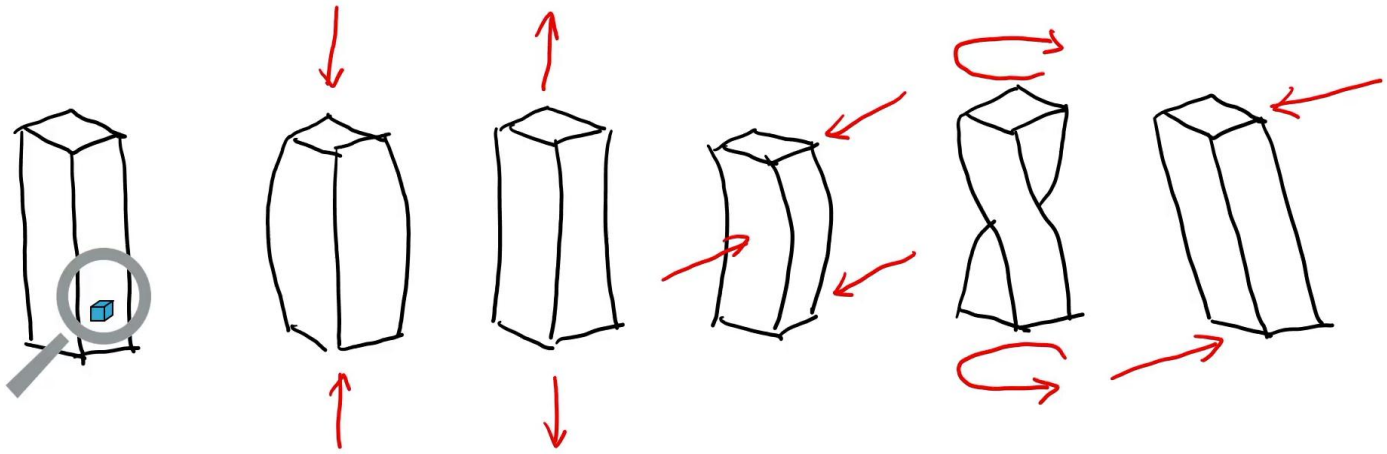
So why are we doing this? Every optomechanical system contains a mechanical element which, so far in this course, has been thought of as a mass on a spring. In reality, the mechanical resonators are freely suspended beams, strings, plates or membranes or even more complex objects. Here you can see, for example, a scanning electron micrograph of a vibrating nanobeam. In this lecture, we will learn to describe the vibrational motion of such objects and we will make a connection to the more generic mass on a spring model.

Notes

Summary

1m 05s

TYPES OF DEFORMATION



Each mechanical vibration of a solid starts with a deformation. So consider, for example, this beam. There are several types of deformation it can be subjected to. Compression, tension, bending, torsion or shear. To describe these deformations we will focus on a single volume element inside our object which will for now be represented by this little cube.

Notes

Summary

1m 45s

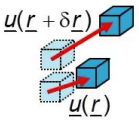
STRESSES AND STRAINS

Deformation of a continuous body:

- Displacement vector $u_i(\underline{r})$

- Strain ε_{ij}:
  Relative differential displacement

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$$



This allows us to describe the deformation of any continuous body in the framework of the theory of elasticity. Each volume element such as our little cube which we assume to sit at position 'r' of the body, is assigned a displacement vector 'u' of 'r'. It describes by how much the volume element is displaced under the deformation. In the following, I will look at the ith component of this displacement vector 'ui' of 'r' which could be along the x, y or z direction. This allows me to express things in the Einstein notation where we have to remember to sum over extra indices. Now we can introduce two important elastic quantities. The strain and the stress. The strain epsilon describes the relative differential displacement at each point of the solid, that is, the difference between the displacement of the volume element at position 'r' plus 'delta r' and that at position 'r' with respect to the equilibrium position. For small 'delta r', I can write this down as derivatives. It is important to notice that the orientation and the displacement of the two volume elements can be in different directions. So generally, the strain is a second rank tensor which I denote 'epsilon ij'.

