

EPFL

OUTLINE

- Brownian motion
- **Langevin equation** of motion
- Formal solution of Langevin equation
- Correlation function from Langevin equation
- Thermal equilibrium
- Power spectral density
- Thermally driven harmonic motion in 1D
- Summary

In this lecture, I will describe Brownian motion which is a random motion of small particles encountered in many physical systems. Here is a brief outline of the contents of this lecture. I will start on Brownian motion in general then I will define the Langevin equation of motion which is the basic tool for describing the physics of Brownian motion. I will derive a formal solution of Langevin equation and then using this formal solution I will derive correlation functions that then can be employed to calculate correlations in thermal equilibrium and to derive power spectral densities for this system. And then I will discuss thermally driven harmonic oscillator in 1D and show its response on random force.

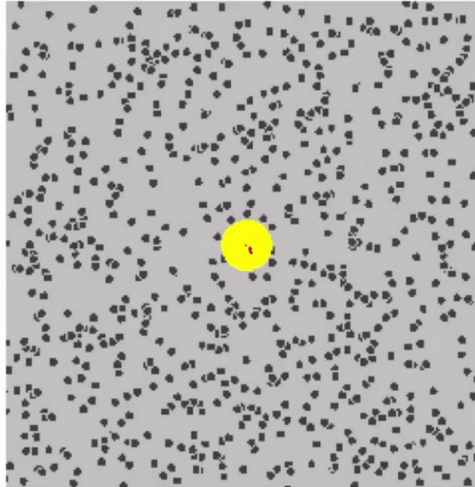
Notes

Summary



BROWNIAN MOTION

- Brownian motion of a big (dust) particle
- Particle collides with a large set of smaller (gas) particles
- Gas molecules move with different velocities in random directions.



Wikipedia



Robert Brown (1773-1858)

Brownian motion is named after the botanist, Robert Brown who first described the phenomenon in 1827 while looking through a microscope at pollen of a plant which was immersed in water. In 1905, Albert Einstein, he studied this problem and published a paper where he modeled the motion of pollen as being moved by individual kicks of water molecules and this is the approach that we are following here so here you can see a yellow particle, dust particle for example, immersed into a fluid, gas or liquid having small particles, black dots here and these black dots particles they move randomly with different velocities different directions and kick the yellow particle and this kind of behavior can be simulated and the random collisions they will move the bigger particle in random fashion through the fluid.

Notes

Summary



1m 02s

LANGEVIN EQUATION OF MOTION

$$m \frac{dv(t)}{dt} = -\gamma v(t) + \xi(t)$$

- since $\xi(t)$ is a stochastic variable, so are $v(t)$ and $x(t)$

- Particle of **mass** m
- Immersed into a fluid of molecules with mass $< m$
- The fluid gives rise to retarding motion by **friction** γ
- Friction force is proportional to **velocity** $v(t)$
- **Random force** $\xi(t)$ due to random density fluctuations owing to molecules undergoing random motion

Dynamic viscosity

$$\gamma = 6\pi\eta R$$

Stokes' law

Here is the equation of motion. For the particle we have the acceleration term, mass of the particle is 'm', v is the velocity and then here we have the friction force dependent on the velocity of the system and then there is a friction coefficient and ξ here that denotes the random force, stochastic random force on the particle. Now in a fluid one can write for the friction coefficient a formula, Stokes' law which relates the friction to the dynamic viscosity and the size of the particle so the bigger is the particle, the bigger is the friction force.

