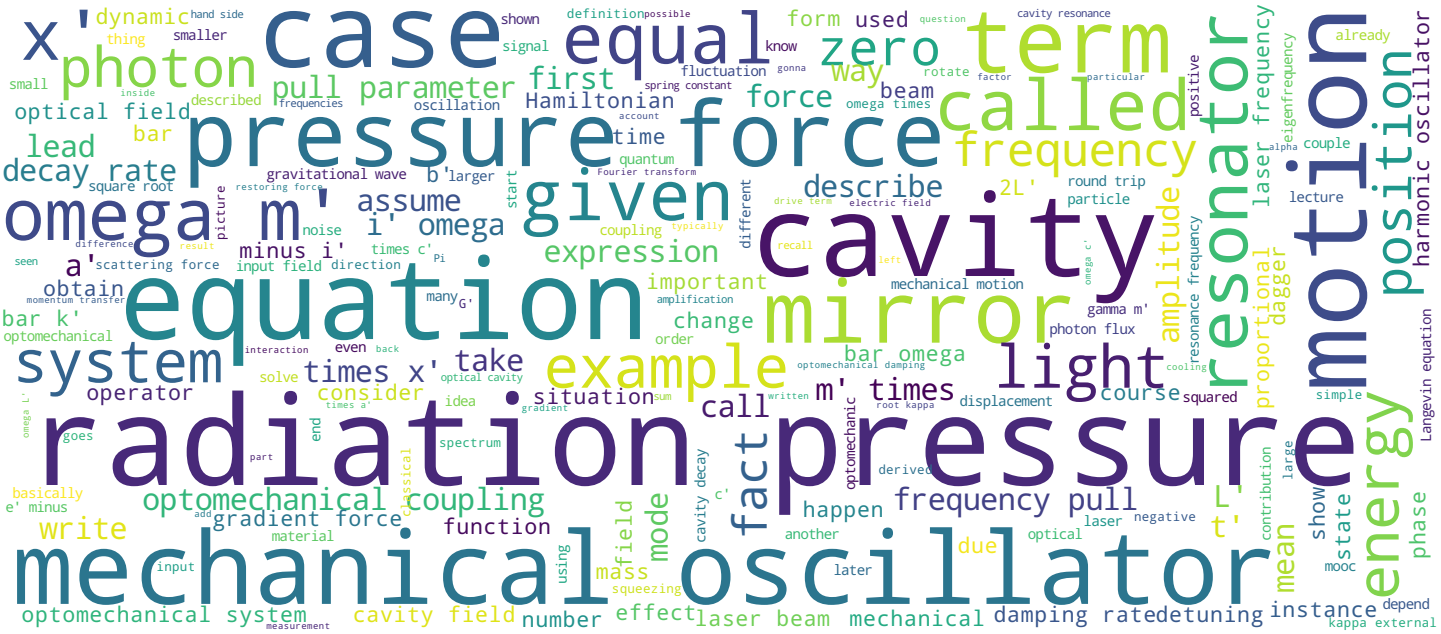
Optomechanical coupling and equations of motion

Cavity Quantum Optomechanics

Tobias Kippenberg

Ecole Polytechnique Fédéral de Lausanne - EPFL

EPFL



Search MOOC

Video

EPFL

OUTLINE

In this lecture, you will learn how to describe the classical dynamics of an optomechanical system:

- **Radiation pressure and optomechanical coupling**
- **Equations of motion for an optomechanical system**

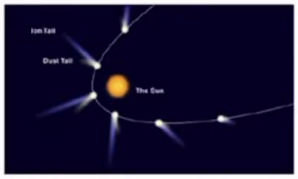
Hello. My name is Tobias Kippenberg. I work at EPFL, the Swiss Federal Institute of Technology in Lausanne, Switzerland and I'd like to talk today about the optomechanical coupling and the corresponding equations of motions that result from that. So in this mooc, what you will learn are how to describe the dynamics of an optomechanical systems and I will discuss radiation pressure and how it leads to optomechanical coupling and I'll describe how you can derive the equations of motion of the optomechanical system, that is, the mechanical oscillator and the optical field.

Notes

Summary

0m 00s

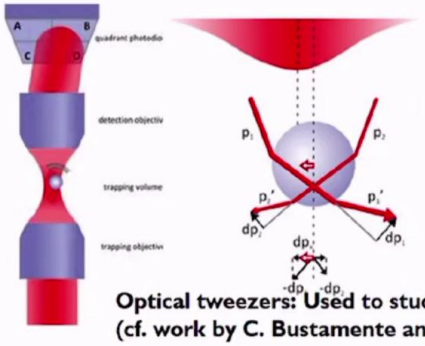
Radiation Pressure



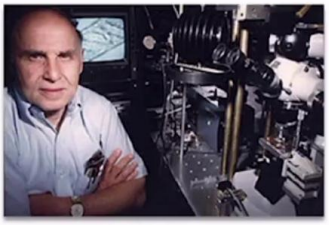
De Cometis 1619

P. Lebedev, 1901, "Untersuchungen über die Druckkräfte des Lichtes", Annalen der Physik, 1901

scattering force



Arthur Ashkin (Bell Labs)



Optical tweezers: Used to study the motion of molecular motors (cf. work by C. Bustamente and Steve Block (Stanford)

Terminology Note: The transverse light forces are called ***gradient*** forces as opposed to the forces in the propation direction (***scattering*** force)

A. Ashkin, "Trapping of Atoms by Resonance Radiation Pressure," Phys. Rev. Lett.40, 729 (1978).
S. Chu et al., "Three-Dimensional Viscous Confinement and Cooling of Atoms by Resonance Radiation Pressure," Phys. Rev. Lett. 55, 48 (1985).

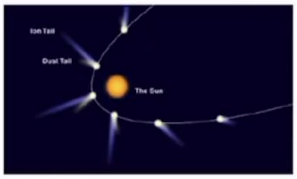
Now as a short introduction the force we are going to discuss today is radiation pressure and so radiation pressure that exists has already been postulated as early as 1619 by De Cometis who observed that a neutral part of a tail from a comet points away from the Sun in contrast to the charged part that has a slightly different angle of deviation and the difference in the two he described it due to the effect of radiation pressure. Later work in the early 1901, in by Lebedev also postulated the existence of radiation pressure. Now seminal experiments that were made by that demonstrated induced radiation pressure were those of Art Ashkin. So Art Ashkin worked at Bell Labs and what he demonstrated in 1970s is that if you take a laser beam and focus it then a particle, in this case a small dielectric, is actually drawn inside the focus of a laser beam. Generally, one describes and differentiates two forms of radiation pressure. One is called scattering force and the scattering force results from the reflection of an photon from, for instance, an object which in our case will be mirror. And so photon has momentum h-bar 'k' so the momentum transfer delta 'p' is just twice h-bar 'k'.

Notes

Summary

0m 33s

Radiation Pressure

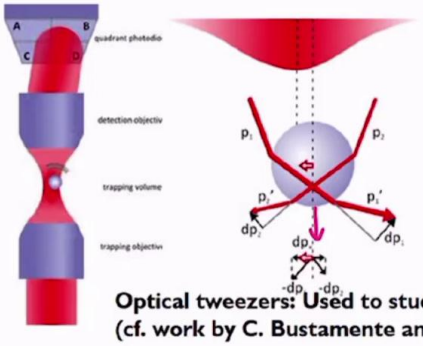


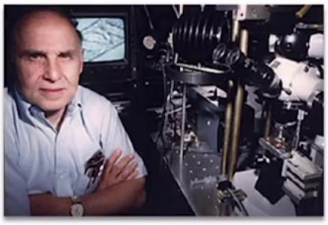
De Cometis 1619

P. Lebedev, 1901, "Untersuchungen über die Druckkräfte des Lichtes", Annalen der Physik, 1901

scattering force
gradient force

Arthur Ashkin (Bell Labs)





Optical tweezers: Used to study the motion of molecular motors (cf. work by C. Bustamente and Steve Block (Stanford)

Terminology Note: The transverse light forces are called ***gradient*** forces as opposed to the forces in the propation direction (***scattering*** force)

Scattering force

$\hbar k$

$\Delta p = 2 \cdot \hbar k$

$F = \Phi \cdot 2\hbar k$

$= \left(\frac{P}{\hbar\omega}\right) 2\hbar k$

$= \frac{P}{\omega} \cdot k$

$\boxed{F = \frac{P}{c}}$

$(\omega = kc)$

A. Ashkin, "Trapping of Atoms by Resonance Radiation Pressure," Phys. Rev. Lett.40, 729 (1978).
S. Chu et al., "Three-Dimensional Viscous Confinement and Cooling of Atoms by Resonance Radiation Pressure," Phys. Rev. Lett. 55, 48 (1985).

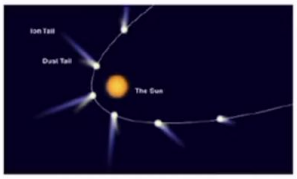
So the total force 'F' on a mirror is nothing else, than the photon flux impinging phi times two h-bar 'k' and so if the photon flux is 'p h' over h-bar omega, 'p' is in some power h-bar omega, is just the energy of the photon so h-bar the Planck constant, omega the frequency times h-bar 'k' one obtains radiation pressure force or simplified 'p' over omega times 'k'. And if you now also recall that our dispersion relation for free space where omega is equal to 'k' times 'c', that's our free space dispersion relation linking the frequency of light to the wave vector. Then one can simplify this expression simply as 'p' divided by 'c' so this is the photon. This is the radiation pressure force that a photon or laser beam of a given power exerts on a mirror. This is called the scattering force. Now in addition to the scattering force, there's also a so-called gradient force. And the gradient force is exactly what's at play here in the context of trap particles because if one has, for instance a laser beam, is one had force that pushes the particle downwards. This is the scattering force. This is due to photons that are reflected from the particle.

