POWER SPECTRAL DENSITY AND NOISE CORRELATIONS

Cavity Quantum Optomechanics

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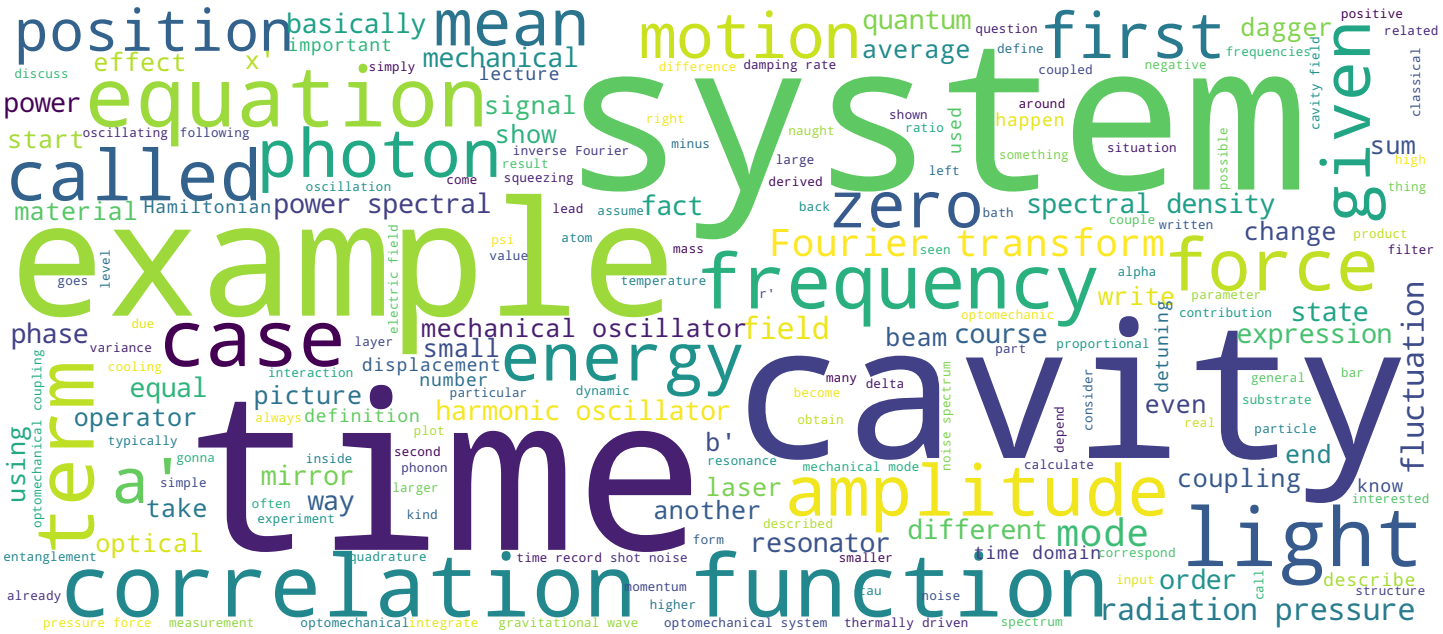