Notes

Summary

2m 21s

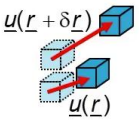
STRESSES AND STRAINS

Deformation of a continuous body:

- Displacement vector $u_i(\underline{r})$

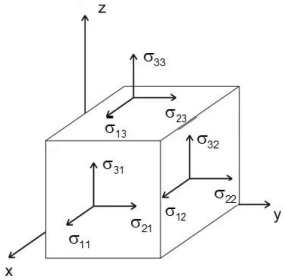
- Strain ε_{ij}:
  Relative differential displacement

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$$



- Stress σ_{ij}:
  Force per cross sectional area

$$F_i / A = \sigma_{ij} \underline{e}_j$$



'Epsilon ij' is given by the sum of the derivative of the ith component of the displacement vector with respect to the 'j' direction and the jth component of the displacement vector with respect to the 'i' direction. I neglect higher order terms but this linearized definition is generally justified for small deformations. To avoid double counting I have to multiply this with a factor of one-half. This is clearly seen when assuming 'i' equals 'j' where 'epsilon ii' is the derivative of 'ui' over 'ri' which describes a simple length change of our solid. Now the stress. The stress represents the force acting per cross-sectional area of our body and again, the force does not have to be normal to the area but can point in any direction. As illustrated in the little schematic, we can have combinations of normal and shear forces acting on each of the three facets of the cube. So also this stress 'sigma ij' is a second rank tensor. Multiplying the stress tensor by the three unit vectors 'ej', don't forget to sum over the index 'j', produces the ith component of the force per area 'A'. In the framework of elasticity, we always assume an elastic solid which will return to its initial shape once the force causing the deformation has been switched off.

Notes

Summary

3m 50s

STRESSES AND STRAINS

Deformation of a continuous body:

- Displacement vector $u_i(\underline{r})$

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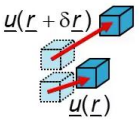
  Relative differential displacement

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$$



- Stress σ_{ij}:
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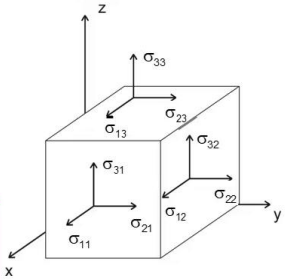
$$F_i / A = \sigma_{ij} \underline{e}_j$$



Elastic solid:

Hooke's law (microscopic version):

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{w/ stiffness tensor } C_{ijkl}$$

Then we have a famous relation between the stress and the strain which are connected by the fourth rank stiffness tensor 'Cijkl'. This is the microscopic version of Hooke's law, a very important result.

Notes

Summary

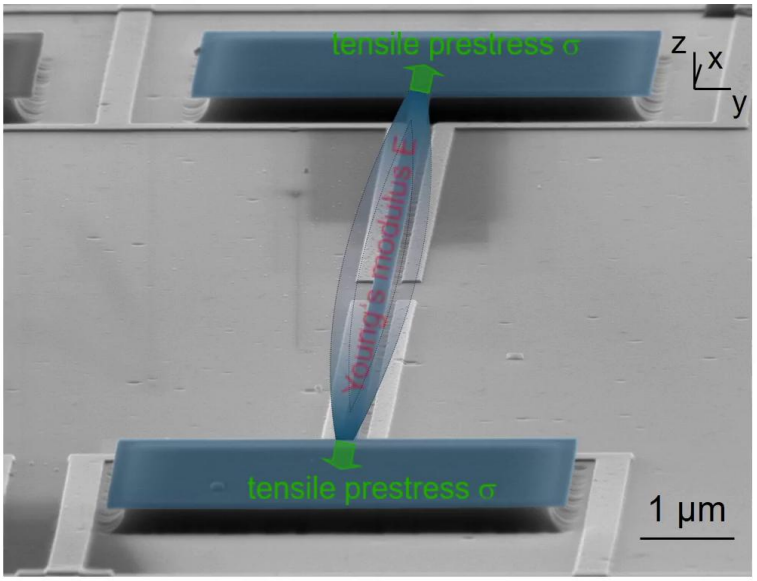
5m 25s

EULER-BERNOULLI BEAM THEORY

Assumptions:

1. homogeneous, isotropic and elastic material (Hooke‘s law)
2. transverse (flexural) vibrations
3. long and thin beam (L >> W, H): only bending & elongation , no shear & torsion

   ⇔ neutral (stress-free) axis (*Euler-Bernoulli hypothesis*)
4. uniaxial stress: only $\sigma_{xx} \neq 0 \quad \Rightarrow \quad \sigma_{xx} = E \cdot \varepsilon_{xx}$



Now we're going to apply these findings to a one-dimensional nanomechanical resonator such as this little beam shown here. We make the following assumptions. First, we consider a homogeneous isotropic and elastic material for which Hooke’s law holds. Second, we focus on transverse that is flexural vibrations such as the one schematically indicated here. Third, we assume that the beam is long and thin with a length exceeding width and thickness by at least about an order of magnitude. This ensures that only two types of deformation are at play. The bending and possibly if the beam is under a certain initial pre-stress sigma, the elongation. All other deformations such as torsion and shear can be neglected. In that case we find that the beam has a stress-free so-called neutral axis running through its center on which all forces compensate. Importantly cross sections of the beam which are initially perpendicular to the neutral axis, remain so when the beam is deflected. This is the so-called Euler-Bernoulli hypothesis. In force, we assume that the deformation of the beam only produces uniaxial stress, that is, only the 'xx' component of the stress tensor is non-zero for a beam oriented along the x-direction.

Notes

Summary

5m 41s

EULER-BERNOULLI BEAM THEORY

Assumptions:

1. homogeneous, isotropic and elastic material (Hooke‘s law)
2. transverse (flexural) vibrations
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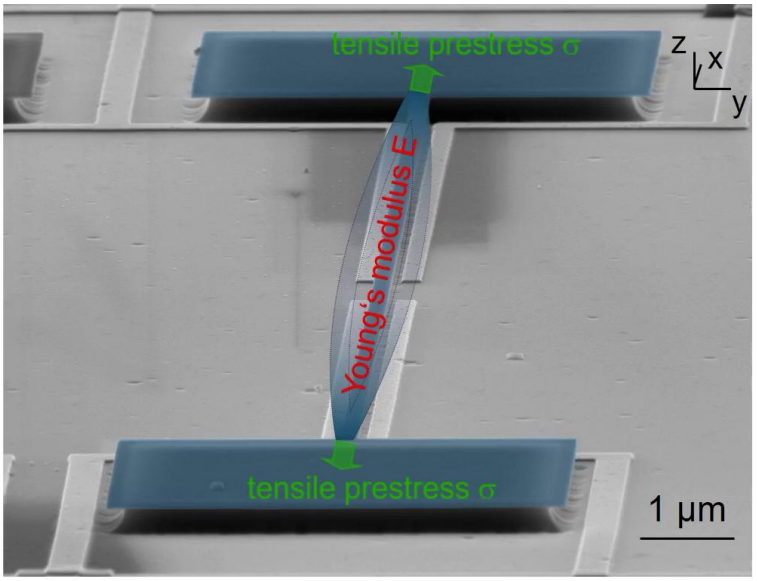


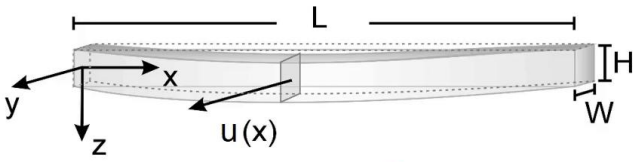
In that case, Hooke's law greatly simplifies to a scalar equation which states that sigma-xx is proportional to epsilon-xx. The proportionality constant 'E' is denoted Young's modulus and describes the bending motion of the beam.