Notes

Summary

1m 53s

Radiation Pressure



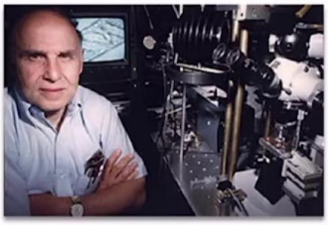
De Cometis 1619

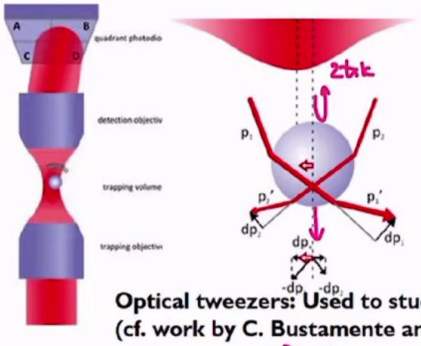
P. Lebedev, 1901, "Untersuchungen über die Druckkräfte des Lichtes", Annalen der Physik, 1901

Scattering force

gradient force

Arthur Ashkin (Bell Labs)





Optical tweezers: Used to study the motion of molecular motors (cf. work by C. Bustamente and Steve Block (Stanford)

Terminology Note: The transverse light forces are called ***gradient*** forces as opposed to the forces in the propation direction (***scattering*** force)

$$F_g = -\nabla U = -\nabla\left[-\left(\alpha \vec{E}\right)\cdot\vec{E}\right] = \frac{1}{2}\nabla\left[|\vec{E}|^2\alpha\right] = -\frac{1}{2}\alpha\nabla|E|^2$$

$\vec{P}$

$\Delta p = 2\cdot\hbar k$

$F = \Phi \cdot 2\hbar k$

$= \left(\frac{P}{\hbar\omega}\right) 2\hbar k$

$= \frac{P}{\omega}\cdot k$

$F = \frac{P}{c}$

$(\omega = kc)$

A. Ashkin, "Trapping of Atoms by Resonance Radiation Pressure," Phys. Rev. Lett.40, 729 (1978).
S. Chu et al., "Three-Dimensional Viscous Confinement and Cooling of Atoms by Resonance Radiation Pressure," Phys. Rev. Lett. 55, 48 (1985).

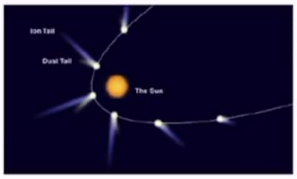
It can lead to a two h-bar 'k' momentum transfer but in addition, there's also a transverse force and this transverse force is called the gradient force and one can show that this force is proportional to the gradient of the field intensity. And in also optomechanical systems this row plays a role. And how one can drive in this case is the gradient force. The gradient force is the gradient of the energy that the particle stores and if the particle has a polarizability that's given by alpha, alpha is the polarizability, then one has this is p, the induced dipole. 'E' is the electric field, 'E' is the field of the laser. Okay, so in this case that give now averages over a cycle that one has one-half E-squared, in this case, times alpha with a minus here and this is the energy stored in the particle if you now also act with this with the gradient then we obtain a force. And so the gradient force in this case will be one-half alpha minus times the gradient of E-squared. And we can see now if there is a gradient in this case in a transverse direction so let's call this direction in this case 'z'.

Notes

Summary

3m 09s

Radiation Pressure

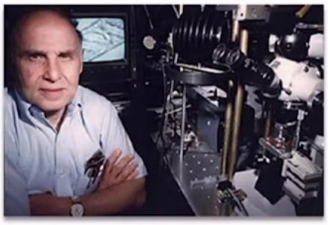


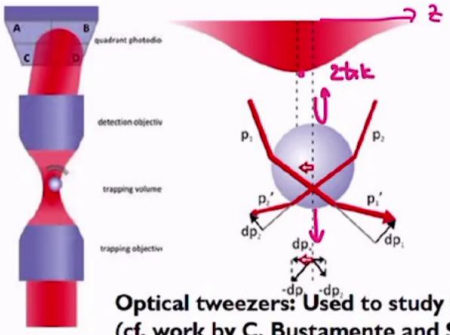
De Cometis 1619

P. Lebedev, 1901, "Untersuchungen über die Druckkräfte des Lichtes", Annalen der Physik, 1901

- scattering force
- gradient force

Arthur Ashkin (Bell Labs)





Optical tweezers: Used to study the motion of molecular motors (cf. work by C. Bustamente and Steve Block (Stanford)

Terminology Note: The transverse light forces are called ***gradient*** forces as opposed to the forces in the propation direction (***scattering*** force)

Scattering force

$\hbar k$

$\Delta p = 2\hbar k$

$F = \Phi \cdot 2\hbar k$

$= \left(\frac{P}{\hbar\omega}\right) 2\hbar k$

$= \frac{P}{\omega} \cdot k$

$F = \frac{P}{c}$

$(\omega = kc)$

$F_g = -\nabla U = -\nabla\left[-\left(\alpha \vec{E}\right)\cdot\vec{E}\right] = \frac{1}{2}|\vec{E}|^2\alpha$

$= -\frac{1}{2}\alpha\nabla|E|^2$

A. Ashkin, "Trapping of Atoms by Resonance Radiation Pressure," Phys. Rev. Lett.40, 729 (1978).
S. Chu et al., "Three-Dimensional Viscous Confinement and Cooling of Atoms by Resonance Radiation Pressure," Phys. Rev. Lett. 55, 48 (1985).

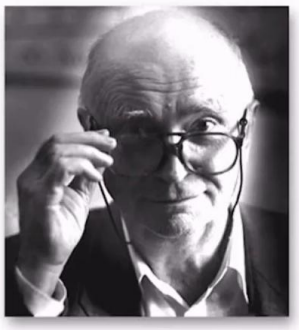
Then this gradient force vanishes exactly at the center of the beam and it will push the particle towards the center if the particle is off axis and this force can be used to trap particles, a technique that's called optical tweezers. That was first demonstrated by Art Ashkin and is still used successfully today, for instance, to describe the motion of or to observe the motion of molecular motors where this for a dielectric particle is attached to a molecule and can be observed by trapping its inside laser foci and also by measuring the motion that it entails, the presence of molecules. So these two forces, scattering and gradient forces appear when laser light interacts with polarizable particles, either in the form of mirrors or in the form of small particles trapped in a laser beam.

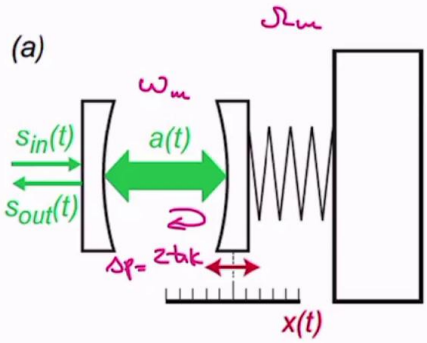
Notes

Summary

4m 23s

Measuring motion with optomechanical coupling





Central question of Braginsky: What is the influence of radiation pressure in a parametric transducer?

$$F_{\mathrm{RP}} = \frac{P}{\hbar\omega} 2\hbar k = 2\frac{P}{c} \qquad P = 1Watt \rightarrow F_{\mathrm{RP}} \approx 6nN$$

Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

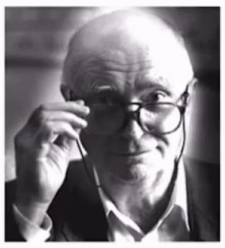
So the next question I'd like to answer in this mooc is what is the Hamiltonian or the parametric equations of motion to describe an optomechanical system and so our optomechanical system contains on one hand the mechanical oscillator with some eigenfrequency omega 'm'. It contains also a cavity field with some eigenfrequencies which will develop, which we'll call here omega also omega 'm' where 'm' may be some index and it's a different index now that's used for the mechanical oscillator. Omega 'm' here, in fact, describes the series of resonances that this resonator can have and what we're interested in now is to understand how to describe the dynamics of a laser field interacting with the mechanical oscillator via radiation pressure so via the fact that the photon on each upon transfer upon a reflection transfer here a momentum delta 'p' per round per reflection of two h-bar 'k'. And so the central question now that we like to understand in this mooc is, what is the influence of the radiation pressure on the mirror and how can it be described? And in this context it should be noted that many of the ideas of optomechanics in the first treatment was done by Vladimir Braginsky, a Russian experimentalist, who already in the 70s and 60s was conceiving the idea of using such an arrangement a mirror coupled to a optical cavity field in the context of gravitational wave detection.

Notes

Summary

5m 10s

Prediction of radiation pressure cooling of mechanical oscillator



V.B. Braginsky

INVESTIGATION OF DISSIPATIVE PONDEROMOTIVE EFFECTS OF ELECTROMAGNETIC RADIATION

V. B. BRAGINSKIĬ, A. B. MANUKIN, and M. Yu. TIKHONOV
Moscow State University
Submitted October 17, 1969

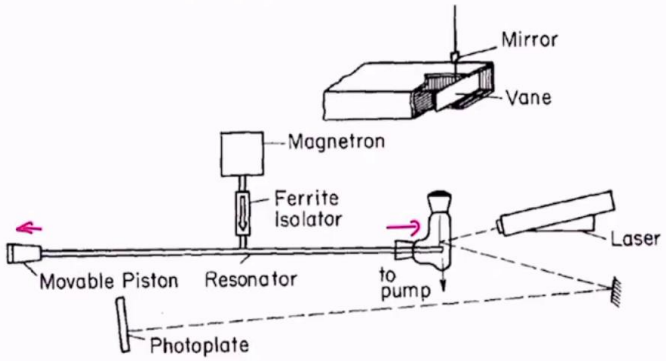


Fig. 10. Torsional oscillator apparatus for measuring the back-reaction influence at microwave frequencies. Displacement was measured optically with a laser.

Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

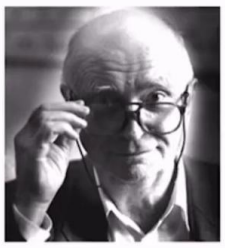
And so in fact, more precisely already in 1969 even before the advent of atomic laser cooling, Vladimir Braginsky predicted in a seminal paper investigation of dissipative ponderomotive effects of electromagnetic radiation that the radiation pressure associated with optical fields, in this case not optical fields but so called microwave fields, can lead to a motion of a modification of the motion of the end mirror and can be used to both amplify and cool mechanical motion. Amplification implying that energies transfers in the optical field to the mechanical oscillator and cooling implying that energy is taken out and his arrangement what he was considering is a microwave cavity so shown here, where the microwaves were reflected of a vane that was suspended here on a very small little thread. So it was mobile and could be could move and what he measured with a laser on reflection was the modification of the intrinsic motion, the oscillations. They were taking place modified by the presence of microwaves that were generated by a magnetron. So a microwave source.