Notes

Summary



FORMAL SOLUTION OF LANGEVIN EQUATION

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) + \frac{1}{m}\xi(t)$$

Initial conditions

$$x(0) = 0; \quad v(0) = v_0$$

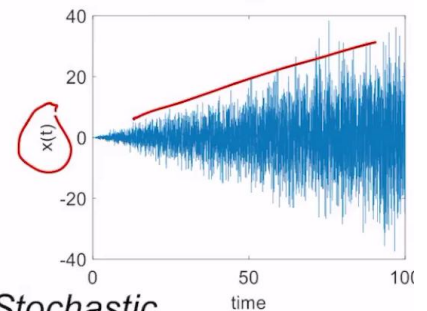
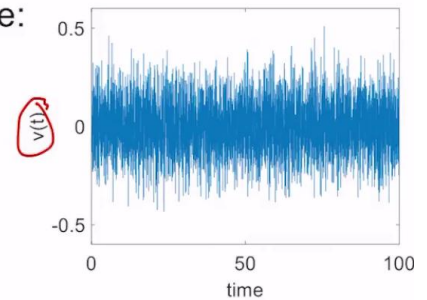
Properties of random force:

$$\langle \xi(t) \rangle_{\xi} = 0$$

$$\langle \xi(t_1)\xi(t_2) \rangle_{\xi} = g\delta(t_1 - t_2)$$

$$\Rightarrow v(t) = v_0 e^{-\frac{\gamma}{m}t} + \frac{1}{m} \int_0^t ds e^{-\frac{\gamma}{m}(t-s)} \xi(s)$$

$$x(t) = \frac{m}{\gamma} v_0 \left(1 - e^{-\frac{\gamma}{m}t} \right) + \frac{1}{\gamma} \int_0^t ds \left(1 - e^{-\frac{\gamma}{m}(t-s)} \right) \xi(s)$$



Stochastic functions

To find the formal solution of the Langevin equation we need the properties of the random force so the mean value of the random force is, of course, zero and then at different times the forces are fully uncorrelated so it's fully a random force and the correlation function of the random force is given by strength 'g' times delta function of 't1' minus 't2'. Now the initial conditions that we use are zero for the position and some velocity v_0 for the velocity then one can rather straightforwardly solve the equation. We have a term which depends on the initial velocity and it goes down due to the damping exponentially down and then we have the term which contains the random kicks of the random force and they are damped by the friction and then all these kicks are integrated from zero to time 't'. Similarly one can find the solution for x, the position and, of course, one can easily check by taking the derivative of this form then one gets the upper formula for the velocity. One can also study this graphically so up here we have the velocity as a function of time so it's a fully random looking trace, white noise looking trace whereas for the position, we have a slightly more complicated form where the amplitude goes up with time, the fluctuations go up in time.

Notes

Summary



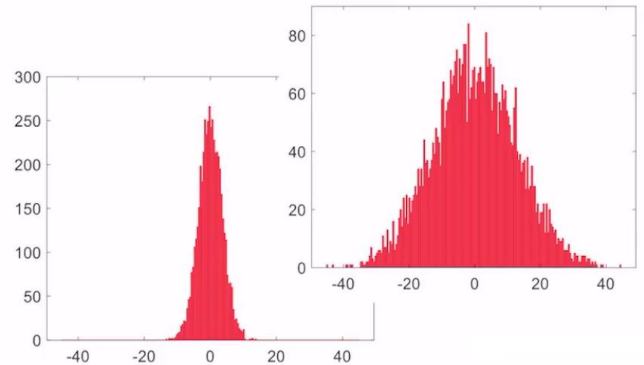
CORRELATION FUNCTION FROM LANGEVIN EQUATION

$$v(t) = v_0 e^{-\frac{\gamma}{m}t} + \frac{1}{m} \int_0^t ds e^{-\frac{\gamma}{m}(t-s)} \xi(s)$$

$$\begin{aligned} \langle v(t_1)v(t_2) \rangle_\xi &= v_0^2 e^{-\frac{\gamma}{m}(t_1+t_2)} + \frac{g}{m^2} \int_0^{t_2} ds_2 \int_0^{t_1} ds_1 \delta(s_2-s_1) e^{-\frac{\gamma}{m}(t_1-s_1)} e^{-\frac{\gamma}{m}(t_2-s_2)} \\ &= \frac{g}{2m\gamma} e^{-\frac{\gamma}{m}(|t_1-t_2|)} + \left(v_0^2 - \frac{g}{2m\gamma} \right) e^{-\frac{\gamma}{m}(t_1+t_2)} \end{aligned}$$