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OUTLINE

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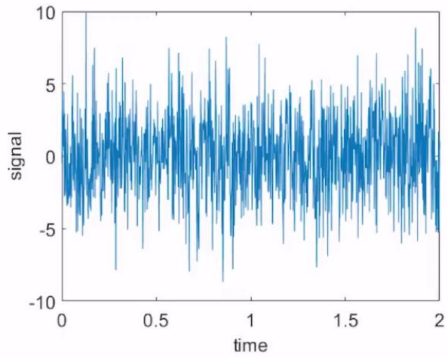
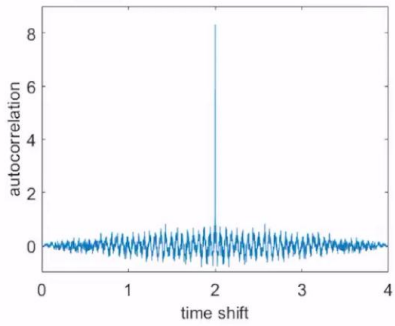
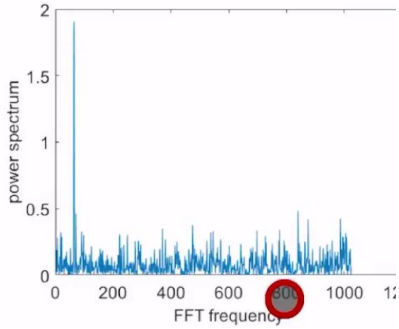
- Variance
- Wiener-Khinchine theorem
- Power spectral density
- Common voltage noise spectra
- Harmonic oscillator example
- Summary







In this lecture, I will discuss correlation functions and power spectral density. Here is a brief outline of my contents. I will start by defining correlations, correlation functions, what they are and how is the relation to variance that is typically used in describing random systems. I will derive Wiener-Khintchine theorem which relates correlation functions with power spectral density. In order to illustrate spectral densities, I will discuss a bit common voltage noise spectra and then in the end, I will discuss briefly thermally driven harmonic oscillator. On this slide I have three pictures which are illustrating what we are going to talk about so the first picture here is time domain signal. It's a sign with lots of Gaussian noise. On top of it we can calculate the the correlation function for this signal and we get this picture. The big peak at in the middle, it is coming from the delta correlated noise random fluctuations and then the oscillations down here they are related to the sinusoidal function. And the last picture here is power spectral density. It indicates a peak for the sinusoidal curve and then the rest is random variation at various frequencies.

Notes

Summary

0m 00s

CORRELATION FUNCTIONS

Correlation function can also be defined by

$$\psi_x(t_1 - t_2) = \overline{\delta x(t_1)\delta x(t_2)} = \lim_{t_m \to \infty} \frac{1}{t_m} \int_{-t_m/2}^{+t_m/2} dt \delta x(t_1 + t)\delta x(t_2 + t)$$

In **ergodic systems** this definitions are equivalent

Quantum effects assumed to be small here:
x variable is a c-number.
In quantum case, *x* replaced by operator $\hat{x}$
using Heisenberg picture

Autocorrelation at $t_1 = t_2$:

$$\psi_x(t,t) = \langle \delta x(t)\delta x(t) \rangle = \sigma^2$$

Variance of the random process

$\psi_x(t_1 - t_2) \to 0$ at large time scales

The system forgets the initial fluctuation on time scales longer than relaxation time; there may be several relaxation times

Let's define the random process x(t) with a mean 'x' in brackets. Now the fluctuation in time is given by the difference of 'x' at time instant 't' from the mean value. The correlation function is a non-random characteristic that shows how fluctuations behave in time on average. Mathematically, it's defined by this expectation where the brackets denote ensemble averaging so we average the product of the fluctuations at time instance ‘t1’ and ‘t2’ and if you write it in full form over different ensemble realizations, over ‘N’ realizations and ‘N’ goes to infinity then we get the correlation function. In stationary systems, the correlation function depends only on the time difference not on the individual times 't1' and 't2'. If you have a very long time record you can also define a correlation function by averaging over a time record. So here this is now the definition with the average in the time domain and here it means that you have a time record of length ‘tm’ and you then average over this time record. Taking this product of the fluctuations and you integrate over ‘t’ here. In ergodic systems, the two definitions are equivalent and the ergodic means that the system when it develops in time it's able to go through all possible states of the system.

Notes

Summary

1m 40s

CORRELATION FUNCTIONS

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These definitions they are now for real variables. In quantum case, 'x' would then be replaced by an operator and this you will see later on in this course. When you set 't1' equal to 't2' then you will get variance. So the autocorrelation function equals the variance of your system. In general, when one deals with long time scales then the correlation function goes to zero. This is rather evident because the correlations they will fluctuate both giving positive and negative values on large scales and then when you do an ensemble averaging these positive and negative values they will just average out. So basically the system forgets about its initial configuration. It forgets about its initial fluctuations. Thus there is relaxation in the system and, in fact, depending on the system you may even have several relaxation times.