Notes

Summary

6m 33s

Prediction of radiation pressure cooling of mechanical oscillator



V.B. Braginsky

INVESTIGATION OF DISSIPATIVE PONDEROMOTIVE EFFECTS OF ELECTROMAGNETIC RADIATION

V. B. BRAGINSKIĬ, A. B. MANUKIN, and M. Yu. TIKHONOV
Moscow State University
Submitted October 17, 1969

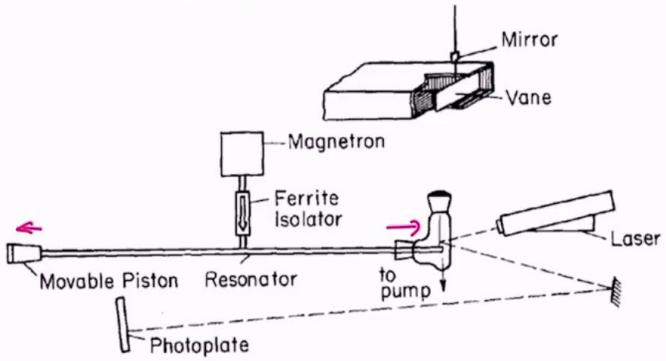


Fig. 10. Torsional oscillator apparatus for measuring the back-reaction influence at microwave frequencies. Displacement was measured optically with a laser.

Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

So this paper really laid the groundwork for what later is now known as cavity optomechanics and it describes provided a framework to describe the interaction of light and mechanical motion and also understand the backaction that's resulting from the radiation pressure both in terms of classical dynamics, in terms of amplification and cooling and the quantum dynamics that relates to the Standard Quantum Limit associated with radiation pressure quantum fluctuations.

Notes

Summary

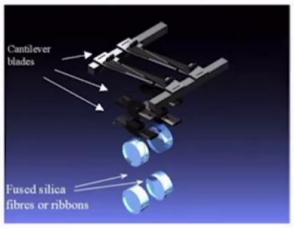
7m 41s

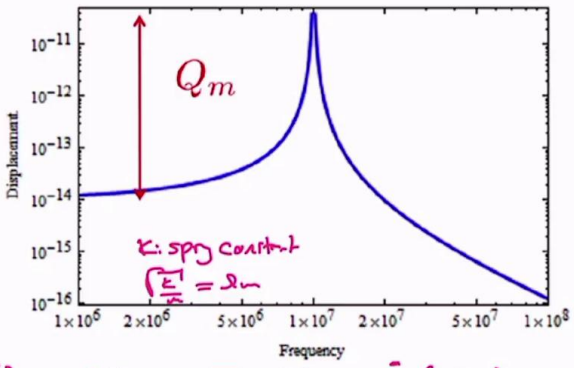
Mechanical oscillators basics

Susceptibility: $\chi(\Omega) = \frac{1/m}{\Omega^2 - \Omega_m^2 + i\Omega\gamma_m}$

Mechanical response: $x(\Omega) = \chi(\Omega) F(\Omega)$

Gravitational wave interferometric detection (VIRGO)





$m\ddot{x} = -kx + F_{ext}(t) + \dot{x}(m\gamma_m)$

restoring force

Damping force

Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

Okay, so let's briefly recall the basis of the mechanical oscillators. So here let's assume that we want to describe the motion of a mechanical oscillator in the form, for instance, of this mirror that's used for gravitational wave detectors a.k.a. LIGO where you have a mirror suspended on a harmonic rod. And so let's recall very briefly the equations of motion so we have just Newton's law, 'm' times acceleration is given by the restoring force 'k' times 'x' that's the restoring force of the mirror. 'k' here is the spring constant of the system. And let's remind ourself that the square root of spring constant over the mass of the system, that's the angular eigenfrequency of the mechanical oscillator. So that's a restoring force then let's also assume you have an external force F-ext. F-ext of 't' plus some damping and the damping force here in this case is a viscous force and so it proportional to velocity x-dot times 'm gamma m' of choosing this just this nomenclature here so I've induced the 'm' here later on. So it cancels and that gamma 'm' becomes a rate so this is the damping force. So now what we can do is we can take the Fourier transform and solve this in the Fourier domain.

Notes

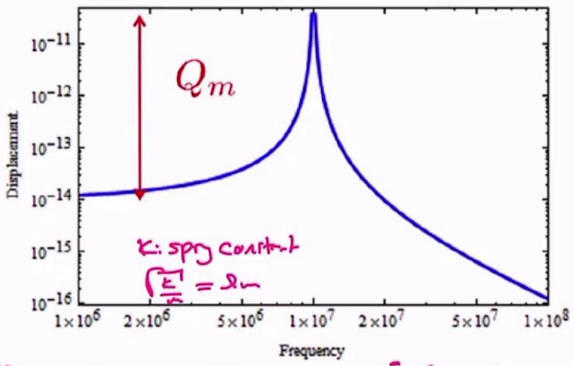
Summary

8m 07s

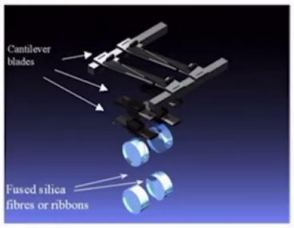
Mechanical oscillators basics

Susceptibility: $\chi(\Omega) = \frac{1/m}{\Omega^2 - \Omega_m^2 + i\Omega\gamma_m}$

Mechanical response: $x(\Omega) = \chi(\Omega)F(\Omega)$



Gravitational wave interferometric detection (VIRGO)



Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

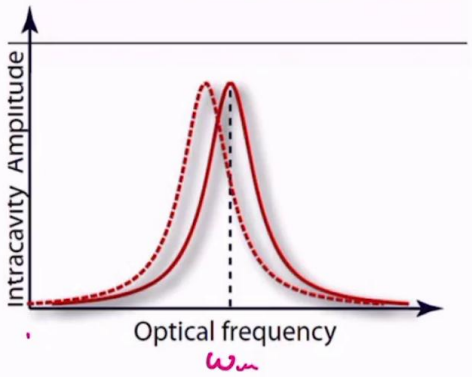
So let's recall that 'x of t' is in this case 'x' of omega so it's our Fourier transform 'e' minus 'i' omega times 't' 'd' omega over two Pi. So this is how we can drive from our time domain signal from the Fourier domain our time domain signal. So then one obtains 'm' times 'x' of omega 'i' omega squared, 'i' omega squared comes from time derivative evidently, plus 'k' times 'x' of omega plus our external force we're gonna put on the other side so minus 'F' external here 'F' external of omega and for viscous form 'i' omega times the mass gamma 'm' times 'x' of omega and now if we divide by 'm' then what we have is minus omega squared times 'x' of omega plus 'k' over 'm' which is the eigenfrequency squared 'x' over 'm' and we get a minus here because beginning from the 'i', here's a plus. then we have plus 'i' omega divided by 'm' so now we have just a gamma 'm'. 'x' of omega rest that is remaining 'F' external of omega divided by 'm'. And if you now so for the so-called susceptibility, so we write 'x' of omega is given by a so-called chi as susceptibility so please differentiate 'x' and chi here. Left is the motion. Chi is susceptibility at link's forced motion and here we have the external Fourier transform of the external drive. And this 'F of x' is precisely the equation that is given up here. So this is the susceptibility of the mechanical oscillator.

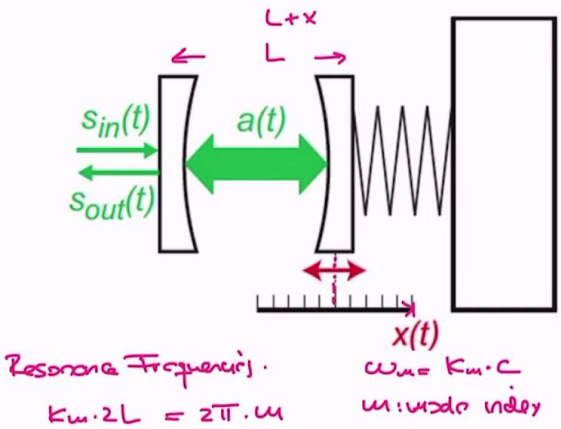
Notes

Summary

9m 18s

Optomechanical coupling in a resonator





Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

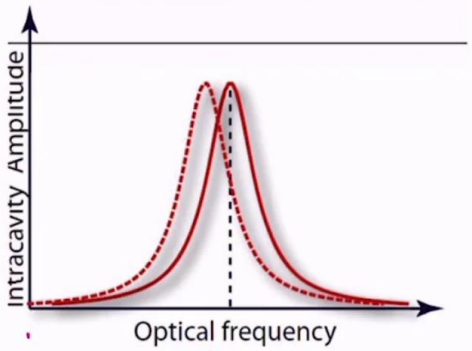
Okay, in this slide, I will talk about optomechanical coupling in a resonator. So we're going to consider is the frequency omega and how it becomes dependent on the position of a mechanical oscillator which we denote here as 'x of t'. Let's assume we have initially a resonator of length 'L' and this resonator can be displaced to some new resonance, so new length that we call 'L' plus 'x' and this is the elongation 'x' that we have due to the motion of the mechanical oscillator. So what we like to compute is how the resonant frequency changes and this can be done quite simply. So let's recollect that to compute the resonance frequencies of the resonator what we have to do, we have to satisfy the following relation that 'km' times '2L' so 'km' is the wave vector, '2L' is the round trip length must be equal to multiple of two Pi and here 'm' is an index that's the mode index. And now let's recall that for free space omega 'm' equal to 'km' times 'c' where 'c' is the speed of light in vacuum. So what we have here in this case is that omega 'm' is equal to 'm' times 'c' over '2L' and 'c' over '2L', this is called the free spectral range of the resonator.