$\langle \xi(t_1)\xi(t_2) \rangle_\xi = g\delta(t_1-t_2)$

$$\begin{aligned} \langle x(t)x(t) \rangle_\xi &= \frac{m^2}{\gamma^2} \left(v_0^2 - \frac{g}{2m\gamma} \right) \left(1 - e^{-\frac{\gamma}{m}t} \right)^2 \\ &\quad + \frac{g}{\gamma^2} \left(t - \frac{m}{\gamma} \left(1 - e^{-\frac{\gamma}{m}t} \right) \right) \end{aligned}$$



Now when we have the formal solution, we can then calculate the correlation function, through the product of the velocity at two time instances 't1' and 't2'. Now in this product we have then a product of two random forces and then, of course, we can use the correlation function of the random forces to simplify the double integral over 's1' and 's2' so we have a delta function due to the correlations of the random forces. So this integral is rather easy to solve to calculate and you get two terms. One is having the difference of 't1' and 't2' and the other one has a sum of 't1' and 't2' in the exponent. Similarly you can work out the correlation function for position. Here we only take the autocorrelation function for the position and now you see very similar in terms as in the velocity-velocity correlation function.

Notes

Summary



FORMAL SOLUTION OF LANGEVIN EQUATION

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) + \frac{1}{m}\xi(t)$$

Initial conditions

$$x(0) = 0; \quad v(0) = v_0$$

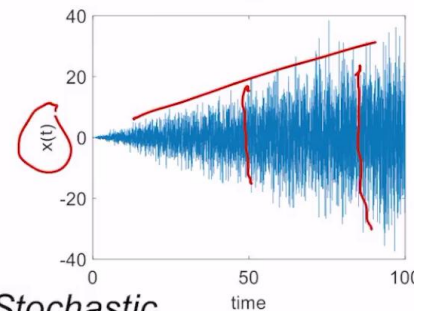
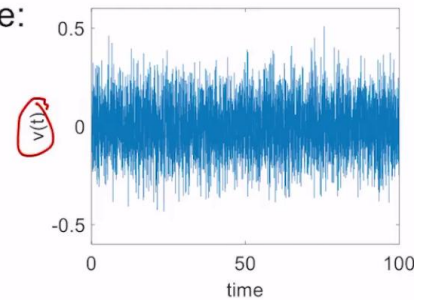
Properties of random force:

$$\langle \xi(t) \rangle_{\xi} = 0$$

$$\langle \xi(t_1)\xi(t_2) \rangle_{\xi} = g\delta(t_1 - t_2)$$

$$\Rightarrow v(t) = v_0 e^{-\frac{\gamma}{m}t} + \frac{1}{m} \int_0^t ds e^{-\frac{\gamma}{m}(t-s)} \xi(s)$$

$$x(t) = \frac{m}{\gamma} v_0 \left(1 - e^{-\frac{\gamma}{m}t} \right) + \frac{1}{\gamma} \int_0^t ds \left(1 - e^{-\frac{\gamma}{m}(t-s)} \right) \xi(s)$$



Stochastic functions

Notes

Summary



6m 34s

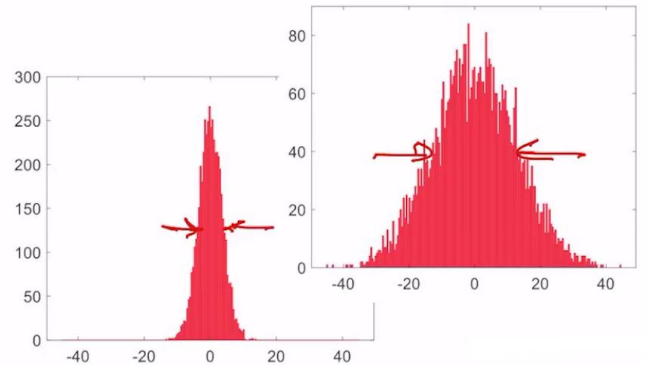
CORRELATION FUNCTION FROM LANGEVIN EQUATION