Notes

Summary

3m 43s

POWER SPECTRAL DENSITY

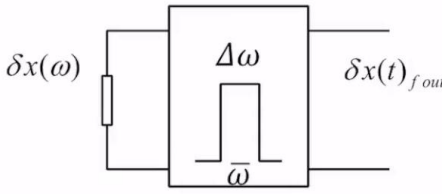
Inverse Fourier transform of correlation function:

$$\psi_x(t_1,t_2) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}\left\langle \delta x(\omega')\delta x(\omega'')\right\rangle e^{-i\omega' t_1 - i\omega'' t_2}$$

$$= \psi_x(t_1 - t_2) \text{ in stationary system}$$

$$\Rightarrow \left\langle \delta x(\omega')\delta x(\omega'')\right\rangle = 2\pi\psi_x(\omega')\delta(\omega' + \omega'')$$

$$\psi_x(t_1 - t_2) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\psi_x(\omega')e^{-i\omega'(t_1 - t_2)} \quad = \text{symmetric in } \omega$$



$$S_{xx}(f)\Delta f = \int_{\bar{\omega}-\Delta\omega/2}^{\bar{\omega}+\Delta\omega/2} \frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}\left[\delta x(\omega')e^{-i\omega' t} + \delta x^*(\omega')e^{i\omega' t}\right]\times\left[\delta x(\omega'')e^{-i\omega'' t} + \delta x^*(\omega'')e^{i\omega'' t}\right]$$

$$S_{xx}(f) = 2\int_{-\infty}^{+\infty} d(t_1 - t_2)\psi_x(t_1 - t_2)e^{-i\omega(t_1 - t_2)} \equiv 2\psi_x(\omega)$$

Wiener-Khintchine theorem

You can represent your fluctuation in Fourier space. So here is the inverse Fourier transform to get your time domain signal from the Fourier domain signal. Now because the variable ‘x’ is real then you can instead of using plus minus infinity you can use only positive frequencies and then you have to integrate over the Fourier component omega and its complex conjugate. To understand how much power we have in the fluctuations, let us analyze its spectral content using a filter. So here we have a filter with a central frequency on the bar, bandwidth delta omega and we are now interested what kind of output signal we get from the input signal which is described by the Fourier spectrum delta 'x' omega. Only frequencies within the band will go through so the output signal in time domain will be a Fourier transform or inverse Fourier transform over the frequencies of the passband of the filter. The power is related to the second power of the signal so this equation here can be used to define the power spectral density per unit bandwidth. You can also write down inverse Fourier transform for the correlation function. We are now here written the correlation function in the omega space.

Notes

Summary

4m 57s

POWER SPECTRAL DENSITY

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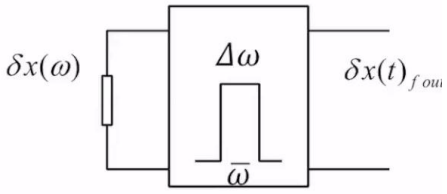
$$= \psi_x(t_1 - t_2) \text{ in stationary system}$$

$$\Rightarrow \langle \delta x(\omega')\delta x(\omega'')\rangle = 2\pi\psi_x(\omega')\delta(\omega' + \omega'')$$

= symmetric in ω

$$\psi_x(t_1 - t_2) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\psi_x(\omega')e^{-i\omega'(t_1 - t_2)}$$



$$S_{xx}(f)\Delta f = \int_{\bar{\omega}-\Delta\omega/2}^{\bar{\omega}+\Delta\omega/2} \frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}\left[\delta x(\omega')e^{-i\omega' t} + \delta x^*(\omega')e^{i\omega' t}\right]\times\left[\delta x(\omega'')e^{-i\omega'' t} + \delta x^*(\omega'')e^{i\omega'' t}\right]$$

$$S_{xx}(f) = 2\int_{-\infty}^{+\infty} d(t_1 - t_2)\psi_x(t_1 - t_2)e^{-i\omega(t_1 - t_2)} \equiv 2\psi_x(\omega)$$