Notes

Summary

10m 59s

Optomechanical coupling in a resonator

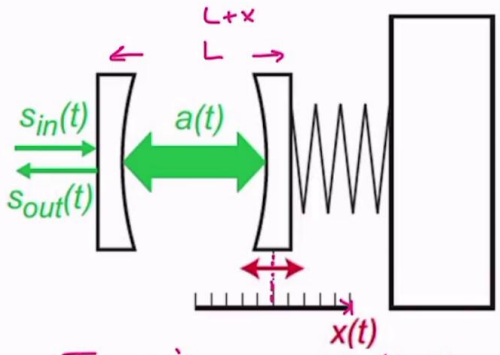


ω_m

$\omega_m = m \cdot \frac{c}{2(L+x)} = m \cdot \frac{c}{2L\left(1+\frac{x}{L}\right)}$

Mit $x \ll L$:

$\omega_m \approx m\left(\frac{c}{2L}\right) \cdot \frac{1}{1+\frac{x}{L}} \approx m\left(\frac{c}{2L}\right)\left(1-\frac{x}{L}\right)$



Resonance Frequency:

$k_m \cdot 2L = 2\pi \cdot m$

$\omega_m = m \cdot \left(\frac{c}{2L}\right)$ Free spectral range

$\omega_m = k_m \cdot c$

m: mode index

$v_m = \frac{\omega_m}{2\pi} = \frac{1}{2\pi}\frac{c}{2L} \cdot m$

Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

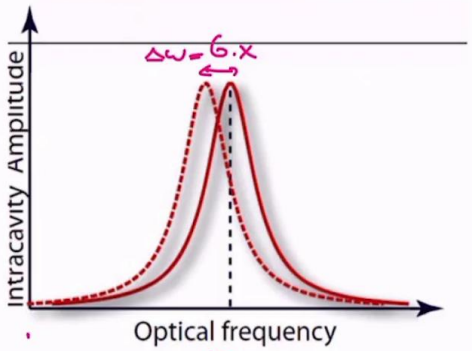
So the eigenfrequency is nu 'm' here, omega 'm' is an angular frequency are divided by two Pi so that is in this case one over two Pi times 'c' over '2L' times 'm'. Those are the eigenfrequencies of the resonator. Now from this expression, it's very easy to compute what your optomechanical coupling is and so for this let's compute omega 'm'. It's again 'm' the mode index times 'c' over '2L'. Now let's assume that our resonator is displaced by some 'x' so we add a little 'x' here that is the additional displacement. What we can do now is we can re-express this as 'c' over '2L' one plus 'x' over 'L' and now noting that 'x' is much much smaller than 'L' so we have a small displacement. We can expand the expression on the right-hand side and we obtain the omega 'm' is approximately equal to 'm' times 'c' over '2L' so the free spectral range times one plus one plus 'x' over 'L' that is approximately 'm' times 'c' over '2L' one minus 'x' over 'L' so here we've done Taylor expansion. And so we have obtained now if we recognize the first part here, this is our original eigenfrequency so let's call this omega 'm' and call this omega 'm' prime so we have omega 'm' prime is equal to omega 'm' times one minus 'x' over 'L'.

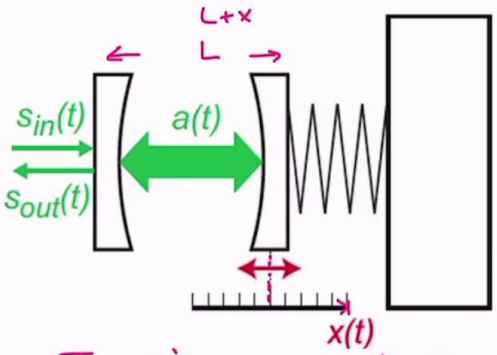
Notes

Summary

12m 29s

Optomechanical coupling in a resonator





$\omega_m = m \cdot \frac{c}{2(L+x)} = m \cdot \frac{c}{2L(1+\frac{x}{L})}$

Natij $x \ll L$:

$\omega_m' \simeq m\left(\frac{c}{2L}\right) \cdot \frac{1}{1+\frac{x}{L}} \simeq \underbrace{m\left(\frac{c}{2L}\right)}_{\omega_m}\left(1-\frac{x}{L}\right)$

$\omega_m' = \omega_m\left(1-\frac{x}{L}\right)$ $\quad \Delta\omega = -\omega_m \frac{x}{L} \equiv G \cdot x$

Resonant Frequency:

$K_m \cdot 2L = 2\pi \cdot m$

$\omega_m = m \cdot \underbrace{\left(\frac{c}{2L}\right)}_{\text{Free spectral range}}$

$\omega_m = K_m \cdot c$

m: mode index

$\nu_m = \frac{\omega_m}{2\pi} = \frac{1}{2\pi}\frac{c}{2L} \cdot m$

G: Frequency Pull parameter

$G = -\frac{\omega_m}{L}$

Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

So we obtain from this that the frequency shift delta omega induced by the mechanical motion is omega 'm' minus omega 'm' times 'x' over 'L'. Okay, so our optomechanical coupling so what we're going to describe, we're going to write this as 'x' times 'G'. 'G' is the frequency pull parameter and this frequency pull parameter and we're gonna see later on that this frequency pull parameter is a very interesting unit because if you multiply it by h-bar then h-bar times 'G' is the force per photon as we will see. And this frequency pull parameter for resonator is just omega 'm' over 'L'. So this is how this is given here. So the shift that we have here of the resonant frequency is just 'G' times 'x'. This is our frequency shift.

Notes

Summary

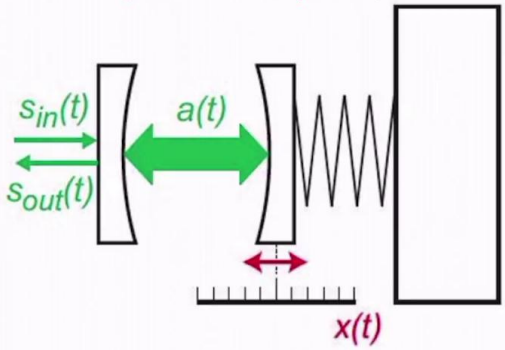
13m 53s

Canonical model for an optomechanical system

H. Haus

$a(t)$: classical field at the cavity $|a(t)|^2$: Energy inside the cavity

$|S_{in}(t)|^2$: optical power



$$\frac{d}{dt}a = -\frac{\kappa}{2}\cdot a - i\omega_m \cdot a + \sqrt{\kappa_{ex}}\cdot S_{in}$$

decay rate

$\kappa = \kappa_0$

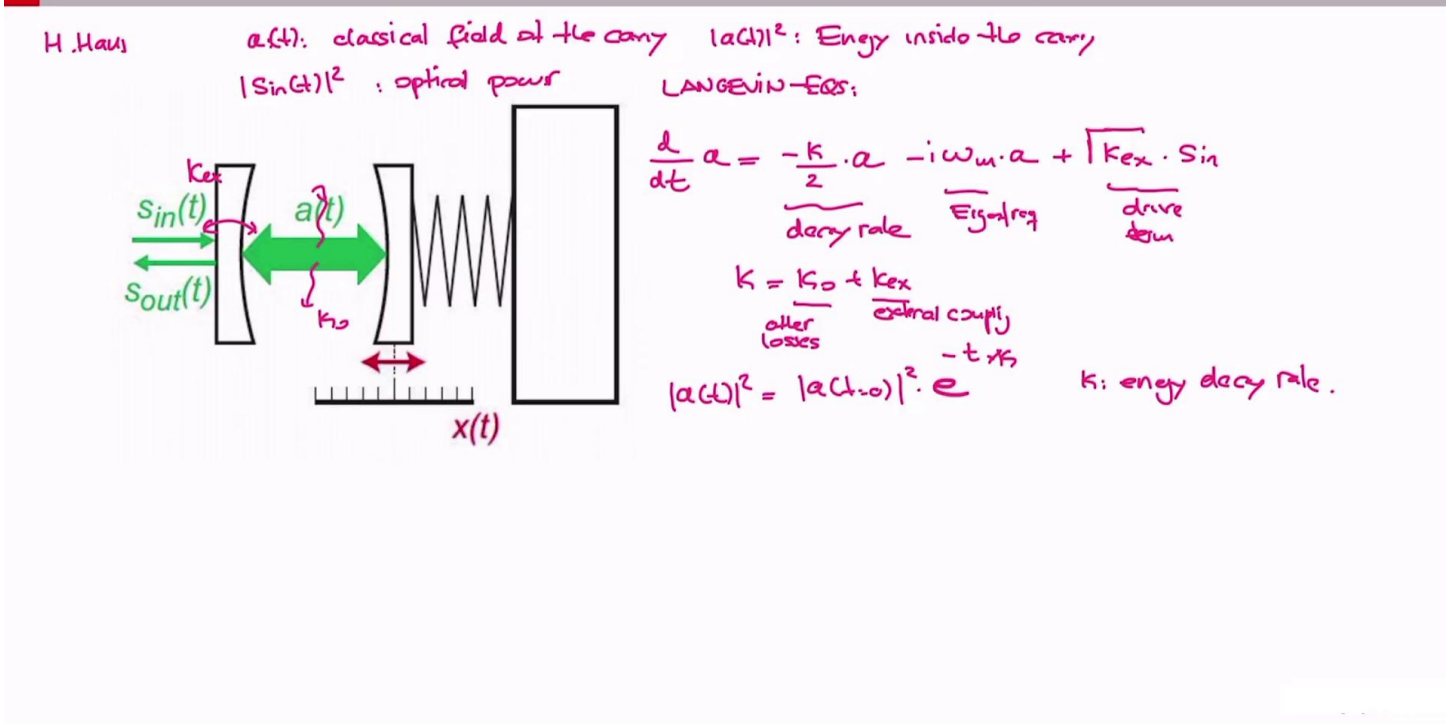
So I haven't derived the frequency pull parameter. We would like to write now the equation of motion for the cavity field. So what we have here is 'a of t' is the classical cavity field and later on we'll look at the quantum mechanical version. And in our notation here which call that of Hermann Haus in fields and waves and optoelectronics. a-squared is the energy inside the cavity. Okay. So with this notion we also have an input field 'S-in of t' and the square of the input field is the optical input power and so note that the input power 'S' and 'a' have different units. So 'S' has the units of square root of power whereas 'a' has the units of square root of energy. And so now what we have to write down is the equation of motion for the field. An equation of motion for the field of a cavity is given by 'd dt' of 'a' is given by a decay kappa over two times 'a' minus 'i' omega 'm' so it's the resonance frequency of the cavity, plus a drive term which is goes like root kappa external times S-in. And so this is the decay rate and this decay has two contributions. Kappa is a contribution from kappa naught. This is intrinsic loss of the cavity and there's a contribution from output coupling due to the fact that the mirror is partially transparent.