$$v(t) = v_0 e^{-\frac{\gamma}{m}t} + \frac{1}{m} \int_0^t ds e^{-\frac{\gamma}{m}(t-s)} \xi(s)$$

$$\begin{aligned} \langle v(t_1)v(t_2) \rangle_\xi &= v_0^2 e^{-\frac{\gamma}{m}(t_1+t_2)} + \frac{g}{m^2} \int_0^{t_2} ds_2 \int_0^{t_1} ds_1 \delta(s_2-s_1) e^{-\frac{\gamma}{m}(t_1-s_1)} e^{-\frac{\gamma}{m}(t_2-s_2)} \\ &= \frac{g}{2m\gamma} e^{-\frac{\gamma}{m}(|t_1-t_2|)} + \left(v_0^2 - \frac{g}{2m\gamma} \right) e^{-\frac{\gamma}{m}(t_1+t_2)} \end{aligned}$$

$\langle \xi(t_1)\xi(t_2) \rangle_\xi = g\delta(t_1-t_2)$

$$\begin{aligned} \langle x(t)x(t) \rangle_\xi &= \frac{m^2}{\gamma^2} \left(v_0^2 - \frac{g}{2m\gamma} \right) \left(1 - e^{-\frac{\gamma}{m}t} \right)^2 \\ &\quad + \frac{g}{\gamma^2} \left(t - \frac{m}{\gamma} \left(1 - e^{-\frac{\gamma}{m}t} \right) \right) \end{aligned}$$



Of course, you can also look at the calculated $x(t)$ so the autocorrelation is the variance so you can then take some positions from here at different times and then make distributions as shown here and then you can see that the variance of these distributions, they grow linearly in time which indicates that basically this term here is the most significant one with the parameters used in these calculations.

Notes

Summary



THERMAL EQUILIBRIUM

$$\left\{ \begin{aligned} \langle v(t_1)v(t_2) \rangle_\xi &= \frac{g}{2m\gamma} e^{-\frac{\gamma}{m}(|t_1-t_2|)} + \left(v_0^2 - \frac{g}{2m\gamma} \right) e^{-\frac{\gamma}{m}(t_1+t_2)} \\ \langle x(t)x(t) \rangle_\xi &= \frac{m^2}{\gamma^2} \left(v_0^2 - \frac{g}{2m\gamma} \right) \left(1 - e^{-\frac{\gamma}{m}t} \right)^2 + \frac{g}{\gamma^2} \left(t - \frac{m}{\gamma} \left(1 - e^{-\frac{\gamma}{m}t} \right) \right) \end{aligned} \right.$$

In thermal equilibrium:

$$\begin{aligned} \langle v_0^2 \rangle_T &= \frac{kT}{m} \\ \Rightarrow g &= 2m\gamma \langle v_0^2 \rangle_T = 2kT\gamma \end{aligned}$$

Correlation functions in thermal equilibrium:

$$\left\{ \begin{aligned} \langle \langle v(t_1)v(t_2) \rangle_\xi \rangle_T &= \frac{kT}{m} e^{-\frac{\gamma}{m}(|t_1-t_2|)} \\ \langle \langle x(t)x(t) \rangle_\xi \rangle_T &= \frac{2kT}{\gamma} \left(t - \frac{m}{\gamma} \left(1 - e^{-\frac{\gamma}{m}t} \right) \right) \rightarrow \frac{2kT}{\gamma} t = Dt \end{aligned} \right.$$

Diffusion with

$$D = \frac{2kT}{\gamma}$$

$$\rightarrow L = \sqrt{Dt}$$

In thermal equilibrium, things simplify a bit. So in thermal equilibrium, the squared velocity depends on temperature so it's 'kT' over 'm' due to equipartition theorem. Using this one can argue that 'g' is equal to 'two kT gamma' so if you take this and plug it in to this equation then you find out that the autocorrelation that you get from here equals exactly this v-zero squared so it's consistent. So the strength of the random force in thermal equilibrium is given by temperature and the friction coefficient gamma. So now if you then plug in 'g' to this formulas you get simple exponential behavior for the velocity-velocity correlation function and for the autocorrelation for the position, you get a formula that at large times is linear in time, grows linearly with time and then it has a diffusion constant in front so we see that the variance grows linearly in time so the spreading distance is a square root of time so the square root of diffusion constant times time and the diffusion constant depends on temperature, it grows with temperature and it goes down with the friction coefficient.