Notes

Summary

7m 15s

EULER-BERNOULLI EQUATION



restoring forces:

$$F_{bend}(x,t) = -\,EI\,\frac{\partial^3 u(x,t)}{\partial x^3}$$

flexural rigidity

$$F_{elong}(x,t) = \sigma A\,\frac{\partial\, u(x,t)}{\partial x}$$

tensile force

use Newton's law:

$$\rho A \cdot dx\,\frac{\partial^2 u(x,t)}{\partial t^2} = F_{tot}(x+dx,t) - F_{tot}(x,t)$$

Euler-Bernoulli equation:

$$\rho A\,\frac{\partial^2 u(x,t)}{\partial t^2} + EI\,\frac{\partial^4 u(x,t)}{\partial x^4} - \sigma A\,\frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

bending — elongation

So we have the two contributions to the restoring force arising from the bending indicated in red which depends on the bending rigidity 'EI' that is the product of the Young's modulus 'E' and the area moment of inertia 'I' and the elongation of the beam depicted in green which is governed by the tensile force acting on the cross-section that is the product of the pre-stress sigma in the area of the beam 'A'. Then there is Newton's law which equates the mass of a thin slice of the beam at position 'x' times its acceleration. The mass being the product of the density rho, the cross-sectional area 'A' and its width 'dx' with the net force acting on this slice which is obtained by considering the forces exerted on the two end facets at 'x' plus 'dx' and at 'x'. Together this allows us to write down the equation of motion of the beam, the so-called Euler-Bernoulli equation. It has besides the dynamic term a characteristic term involving the fourth derivative of the displacement times the bending rigidity accounting for the bending force and, provided sigma is non-zero, a second term containing the second derivative of the displacement times the tensile force parameterizing the elongation of the vibrating beam under tension.

Notes

Summary

7m 35s

DIFFERENT BOUNDARY CONDITIONS

$$\rho A \frac{\partial^2 u(x,t)}{\partial t^2} + EI \frac{\partial^4 u(x,t)}{\partial x^4} - \sigma A \frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

separation in

position

time

1 mode shapes, eigenfrequencies

$$EI \cdot \frac{\partial^4 u(x)}{\partial x^4} + \sigma A \cdot \frac{\partial^2 u(x)}{\partial x^2} - \Omega_m{}^2 \rho A \cdot u(x) = 0$$

2 dynamics

$$\frac{\partial^2 u(t)}{\partial t^2} + \left[\Gamma \cdot \frac{\partial u(t)}{\partial t} \right] + \Omega_m{}^2 \cdot u(t) = 0$$

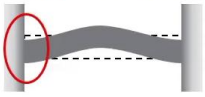
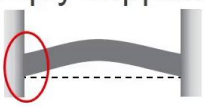
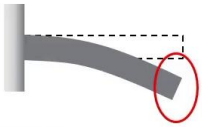
In order to solve this equation we separate the position and the time variable. This gives rise to a position dependent equation which contains the information about the mode shapes and the eigenfrequencies and to a time dependent equation which describes a harmonic vibration and I already included a damping term to be a bit more realistic here. We will now focus on the solution of the first equation.

Notes

Summary

9m 07s

DIFFERENT BOUNDARY CONDITIONS

	x = 0:	x = L:
• Clamped end:  clamped-clamped beam	$u(x=0)=0$ $u'(x=0)=0$	$u(x=L)=0$ $u'(x=L)=0$
• Simply supported end:  simply supported beam	$u(x=0)=0$ $u''(x=0)=0$	$u(x=L)=0$ $u''(x=L)=0$
• Free end:  cantilever	$u(x=0)=0$ $u'(x=0)=0$	$u''(x=L)=0$ $u'''(x=L)=0$

To proceed, we need to know about the boundary conditions of the two ends of the beam at positions zero and ‘L’. A few common boundary conditions are: the clamped condition for which both 'u' and the first derivative of 'u' must be zero at the respective end at zero or at 'L'. The simply supported condition for which 'u' and its second derivative, the curvature is set to zero at zero or at 'L'. And the free condition. For example, for the case of the cantilever, which implies for the free end that the second and third derivative of 'u' to be zero such there are neither transverse forces nor torques acting on the free end. Here we will focus on the simply supported beam with two simply supported ends which leads to a very simple solution.