Notes

Summary

14m 39s

Canonical model for an optomechanical system



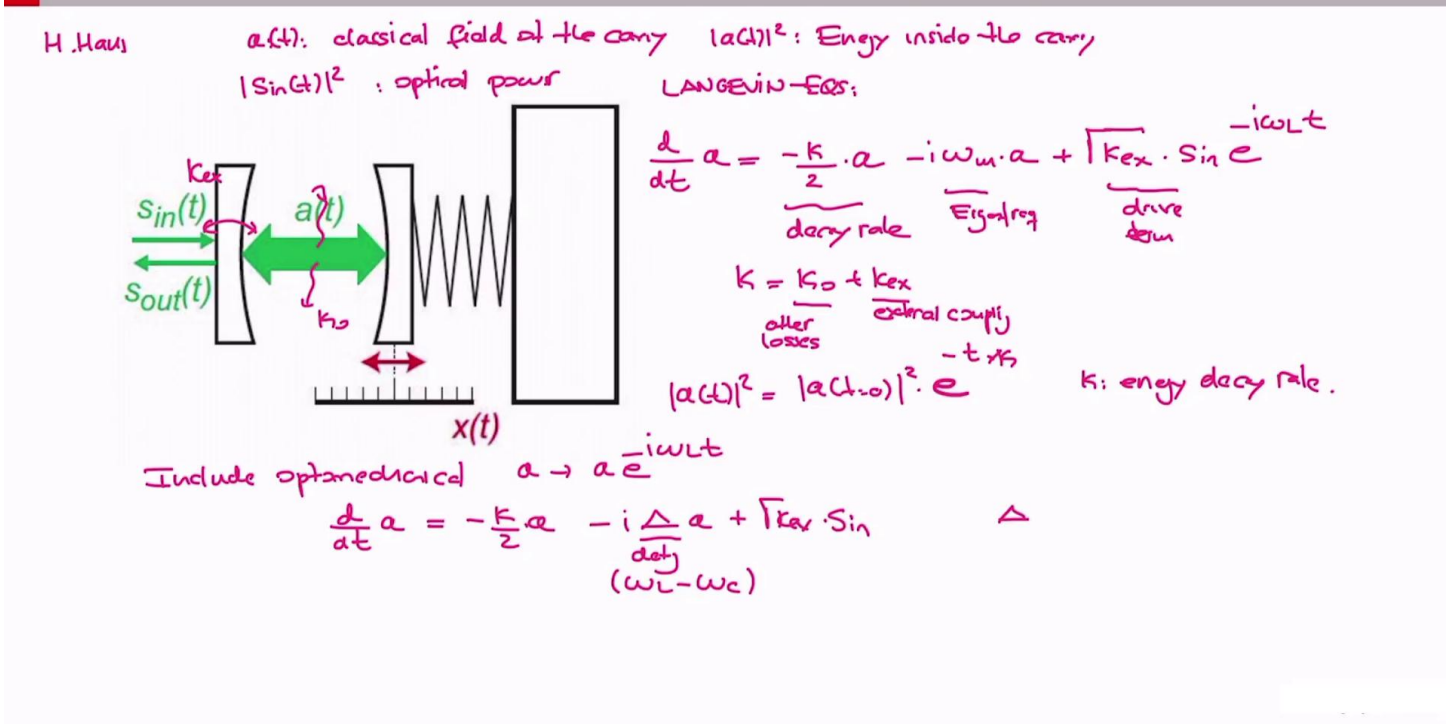
We call this kappa 'x'. This is the external coupling. So this is the external coupling. So output coupling into the direction of the input laser beam and kappa naught are all the other losses, for instance, absorption scattering or losses arising from mode mismatch. Omega 'm' is the eigenfrequency of the cavity and this is the drive term. And I'd like to make a comment on the drive term. The drive term here so this equation is called a Langevin equation. It's called Langevin equation even though there is not yet a noise term. It's a Langevin equation for the cavity field and I like to draw the attention to the fact that there's a root kappa external appearing in front of the S-in because input power S-in and 'a', a-squared being the energy, have different units and this is why we have here this relation. We can also see what happens and solve this equation for the case where the cavity is initially has some initial field 'a' and has in the absence of an external drive and then we find that the solution of 'a of t' squared is just 'a of t' equal to zero squared times minus 't' over times kappa. And so we can clearly see here that kappa is the energy decay rate. Good.

Notes

Summary

16m 11s

Canonical model for an optomechanical system



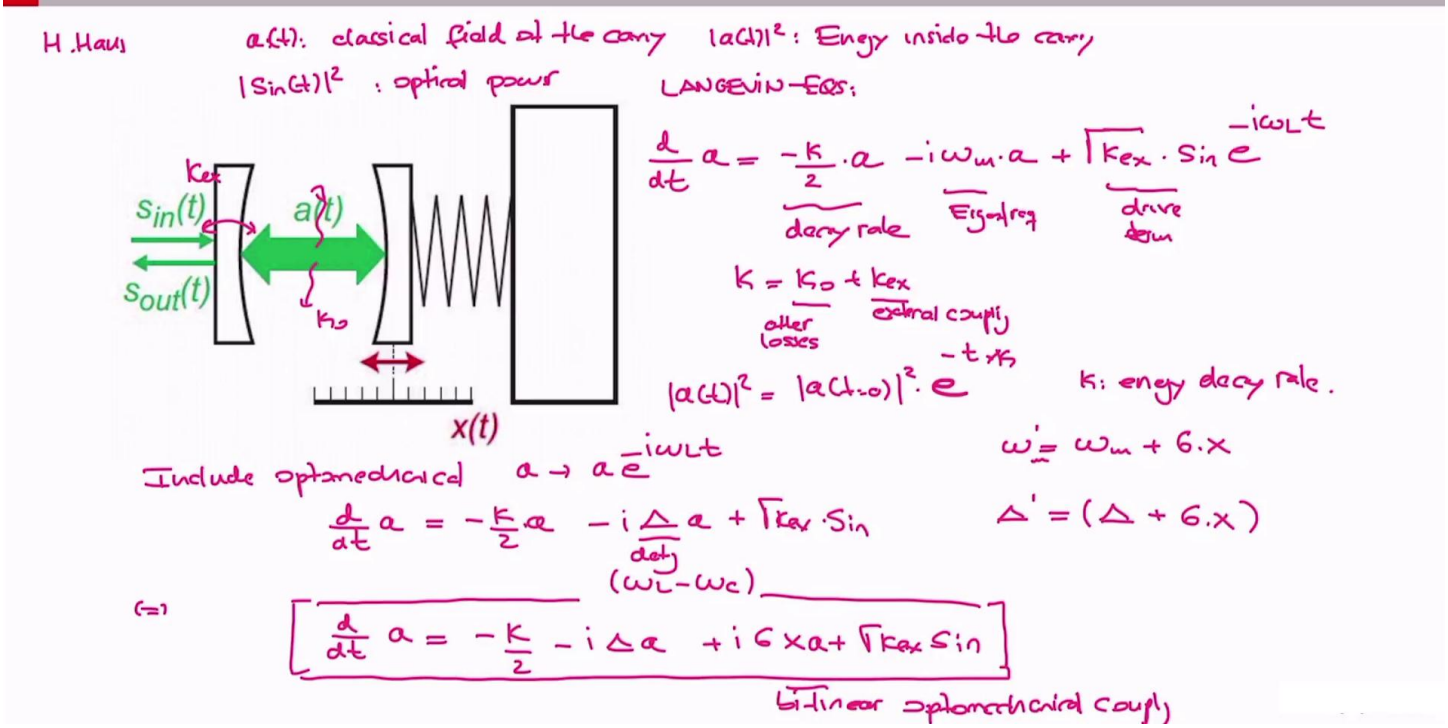
So now we need to modify these equations of motion to include the optomechanical coupling. And to do that I'd like to make several comments. So first of all, what I've neglected here is the time dependence in equations of the input field. So let's assume that we have an input field that goes like 'i' omega 'L' then what we can do is we can rotate, we can transmit a rotating frame and this frame 'd' so we can take 'a'. We can replace by 'e' to the minus 'i' omega laser 't' so we rotate in the frame that rotates with the pump laser frequency. In this case, the 'a' becomes stationary and our equation of motion is modified. We have here minus kappa over two times 'a' then we get a detuning term. So this is now the detuning between the laser frequency and the cavity frequency times 'a' plus root kappa external and then S-in and now S-in itself it becomes just a constant because we are in this rotating frame. Now what we have to do, however now, is to account for the fact that our detuning is in reality dependent now on the optomechanical coupling. So from this what we have just derived in the previous slide is that omega prime is equal to omega 'm' plus our frequency pull parameter times 'x' and so now we can enter this term.

Notes

Summary

17m 34s

Canonical model for an optomechanical system



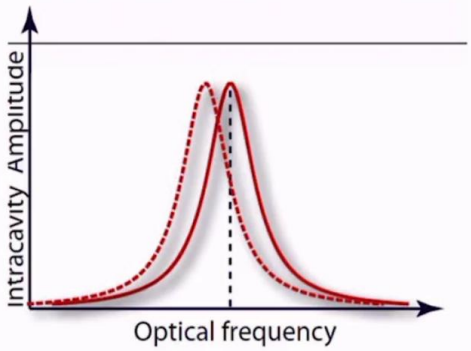
So our detuning in this case gets a modification and detuning is given by our original detuning plus 'G' times 'x'. So we have here the optomechanical coupling and this now leads to the following equation that 'd dt' of 'a' is minus kappa over two minus 'i' delta 'a' and then we have here an additional term of minus or plus depending on the definition now we use of 'iG' times 'x' plus a root kappa external times S-in. So this equation now is our equation for the optical cavity in the presence of an optomechanical optomechanical coupling to the end mirror. And so we've seen here is that our kap, so we have a new term appearing and this term is bi-linear. It's in fact, a nonlinear term and this describes the optomechanical coupling.

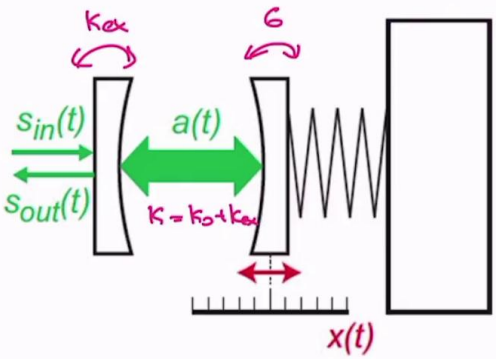
Notes

Summary

19m 01s

Optomechanical coupling in a resonator





$$\frac{da}{dt} = \underbrace{-(\kappa/2)a}_{\text{cavity decay rate}} + \underbrace{i(\Delta + Gx)a}_{\text{position dependent detuning}} + \underbrace{\sqrt{\kappa_{ex}}s}_{\text{pump term}}$$

rotaly frame $e^{-i\omega_L t}$

$$G = \frac{d\omega}{dx} = -\frac{1}{L}\omega_c$$

Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

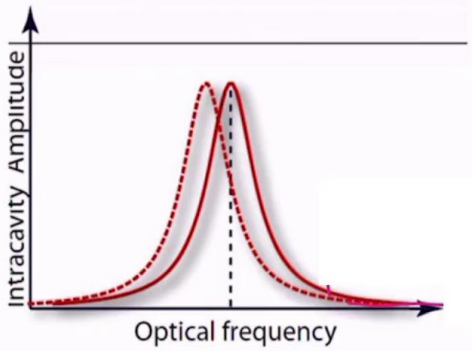
So I summarized again the optomechanical coupling in a resonator by this classical description. So again we have in a rotating frame that rotates with 'e' minus 'i' omega 'L' times 't', the laser frequency. We have here the cavity field 'a' which has a cavity decay rate of factor of two is because we look at if you decay here on the energy. We have a position dependent coupling term and then we have the pump term and as mentioned 'Gx' here is the coupling rate. Kappa 'x' is the coupling rate that is the decay rate associated with mirror losses so it's a finite mirror of the cavity and the term kappa itself so the cavity decay rate will have totally contribution of internal losses plus external losses where the internal losses are losses of they can arise, for instance, from absorption. In addition here we have our optomechanical coupling denoted 'G' that couples the radiation pressure to the mechanical oscillator. So for the final row, the frequency pull parameter is just omega 'c' over 'L' and what we can now also deduce is the equation of motion for the mechanical oscillator so here we can use just again classically, we just used Newton's law so we have 'm' times 'a' is minus 'k' times 'x', 'k' is the restoring force.