Notes

Summary



POWER SPECTRAL DENSITY

$$S_{v,v}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \left\langle \left\langle v(t_1 + \tau)v(t_1) \right\rangle_{\xi} \right\rangle_T \quad \leftarrow \left\langle \left\langle v(t_1)v(t_2) \right\rangle_{\xi} \right\rangle_T = \frac{kT}{m} e^{-(\gamma/m)(|t_1-t_2|)}$$

$$S_{v,v}(\omega) = \frac{2kT}{m} \frac{\gamma/m}{\omega^2 + (\gamma/m)^2}$$

$$S_{v,v}(\omega) = \frac{1}{m^2} \frac{S_{\xi,\xi}(\omega)}{\omega^2 + (\gamma/m)^2}$$

Spectral properties of random force

$$S_{\xi,\xi}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \left\langle \left\langle \xi(t_1 + \tau)\xi(t_1) \right\rangle_{\xi} \right\rangle_T \quad \leftarrow \left\langle \xi(t_1)\xi(t_2) \right\rangle_{\xi} = g\delta(t_1 - t_2)$$

$$S_{\xi,\xi}(\omega) = g = 2\gamma kT$$

$$g = 2kT\gamma$$

When you know the correlation functions you can then calculate the power spectral density by just Fourier transforming the correlation function so for velocity-velocity correlation function by plugging this exponential form into the Fourier transform formula we get for the power spectrum for velocity in Brownian motion this formula and we can see that it's Lorentzian shaped with a cut-off given by gamma over 'm'. Now in the same fashion you can calculate the the spectral properties of the random force so you plug in to your Fourier transform the correlation function of the random force. Now we know that 'g' is 'two kT gamma' and then we get that the power spectral density of the random force is given by 'two gamma kT'. So the larger is the friction, the bigger is the force, the power spectrum of the random force and also the temperature increases the power spectrum of the force.

Notes

Summary



THERMALLY DRIVEN HARMONIC MOTION IN 1D

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m} v(t) - \omega_0^2 x + \frac{1}{m} \xi(t) \quad \frac{dx(t)}{dt} = v(t) \quad \langle \xi(t_1) \xi(t_2) \rangle_\xi = g \delta(t_1 - t_2)$$

$$-i\omega v(\omega) = -\frac{\gamma}{m} v(\omega) - \omega_0^2 x(\omega) + \frac{1}{m} \xi(\omega)$$

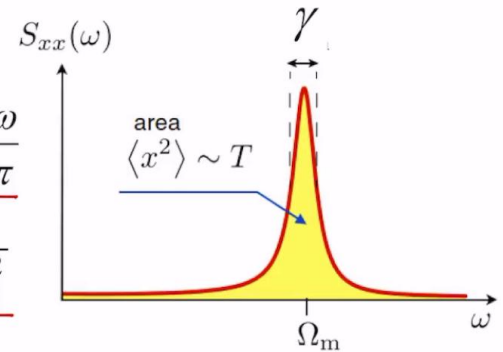
$$-i\omega x(\omega) = v(\omega)$$

$$S_{\xi, \xi}(\omega) = g = 2kT\gamma$$

$$x(\omega) = \frac{1}{m} \frac{\xi(\omega)}{\omega_0^2 - \omega^2 - i\frac{\gamma}{m}\omega}$$

$$S_{x,x}(\omega) = \frac{1}{m^2} \frac{S_{\xi, \xi}(\omega)}{\left| \omega_0^2 - \omega^2 - i\frac{\gamma}{m}\omega \right|^2}$$

$$\int_{-\infty}^{+\infty} S_{xx}(\omega) \frac{d\omega}{2\pi} = \langle x^2 \rangle = \frac{kT}{m\omega_0^2}$$