Notes

Summary

9m 36s

THE SIMPLY SUPPORTED VIBRATING BEAM

- Mode shapes

$$u_n(x) = A \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right)$$

- Eigenfrequencies

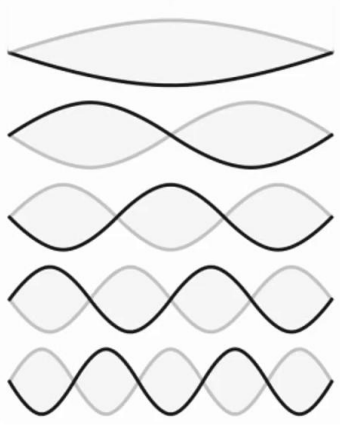
$$\omega_n = \frac{n^2\pi^2}{L^2} \cdot \sqrt{\frac{EI}{\rho A} \cdot \left(1 + \frac{\sigma A L^2}{n^2 E I \pi^2}\right)}$$

For vanishing prestress σ: $\omega_n = \frac{n^2\pi^2}{L^2} \cdot \sqrt{\frac{EI}{\rho A}}$

Euler-Bernoulli beam

For dominating prestress σ: $\omega_n = \frac{n\pi}{L} \cdot \sqrt{\frac{\sigma}{\rho}}$

vibrating string



A simple calculation for that boundary condition will show that the beam has 'n' eigenmodes where 'n' runs from one on. The modes have a sinusoidal shape and the number of nodes increases with 'n' so the fundamental mode for 'n' equals one only has nodes at the two ends of the beam whereas the second harmonic for 'n' equals two has an additional node in the center et cetera. The eigenfrequencies associated with the eigenmodes are described by this equation. Generally, the eigenfrequencies depend on the bending rigidity 'EI' and the tensile pre-stress sigma as well as on the geometry of the beam, that is, its length in cross-sectional area and, of course, a higher eigenfrequency is found for higher harmonics 'n'. The discussion is getting a bit more intuitive in the two limiting cases. For vanishing pre-stress sigma, the eigenfrequency scales as n-squared over L-squared. This is the simple Euler-Bernoulli beam. For a large pre-stress dominating over the bending rigidity, we find the eigenfrequency to scale as 'n' over 'L'. That is the solution of a vibrating string without bending rigidity.

Notes

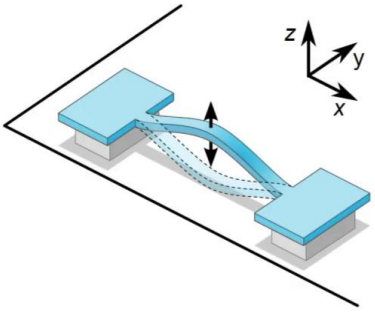
Summary

10m 42s

THE EFFECTIVE MASS

Simplification:

- Each eigenmode of a continuous object (e.g. a nano-mechanical resonator) can be described as a harmonic oscillator (i.e. a point mass) with effective parameters
- This is accomplished by integrating over the deflection profile u(x) of the entire structure:



$$m_{eff} = \frac{1}{u^2(\tilde{x})} \cdot \int_0^L \rho A \cdot u^2(x) dx$$

- A point mass of mass m_{eff} located at the point of maximum deflection of the mode $\tilde{x}$ undergoes the same dynamics as the original resonator with mass m!
- For the simply supported beam: $m_{eff} = 0.5\ m$

We know how to solve the Euler-Bernoulli equation of a beam. However, it seems complicated to always consider the beam as an extended object of position dependent displacement. But there is a solution. Each eigenmode of the resonator can be described by a harmonic oscillator. A simple point mass with effective parameters. This can be seen by integrating over the deflection profile of the entire resonator to integrate out the position coordinate. The effective mass is defined as the integral of rho times 'A' times u-squared integrated over 'x' and normalized by the squared maximum displacement which occurs at position x-tilde. For the example shown here, x-tilde would correspond to the center of the beam. A point mass of mass m-effective located at the point of maximum deflection x-tilde undergoes the same dynamics as the original resonator with mass 'm'. So the simple mass on a spring model correctly describes the resonator provided the effective mass is employed rather than the physical mass and that is, of course, much simpler than always considering the full Euler-Bernoulli theory. For the simply supported beam previously discussed all eigenmodes have the same effective mass which equals half of the physical mass.