Wiener-Khintchine theorem

Now this we are interested in stationary systems so this Fourier transform, inverse Fourier transform, it depends only on the time difference ‘t1’ minus ‘t2’. In order to fulfill this one can look at the exponents here and you realize that omega prime has to be minus omega double prime. So, in fact, your Fourier space correlation function has to fulfill this that there's a delta function relating omega prime and omega double prime and the correlation function depends only on one frequency. Utilizing this one can write for the inverse Fourier transform of the correlation function this equation here. Now using the above information, we can then look at the power spectral density that we derived on the previous slide and then we can see that only the cross-products here which have different signs, they will then be relevant for the spectral density. Thus we arrive that the power spectral density is two times the Fourier transform of the correlation function and this is called Wiener-Khintchine theorem.

Notes

Summary

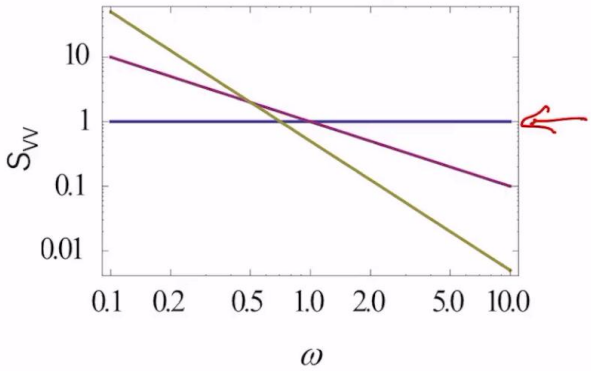
6m 50s

VOLTAGE NOISE EXAMPLES

$$S_{VV}(\omega) = \int_{-\infty}^{+\infty} d\tau\, e^{i\omega\tau} \langle v(\tau)v(0)\rangle \qquad v(t) = V(t) - \langle V \rangle$$

White noise spectrum

$$\langle v(t)v(0)\rangle \propto \delta(t)$$



Lorentzian power spectrum $\tau/(1+(\omega\tau)^2)$

$$\langle v(t)v(0)\rangle \propto \exp(-t/\tau)$$

Pink noise power spectrum (1/*f*)

$$\langle v(t)v(0)\rangle \propto \sum_k a_k \exp(-t/\tau_k)$$

Here are some examples about voltage noise. Here I'm now looking at fluctuations of voltage which is the difference between instantaneous voltage from the mean, the Fourier transform of the correlation function now without two and here it means that you have to take separately negative and positive frequencies and this is the way people very often do in quantum physics. The first example here is white noise that you can get for a delta correlated noise. For example, if you have electron emission fully randomly then you will have shot noise and the shot noise will have a wide spectrum, constant spectrum in frequency. The other very common example is Lorentzian noise spectrum that is, for example, if you have a fluctuating system, two-level fluctuator and that governs two different values for your system. The cutoff for the spectrum is given by the transient rate between these levels so if the rate is one over tau then you get this spectrum that is written down here. In time domain the same transient rate enters the correlation function which is then exponential minus 't' over tau. When you are above the cutoff then your frequency noise will go down as frequency squared.