As you saw, it's a bit delicate to deal in time domain. So now in this application of this Brownian motion inbound harmonic particle. I will switch to Fourier treatment so highlighted with yellow here you have the basic formulas. Now omega zero is the Eigen frequency of the harmonic oscillator. Without damping, we can Fourier transform all these three equations and we get these three equations shown here on the left. It's easy to solve now the the displacement components at frequency omega by solving these two equations. On the left you see that it depends now on the random force and then the response function of your resonator. From here by taking the square you can then get the power spectrum of the fluctuations and the random force component gives now the power spectrum of the random force here and this we know, this is now 'two kT gamma'. Now if you integrate this formula, you will get that the mean square displacement is 'kT' over 'm' omega zero squared. This now, of course, assumes that dissipation is not large. That the frequency, Eigen frequency is roughly omega zero even though there is some dissipation in the system.

Notes

Summary



THERMALLY DRIVEN HARMONIC MOTION IN 1D

$$S_{x,x}(\omega) = \frac{1}{m^2} \frac{2kT\gamma}{(\omega_0^2 - \omega^2)^2 + \frac{\gamma^2}{m^2} \omega^2}$$

Fourier transform of derivative:

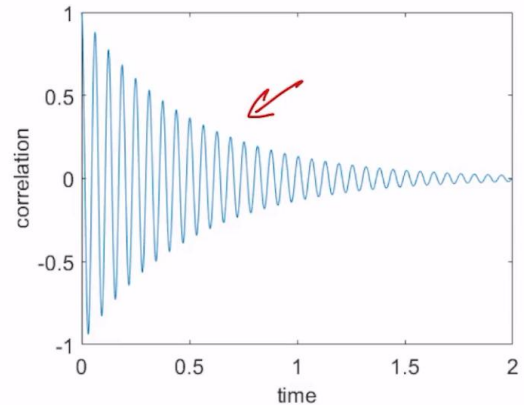
$$v = \frac{dx}{dt} \rightarrow S_{v,v}(\omega) = \omega^2 S_{x,x}(\omega)$$

inverse **Wiener-Khintchine theorem**

$$\psi_x(\tau) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} S_{x,x}(\omega') e^{-i\omega'\tau}$$

$$\psi_x(\tau) = \frac{kT}{m\omega_0^2} e^{-\frac{\gamma}{2m}\tau} \left\{ \cos \omega_1 \tau + \frac{\gamma}{2m\omega_1} \sin \omega_1 \tau \right\}$$

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}} \approx \omega_0 \quad \frac{\gamma}{m} \ll \omega_0$$



Now when you know the power spectrum of fluctuations then you can use that to, for example, calculate the velocity-velocity spectrum. So you just use the correlation, the connection between velocity and position, and then you get a term, omega squared, multiplying your displacement power spectrum. Also you can use the power spectrum to derive the correlation function. You use the inverse Wiener-Khintchine theorem and you Fourier transform your power spectrum into a time domain correlation function. By this, one obtains a formula where this omega one here, it's now the basic non-dissipative Eigen frequency minus a term depending on the dissipation. But we are interested mostly in systems where this dissipation is small so this is roughly omega zero and this is valid when this condition is applicable. In this case we can then neglect this one and we have a correlation function where we have the exponentially decaying part and then we have the oscillating cosine part and that's indicated in this graph here.