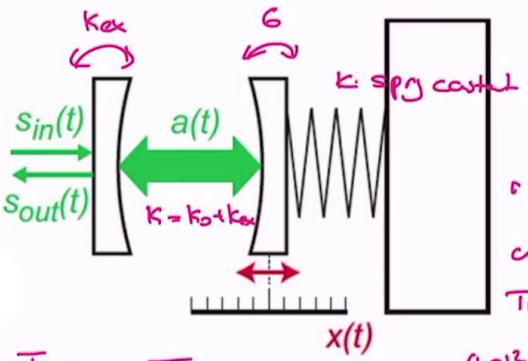
Notes

Summary

19m 59s

Optomechanical coupling in a resonator





$\omega_c = k \cdot c$

$T_{rt} = 2L/c$

$m \cdot \frac{d^2x}{dt} = -kx - \underbrace{\Gamma_m \cdot \dot{x}\, m}_{\text{damping}} + F_{rad}$

$F_{rad} = (2\hbar k)\Phi = 2\hbar k\left(\frac{|a|^2}{T_{rt}} \frac{1}{\hbar\omega}\right) = \frac{|a|^2}{L}\hbar$

rotating frame $e^{-i\omega_L t}$

$$\frac{da}{dt} = \underbrace{-(\kappa/2)a}_{\text{cavity decay rate}} + \underbrace{i(\Delta + Gx)a}_{\text{position dependent detuning}} + \underbrace{\sqrt{\kappa_{ex}}s}_{\text{pump term}}$$

$$G = \frac{d\omega}{dx} = -\frac{1}{L}\omega_c$$

Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

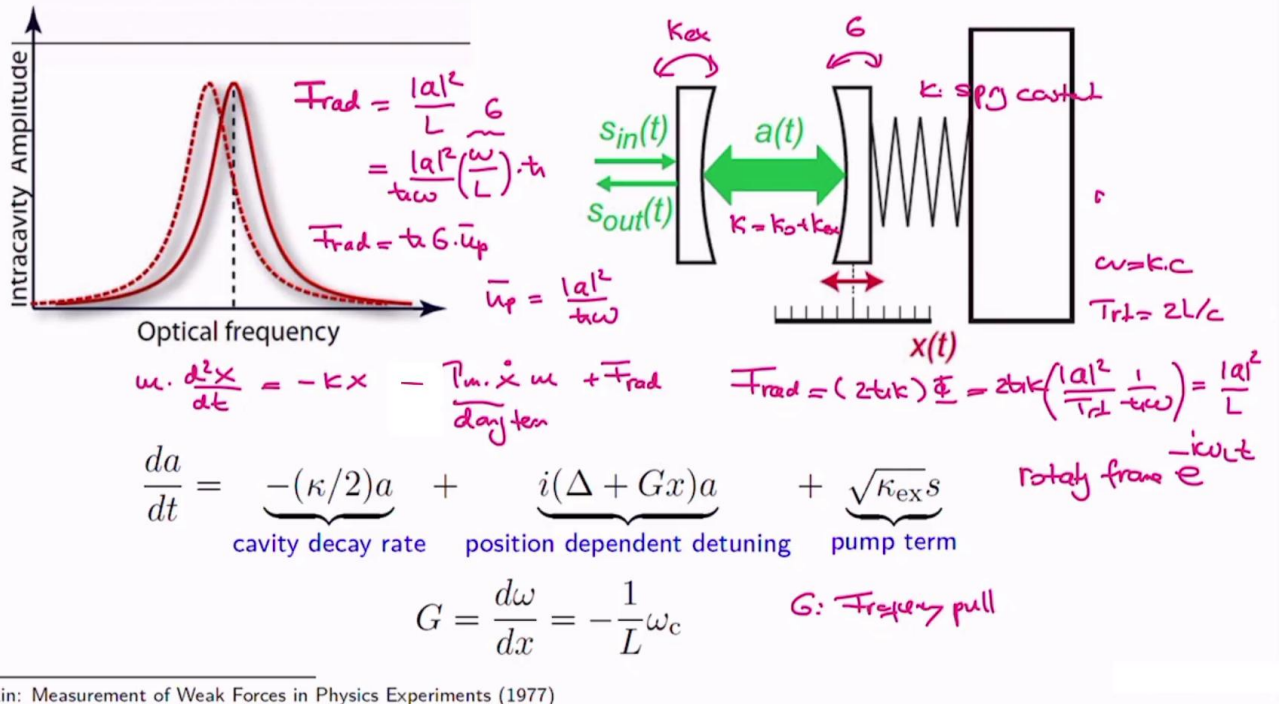
So it is this spring constant of harmonic oscillator. We have again as mentioned before a viscous term so you have a gamma 'm' times x-dot times the mass. This is our damping term and we have here an additional term now that is due to radiation pressure. So if we head for the radiation pressure, this is equal to two h-bar 'k', the momentum transfer parameter times the photon flux. What is the photon flux? The photon flux in the cavity is just a-squared divided by 't' round trip. 't' round trip is a round trip time so that this '2L' over 'c'. Then we have to multiply this expression by one over h-bar omega because that's the photon flux then. So this is the number of photons per second impinging on the mirror each giving a two h-bar 'k' momentum transfer. And if we look at the expression and enter round trip time here, what we see is that the only thing that is left over then is a-squared so the energy in the cavity divided by 'L' multiplied in this case by 'c' then also note that we have a 'k' here and omega is equal to 'kc'. That's our free space dispersion relation. So 'kc' is omega and that implies that we have omega of omega so we just have the expression here as shown just divided by 'L'.

Notes

Summary

21m 16s

Optomechanical coupling in a resonator



Braginsky, Manukin: Measurement of Weak Forces in Physics Experiments (1977)

So that means that in summary we have that the radiation pressure force F-rad is equal to a-squared divided by 'L' and this we can also write a slight different form. We can write this as a-squared times omega over 'L', omega over 'L' as our frequency pull parameter divided by omega. So we see is that radiation pressure force is again proportional to 'G', the frequency pull parameter. Now what we can also now write and recognize is the following aspect that the radiation pressure force can also be written in a different form. It can also be written in the form h-bar 'G' times 'n' where 'Np' is the number of photons in the cavity and this we can immediately see by dividing the expression here multiplying by h-bar, dividing by h-bar here multiplying by h-bar and then recognizing that number of photons in the cavity is just a-squared divided by h-bar omega. So this means that the 'G' has a very natural kind of interpretation. The frequency pull parameter is just the force per photon so 'G' is the frequency pull parameter and it is just the force per photon.

Notes

Summary

22m 37s

Classical optomechanical coupled equations of motion

$$\ddot{x} + \frac{1}{\tau m}\dot{x} + \Omega_{\mathrm{m}}^2 x = \frac{F_{\mathrm{RP}}}{m} = \frac{G}{m}\frac{|a|^2}{\omega}$$

$$\dot{a} = i(\Delta - Gx)a - \frac{\kappa}{2}a + \sqrt{\kappa}s$$

Equation for the mechanical oscillator

Equation for the optical cavity field

Radiation pressure force:

$$F_{\mathrm{RP}} = G\frac{|a|^2}{T_{\mathrm{cav}}\hbar\omega}2\hbar k = G\frac{|a|^2}{\omega}$$

Intracavity energy $|a|^2$ Cavity round trip time T_{cav} Photon momentum $\hbar\kappa$

coupled mode equations can be solved perturbatively.

Kippenberg, T. J. & Vahala, K. J. Cavity optomechanics. Optics Express 15 (2007).
Kippenberg, T. J., Rokhsari, H., Carmon, T., Scherer, A. & Vahala, K. J. Analysis of Radiation-Pressure Induced Mechanical Oscillation of an Optical Microcavity. Physical Review Letters 95, 033901 (2005).

Okay, so now having derived the force per photon and frequency pull parameter, we can now write down the classical optomechanical coupled equations of motion and so now here I repeat one more time. We have the equation for the mechanical oscillator and we have here again appearing the the force, the radiation pressure force F-rad divided by 'm' that are introduced on the previous slide and we have the equation for the optical fields. And again this is the Langevin equation. It has a damping and detuning the frequency pull parameter times times 'x' and then the input term 'S' for the coupling to the external external drive line typically this is denoted as kappa external. So these are the classical equations of motion. An important comment is in place. The equations themselves are non-linear. We can see this immediately by noting that there's a product appearing of 'x' and 'a' here as well as the amplitude squared of the optical fields and so generally to, as in the other lectures here of this mooc, to solve this equation you need to use perturbative analysis and one way, for instance, to do that is to assume that the cavity field 'a' is composed of a drive 'e' minus 'i' omega laser field 't' and then plus sidebands, let's call them 'a1' 'e' minus 'i' omega 'L' plus omega 'm' times 't' as well as a sideband that rotates with a frequency minus omega 'mt'.