Notes

Summary

8m 22s

VOLTAGE NOISE EXAMPLES

$$S_{VV}(\omega) = \int_{-\infty}^{+\infty} d\tau\, e^{i\omega\tau} \langle v(\tau)v(0)\rangle \qquad v(t) = V(t) - \langle V \rangle$$

White noise spectrum

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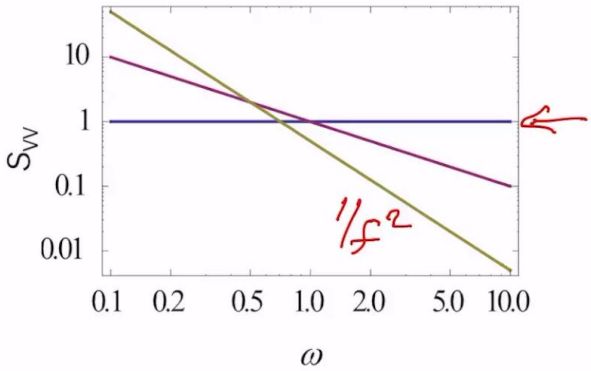


Lorentzian power spectrum $\tau/(1+(\omega\tau)^2)$

$$\langle v(t)v(0)\rangle \propto \exp(-t/\tau)$$

Pink noise power spectrum (1/*f*)

$$\langle v(t)v(0)\rangle \propto \sum_k a_k \exp(-t/\tau_k)$$

Third very common example is 'one over f' noise, pink noise. The most common theories for this noise is that it's a sum over many fluctuators each having this exponential correlation function. The rates one over tau-k they are broadly distributed and then you get noise power which is close to 'one over f'.

Notes

Summary

10m 05s

EXAMPLE: HARMONIC OSCILLATOR

$$\begin{cases} x(t) = x(0)\cos(\Omega t) + p(0)\dfrac{1}{M\Omega}\sin(\Omega t) \\ p(t) = p(0)\cos(\Omega t) - x(0)M\Omega^2\sin(\Omega t) \end{cases}$$

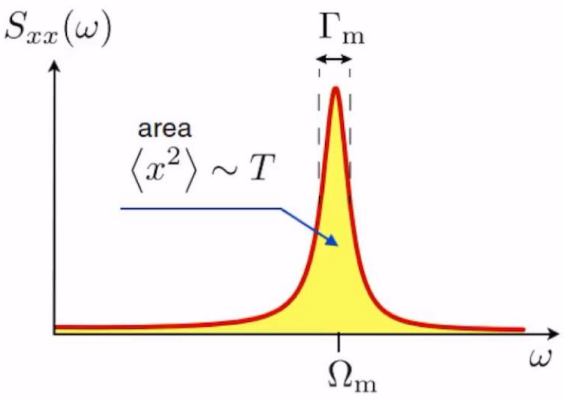
$$\psi_x(t) = \langle x(0)x(0)\rangle\cos(\Omega t) + M^{-1}\Omega^{-1}\langle p(0)x(0)\rangle\sin(\Omega t)$$

Equipartition theorem

$$\frac{1}{2}M\Omega^2\langle x^2\rangle = \frac{1}{2}k_B T$$

$$S_{xx}(\omega) = \pi\frac{k_B T}{M\Omega^2}\left[\delta(\omega-\Omega)+\delta(\omega+\Omega)\right]$$

Indeed symmetric in frequency



$$\int_{-\infty}^{+\infty} S_{xx}(\omega)\frac{d\omega}{2\pi} = \langle x^2\rangle = \frac{k_B T}{M\Omega^2}$$

Finally, harmonic oscillator thermally driven. It's easy to write down the apertures for your oscillator 'x' and 'p', position and momentum. Now if you multiply the above equation with x-zero and you integrate over time realizations then you get the correlation function. The latter term that averages out and you're left with the first term here. Now in order to evaluate the pre-factor, we can use the Equipartition theorem that relates the thermal energy to the mean squared fluctuation of your system so basically this is the potential energy of your thermally driven harmonic oscillator. So now if you then Fourier transform the cosine omega 't', plug in the pre-factor, you get the power spectral density in the square box and you can see that it has both omega and minus omega for the rest frequency so this is indeed symmetric in frequency. Now if you integrate this power spectral density, you get the mean square displacement in your device which is then related with the temperature. Now in the picture above we have added some damping to broaden the peak but, of course, in this case the area under the curve is the same.