Notes

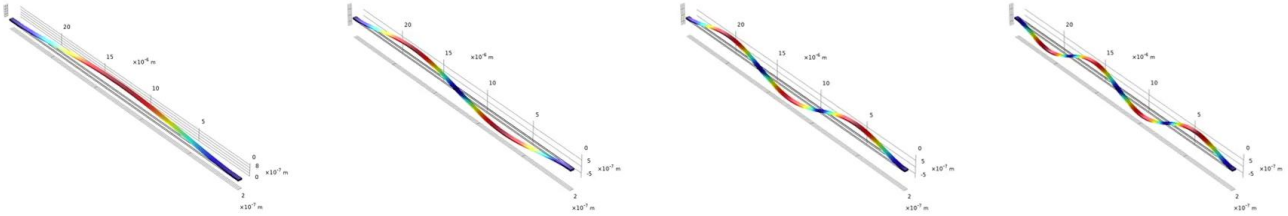
Summary

12m 08s

FURTHER EXAMPLES OF MODE SHAPES

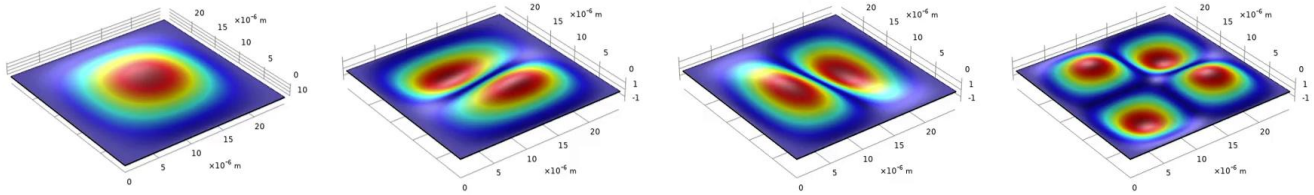
- Clamped-clamped beam:

$\omega_1/2\pi$ = 1.48 MHz $\omega_2/2\pi$ = 4.08 MHz $\omega_3/2\pi$ = 7.99 MHz $\omega_4/2\pi$ = 13.20 MHz



- Square membrane:

$\omega_{1,1}/2\pi$ = 2.44 MHz $\omega_{1,2}/2\pi$ = 4.98 MHz $\omega_{2,1}/2\pi$ = 4.98 MHz $\omega_{2,2}/2\pi$ = 7.35 MHz



- Finite Element simulations with Comsol Multiphysics®

Finally, I would like to show you some further examples beyond that of the simply supported beam while the Euler-Bernoulli equation for beams of different boundary conditions still can be solved. I chose to show you some numerical solutions here which have been obtained by finite element simulations. In the top row, you see the first four harmonics of a clamped-clamped beam of a certain geometry. And in the bottom row, the first eigenmodes of a square membrane are depicted. These results were obtained using the commercial simulation package Comsol Multiphysics. This can also be used to solve for much more complicated resonator shapes. You can hear more about that in lecture number 30 by Peter Degenfeld-Schonburg which discusses finite element simulations in detail.

Notes

Summary

13m 45s

CONCLUSION / SUMMARY

In this lesson, you learned how to describe the vibrational modes of (nano)mechanical resonators using the theory of elasticity:

- **Hooke's law relates stresses and strains in a deformed elastic solid**
- **The Euler Bernoulli equation describes the vibration of beams**
- **Boundary conditions and geometry determine its solution**
- **Introducing an effective mass allows to treat the dynamics of the resonator as that of a simple point mass**

Further reading: *Cleland „Foundations of Nanomechanics", Landau & Lifshitz "Theory of Elasticity", Schmid et al., „Fundamentals of Nanomechanical Resonators", Timoshenko et al. "Vibration Problems in Engineering", etc.*

So let me conclude. In this lecture, you learned how to describe the vibrational modes of a mechanical resonator using the theory of elasticity. In particular, you saw that Hooke's law relates stresses and strains in a deformed elastic solid. That the Euler-Bernoulli equation describes the vibrations of beams. And that the boundary conditions and geometry of the beam determine its solution. Finally, the concept of the effective mass allows to map the dynamics of the resonator to a simple mass on a spring model. If you want to read up more on this topic, there is a range of textbooks providing more information such as, for example, the books of Cleland, Landau and Lifshitz, Schmid et al. or Timoshenko.

Notes

Summary

14m 37s