Notes

Summary

23m 59s

Classical optomechanical coupled equations of motion

$$\ddot{x} + \frac{1}{\tau m}\dot{x} + \Omega_{\mathrm{m}}^2 x = \frac{F_{\mathrm{RP}}}{m} = \frac{G}{m}\frac{|a|^2}{\omega}$$

$$\dot{a} = i(\Delta - Gx)a - \frac{\kappa}{2}a + \sqrt{\kappa}s$$

Equation for the mechanical oscillator

Equation for the optical cavity field

Radiation pressure force:

$$F_{\mathrm{RP}} = G\frac{|a|^2}{T_{\mathrm{cav}}\hbar\omega}2\hbar k = G\frac{|a|^2}{\omega}$$

Intracavity energy $|a|^2$ Cavity round trip time T_{cav} Photon momentum $\hbar\kappa$

coupled mode equations can be solved perturbatively.

Kippenberg, T. J. & Vahala, K. J. Cavity optomechanics. Optics Express 15 (2007).
Kippenberg, T. J., Rokhsari, H., Carmon, T., Scherer, A. & Vahala, K. J. Analysis of Radiation-Pressure Induced Mechanical Oscillation of an Optical Microcavity. Physical Review Letters 95, 033901 (2005).

And we can assume then that for 'x', the position of a mechanical oscillator this is proportional to x-naught times 'e' minus 'i' omega 'mt'. So that's the motion of the mechanical oscillator here in complex notation and so if you assume this type of ansatz then what you will recognize is that the equations of motion is nonlinear set of equations of motion can be broken down into equations for a-zero a-one 'a' minus one and then the amplitude of the mechanical oscillator and will give rise to a close set of equations that can be linearized and solved and a certain approximations with higher-order sidebands are measurable. And this will be done in a later lecture and gives rise to dynamics such as dynamical instabilities or cooling. Again I'd like to emphasize one more time that radiation pressure force here is just 'G' a-squared over omega and that is just given by h-bar 'G' times a-squared divided by h-bar omega where the last quantity here is just the number of photons inside the cavity.

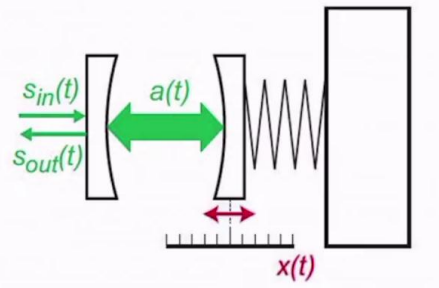
Notes

Summary

25m 28s

Radiation pressure dynamical backaction

Optical field responds on the mechanical motion with delay($\tau \approx \kappa^{-1}$)



$$\frac{d^2}{dt^2}x + \Gamma_m \frac{dx}{dt} + \omega_m^2 x = \frac{F_{rp}}{m_{eff}}$$

Amplification **Blue detuning**

Dynamical backaction induced Parametric instability (JETP 1969, Braginsky)

Cooling **Red detuning**

(JETP 1969, Braginsky)

Braginsky, V. B. Measurement of Weak Forces in Physics Experiments. (University of Chicago Press, 1977).
Braginsky, V. B. & Khalili, F. Y. Quantum Measurement. (Cambridge University Press, 1992).
Braginsky, V. B., Strigin, S. E. & Vyatchanin, S. P. Analysis of parametric oscillatory instability in power recycled LIGO interferometer. Physics Le
111-124 (2002).

Okay. I would like to briefly analyze how one can really gain insights into the optomechanical dynamics not by solving the couple of mode equations for the optical field and mechanical oscillator in a perturbative fashion but by a simple analysis of the radiation pressure acting on a mechanical oscillator and this analysis I'm going to present here has been originally carried out by Vladimir Braginsky in his book Measurement of Weak Force in Physics Experiments, in 1977 and it was earlier part of a publication in 69' on the dissipative ponderomotive effects of electromagnetic radiation published in a Russian Journal of Physics. And so the idea is to consider the mechanical oscillator so we have here the second derivative of 'x' the motion of the mirror here. Then we have some damping rate that acts on the mirror so there's a viscous term then there's the eigenfrequency omega 'm' squared times 'x' and then there's a radiation pressure force 'rp' divided by the effective mass or the mass of the mirror. And now the key insight that Braginsky brought forward was to notice that radiation pressure force depends on a mirror position but it does not depend on an on the position in an instantaneous way but with a delayed fashion and this is a completely heuristic modification that he made to the equation of motion but what it accounts for is the cavity delay.

Notes

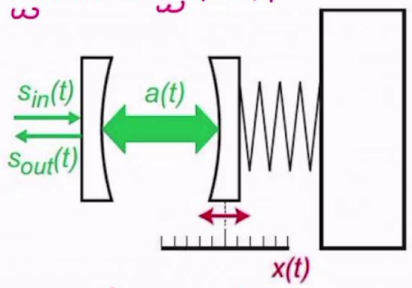
Summary

26m 35s

Radiation pressure dynamical backaction

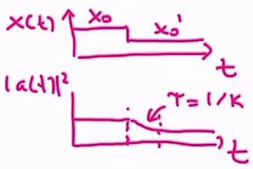
Optical field responds on the mechanical motion with delay($\tau \approx \kappa^{-1}$)

$F_{rp} = \frac{G}{\omega}|a|^2 = \frac{G}{\omega}|a(t)|^2$



$\frac{d^2}{dt^2}x + \Gamma_m \frac{dx}{dt} + \Omega_m^2 x = \frac{F_{rp}}{m_{eff}} = \frac{F_{rp}(x(t-\tau))}{m_{eff}}$

Photon decay rate $\kappa = 1/\tau$



Amplification **Blue detuning**

Dynamical backaction induced Parametric instability (JETP 1969, Braginsky)

Cooling **Red detuning**

(JETP 1969, Braginsky)

Braginsky, V. B. Measurement of Weak Forces in Physics Experiments. (University of Chicago Press, 1977).
Braginsky, V. B. & Khalili, F. Y. Quantum Measurement. (Cambridge University Press, 1992).
Braginsky, V. B., Strigin, S. E. & Vyatchanin, S. P. Analysis of parametric oscillatory instability in power recycled LIGO interferometer. Physics Le
111-124 (2002).

So let's consider again the radiation pressure force so the radiation pressure force is just given by 'G' over omega times a-squared in our previous treatment and this we can write as 'G' over omega times 'a of t' squared. So now we can ask ourselves a question. What is the influence of the photon decay rate? So we have a photon decay rate of the cavity that is denoted by kappa which is one over tau and in this situation here what it implies is if the mechanical oscillator so if we draw, for instance, a coordinate here so this is say, the time this is 'x of t' and if we imagine that the mechanical oscillator's entire position changes abruptly from some initial value, call it x-zero to some new value x-zero prime then the intracavity field 'a of t' will also change but 'a of t' will not change instantaneously but actually has a a delay associated. And this delay comes from the fact that again the cavity has a ring-up time and ring-down time so stating so this is the characteristic time scale here that is given by tau which is one over kappa. And what this now implies is that one can model the radiation pressure force as being instantaneous in the position but now having this delay term tau from the cavity decay rate.

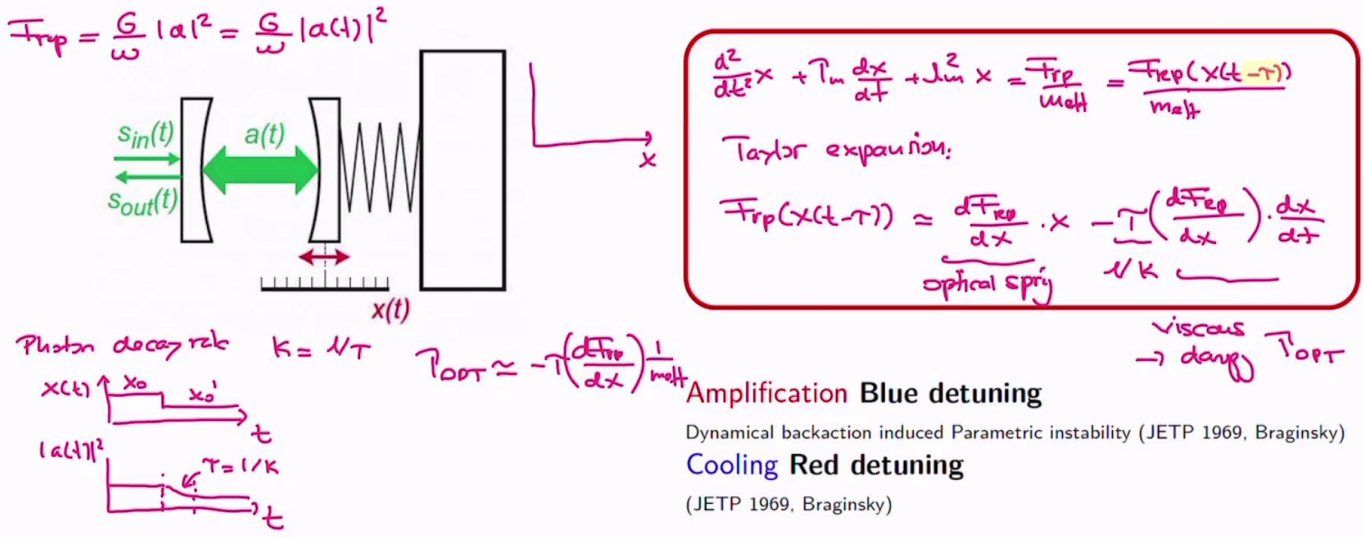
Notes

Summary

27m 59s

Radiation pressure dynamical backaction

Optical field responds on the mechanical motion with delay($\tau \approx \kappa^{-1}$)



Amplification **Blue detuning**

Dynamical backaction induced Parametric instability (JETP 1969, Braginsky)

Cooling **Red detuning**

(JETP 1969, Braginsky)

Braginsky, V. B. Measurement of Weak Forces in Physics Experiments. (University of Chicago Press, 1977).
Braginsky, V. B. & Khalili, F. Y. Quantum Measurement. (Cambridge University Press, 1992).
Braginsky, V. B., Strigin, S. E. & Vyatchanin, S. P. Analysis of parametric oscillatory instability in power recycled LIGO interferometer. Physics Le
111-124 (2002).

And what you can do now is a Taylor expansion of the right-hand side and so if you do Taylor expansion, what you find is that the radiation pressure force 'rp' of 'x' 't' minus tau so layer counts for this final photon lifetime. This is approximately equal to 'dF' radiation pressure 'dx' times 'x' plus a term now which is proportional just to delay so the inverse cavity decay rate which proportional to the slope of the radiation pressure force with position times x-dot so 'dx dt'. So this term here is called leads to an optical spring. Why? Because now this term adds to the eigenfrequency of the mechanical oscillator. The second term here, this is a viscous force that adds or subtracts to the damping and this leads to a so-called optomechanical damping rate and so this optomechanical damping rate gamma-opt is approximately in this case minus tau times the radiation pressure force 'dx' the derivative divided by the effective mass. So now let's take a look at the radiation pressure force and how it varies with the position of the mirror. So if you have a cavity this is position. We have the radiation pressure force then the radiation pressure force, let's assume that at some position that's called x-tilde.