Notes

Summary

10m 34s

SUMMARY

- Correlation function: in ensemble & time domain $\psi_x(t_1,t_2)=\langle \delta x(t_1)\delta x(t_2)\rangle$
- Stationary system: only time difference dependence $\psi_x(t_1,t_2)=\psi_x(t_1-t_2)$
- "Power spectral density is (twice) the Fourier transform of the autocorrelation function"
  = ***Wiener-Khintchine theorem*** $S_{xx}(f)=2\int_{-\infty}^{+\infty} d(t_1-t_2)\psi_x(t_1-t_2)e^{-i\omega(t_1-t_2)}$
- Classical noise:
  real function – symmetric spectral density
- Major noise categories:
  white noise, fluctuator noise (Lorentzian spectrum), 1/*f* noise
- Thermal noise in a harmonic oscillator $S_{xx}(\omega)=\pi\frac{k_B T}{M\Omega^2}\left[\delta(\omega-\Omega)+\delta(\omega+\Omega)\right]$

To sum up, we have discussed correlation functions both in ensemble of systems and in time domain. We have mostly been dealing with stationary systems. We derived the Wiener-Khintchine theorem in these systems. We briefly discussed major noise examples. White noise, fluctuator noise and 'one over f' noise and in the end we derived the power spectral density for position in a thermally driven harmonic oscillator.

Notes

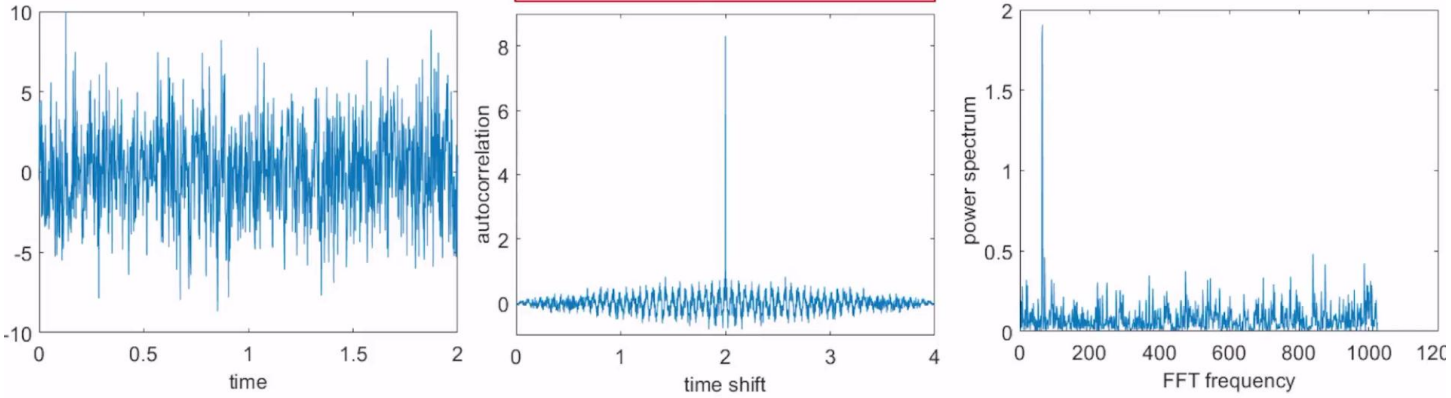
Summary

12m 27s

TAKE HOME MESSAGE

SIGNAL ➡ AUTOCORRELATION ➡ POWER SPECTRUM

$$\psi_x(t_1,t_2) = \langle \delta x(t_1)\delta x(t_2) \rangle$$



$$S_{xx}(f) = 2\int_{-\infty}^{+\infty} d(t_1 - t_2)\psi_x(t_1 - t_2)e^{-i\omega(t_1-t_2}$$

With this take-home message slide, I will thank you for your attention.

Notes

Summary

13m 04s