Notes

Summary



SUMMARY

- Brownian motion $\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) + \frac{1}{m}\xi(t)$ $x(t) = \frac{1}{\gamma} \int_0^t ds \left(1 - e^{-(\gamma/m)(t-s)}\right) \xi(s)$
- Langevin equation
- Formal solution of Langevin equation $L = \sqrt{D\tau}$ $D = \frac{2kT}{\gamma}$ $S_{\xi,\xi}(\omega) = 2\gamma kT$
- Correlation function from Langevin equation
- Thermal equilibrium $S_{v,v}(\omega) = \frac{1}{m^2} \frac{2kT\gamma}{\omega^2 + (\gamma/m)^2}$
- Power spectral density
- Thermally driven harmonic motion in 1D $S_{x,x}(\omega) = \frac{1}{m^2} \frac{2kT\gamma}{(\omega_0^2 - \omega^2)^2 + \frac{\gamma^2}{m^2}\omega^2}$

To sum up, we have used Langevin equation to describe Brownian motion. We derived a formal solution of the Langevin equation and we used that to derive the correlation functions for Brownian motion. In thermal equilibrium we derived that the diffusion length is square root of diffusion constant times time and the diffusion constant is related to temperature and the friction force gamma. In thermal equilibrium we also derived that the power spectrum of the random force is 'two gamma kT'. For the velocity fluctuations, the power spectrum of them for Brownian motion, we derived an equation which has Lorentzian form and it has a cut-off gamma over 'm'. For the thermally driven harmonic motion, we derive the displacement power spectrum and found that it's related to the spectrum on the random force and the response function of the oscillator.

Notes

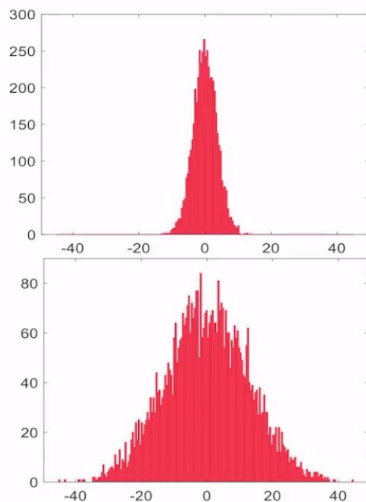
Summary



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TAKE HOME MESSAGE

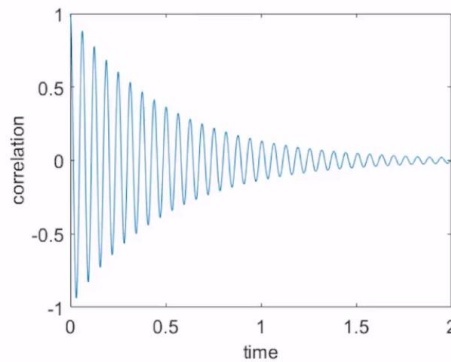
Brownian Diffusion



$$D = \frac{2kT}{\gamma}$$

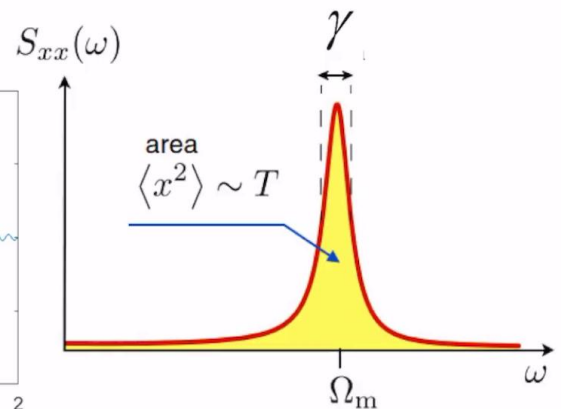
$$L = \sqrt{D\tau}$$

Correlation function for harmonic particle



$$\psi_v(\tau) \propto e^{-\frac{\gamma}{m}|\tau|} \cos(\omega_0 \tau)$$

Power spectrum



$$S_{x,x}(\omega) = \frac{1}{m^2} \frac{S_{\xi,\xi}(\omega)}{\left| \omega_0^2 - \omega^2 - i \frac{\gamma}{m} \omega \right|^2}$$

With this take-home message slide, I want to thank you for your attention.

Notes

Summary

15m 11s