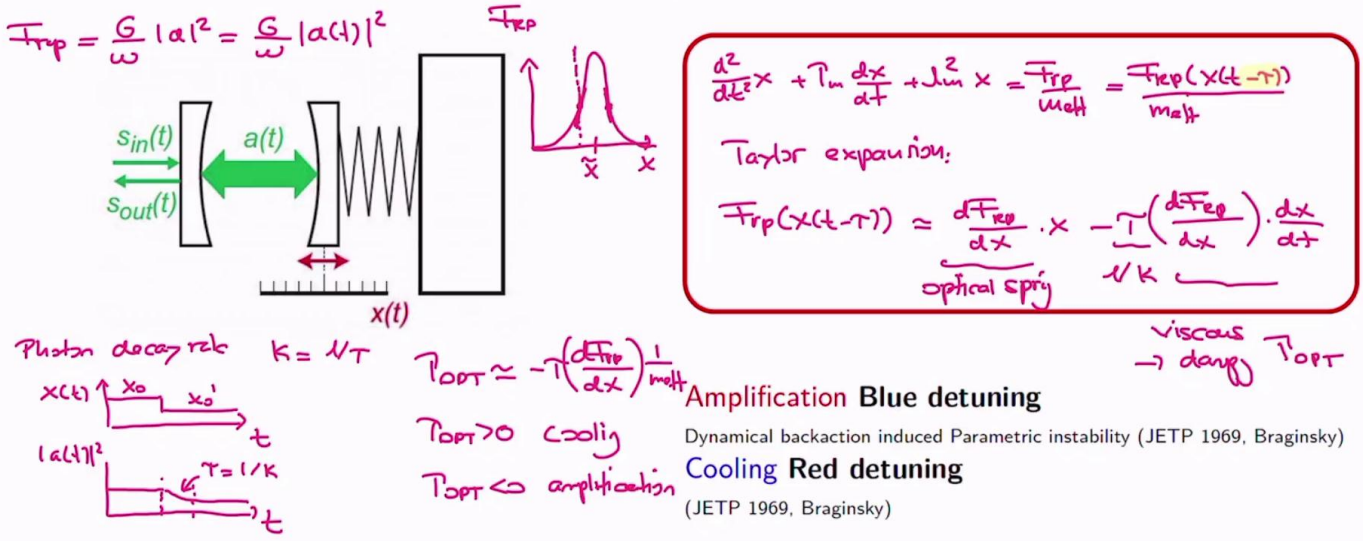
Notes

Summary

29m 24s

Radiation pressure dynamical backaction

Optical field responds on the mechanical motion with delay($\tau \approx \kappa^{-1}$)



Amplification **Blue detuning**

Dynamical backaction induced Parametric instability (JETP 1969, Braginsky)

Cooling **Red detuning**

(JETP 1969, Braginsky)

Braginsky, V. B. Measurement of Weak Forces in Physics Experiments. (University of Chicago Press, 1977).
Braginsky, V. B. & Khalili, F. Y. Quantum Measurement. (Cambridge University Press, 1992).
Braginsky, V. B., Strigin, S. E. & Vyatchanin, S. P. Analysis of parametric oscillatory instability in power recycled LIGO interferometer. Physics Le
111-124 (2002).

The mirror is in such a way that the laser whose frequency is fixed is on resonance then we have a simple cavity Lorentzian for the cavity build up and so we have a region where on the right-hand side or left-hand side of the resonance where 'dF dx' is either positive or negative so depending on where the laser sits with respect to the cavity resonance, we have a different dynamics and either we have a positive sign or a negative sign. If gamma-opt is positive, we have a cooling so we'll show this later but we obtain a dynamics in which the radiation, the retarded radiation pressure force extracts energy from the mechanical oscillator so it always opposes the motion of the mirror. In contrast, if gamma-opt is negative then we have amplification. And in this situation what happens is that the radiation pressure force always pushes in the same direction as the motion of the mirror so we have an amplification. And if gamma-opt plus gamma 'm' is equal to zero, we have what is called a parametric instability.

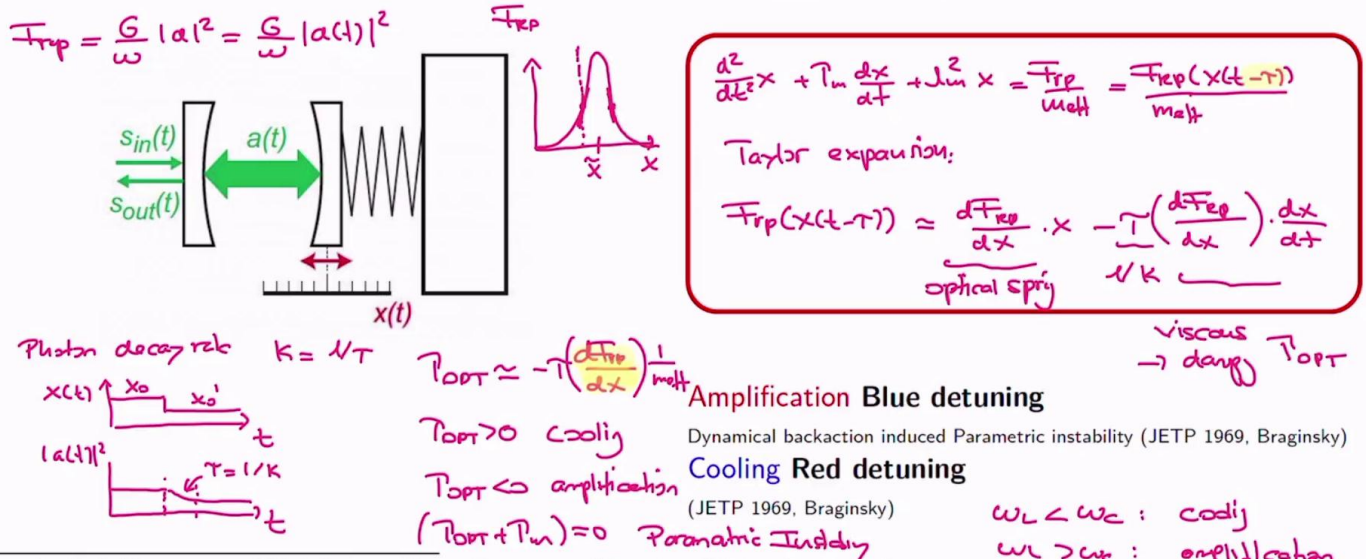
Notes

Summary

30m 50s

Radiation pressure dynamical backaction

Optical field responds on the mechanical motion with delay($\tau \approx \kappa^{-1}$)



Braginsky, V. B. Measurement of Weak Forces in Physics Experiments. (University of Chicago Press, 1977).
Braginsky, V. B. & Khalili, F. Y. Quantum Measurement. (Cambridge University Press, 1992).
Braginsky, V. B., Strigin, S. E. & Vyatchanin, S. P. Analysis of parametric oscillatory instability in power recycled LIGO interferometer. Physics Le
111-124 (2002).

And this is the phenomena that Braginsky already in 1969 analyzed in his JETP paper and desirable the contrast gravitational wave detectors and it occurs when the laser is detuned to the blue so these two situations where the radiation pressure force 'dF dx' so the slope it has a positive negative, this occurs for the laser frequency either it's being smaller than the cavity resonance frequency, this leads to a cooling whereas omega 'L' larger than omega 'c' leads to an amplification.

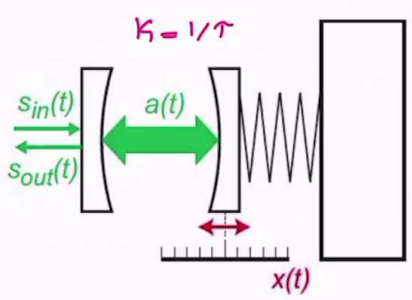
Notes

Summary

31m 53s

Radiation pressure dynamical backaction

Optical field responds on the mechanical motion with delay($\tau \approx \kappa^{-1}$)



$$\frac{d^2x}{dt^2} + \Gamma_m \frac{dx}{dt} + \Omega_m^2 x = \frac{F_{RP}(x(t-\tau))}{m_{eff}}$$

Taylor expansion yields:

$$F_{RP}(x(t-\tau)) \approx \frac{dF}{dx}x - \tau \frac{dF}{dx}\frac{dx}{dt}$$

optical spring — optomechanical damping rate

$$\Gamma_{opt} \approx -\tau \frac{dF}{dx}\frac{1}{m_{eff}}$$

$\Delta > 0$ $\omega_L > \omega_c$ Amplification **Blue detuning**

Dynamical backaction induced Parametric instability (JETP 1969, Braginsky)

$\Delta < 0$ $\omega_L < \omega_c$ Cooling **Red detuning**

(JETP 1969, Braginsky)

Braginsky, V. B. Measurement of Weak Forces in Physics Experiments. (University of Chicago Press, 1977).
Braginsky, V. B. & Khalili, F. Y. Quantum Measurement. (Cambridge University Press, 1992).
Braginsky, V. B., Strigin, S. E. & Vyatchanin, S. P. Analysis of parametric oscillatory instability in power recycled LIGO interferometer. Physics Le
111-124 (2002).

So I summarize here one more time on different slides now typed the analysis that I've just hand-written on the previous slides. So we have the equation of the mechanical oscillator. We have the decay time tau that was introduced heuristically by Braginsky to account for the fact that the cavity has a fine response time kappa one over tau which describes the rate at which photon escape from the cavity and I've had two more time the Taylor expansion that leads to the so-called optical spring effect the change in the eigenfrequency and it leads to a change in the damping so this is the optomechanical damping rate. And we have two dynamics and the dynamics that we obtain depends on how the laser is detuned. We differentiate between a blue detuning where the laser frequency is larger than the cavity frequency or a red tuning where the laser frequency is smaller than the cavity resonance frequency. These are designated as delta larger than zero and delta smaller than zero so this is just a definition that we choose here so a positive detuning corresponds to a laser frequency larger than the cavity, a negative detuning negative detuning corresponds to laser frequency smaller than the cavity resonance frequency and this gives rise to two different effects of amplification and cooling.

Notes

Summary

32m 28s