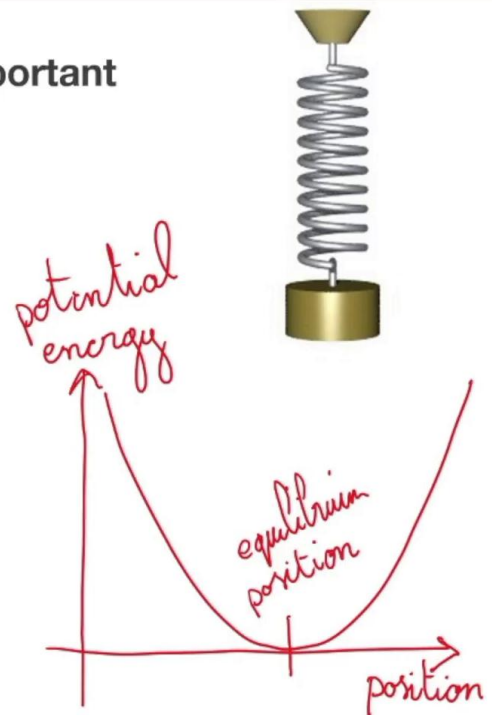


EPFL

OUTLINE

In this lesson, we will solve one of the most important problems in quantum mechanics: the quantum harmonic oscillator

- Quantization of the harmonic oscillator
- The quantum ground state
- The operator method: annihilation and creation operators
- Electromagnetic field quantization



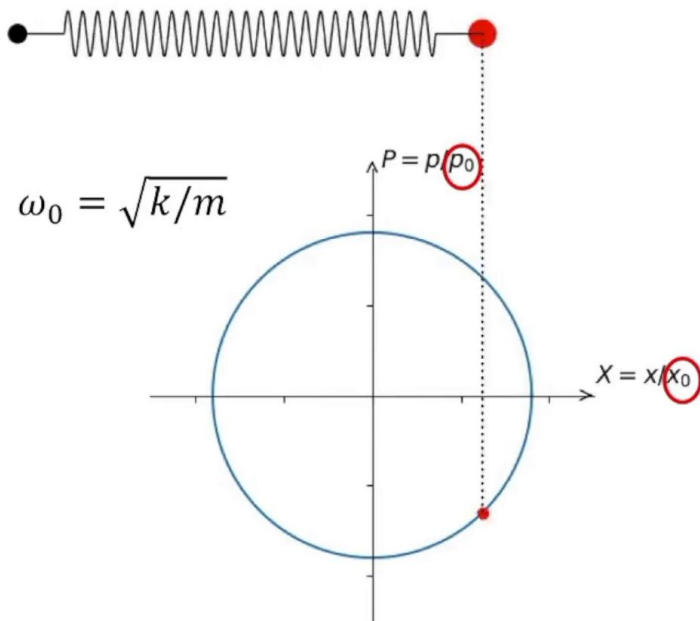
Welcome to this video on the harmonic oscillator and field quantization. I'm Samuel Deléglise from Laboratoire Kastler Brossel and this video belongs to the OMT mooc and cavity optomechanics. In this lesson, you will learn how to solve one of the most important problems in quantum mechanics, the quantum harmonic oscillator. One of the most simple example of a harmonic oscillator is this mass attached to a spring where potential energy of the system depends quadratically on the position of the mass. So this is the definition of a harmonic oscillator and in this lecture, we will see how to formulate this problem in a quantum mechanical language. And we will, in particular, see that because of Heisenberg's uncertainty principle the mass cannot remain strictly at rest at its equilibrium position. This leads to the existence of a quantum ground state which is the state with the minimum energy allowed by quantum mechanics and we will then introduce a very powerful method derived by Dirac to construct iteratively all the other energy eigenstates starting from the quantum ground state. Finally, we will see that this formalism is not limited to the mass-spring system. In fact, a very important example of harmonic oscillator is the electromagnetic field itself.

Notes

Summary



The classical harmonic oscillator



Complex amplitude
 $\alpha = X + iP$

Evolution:
 $\alpha = \alpha_0 e^{-i\omega_0 t}$

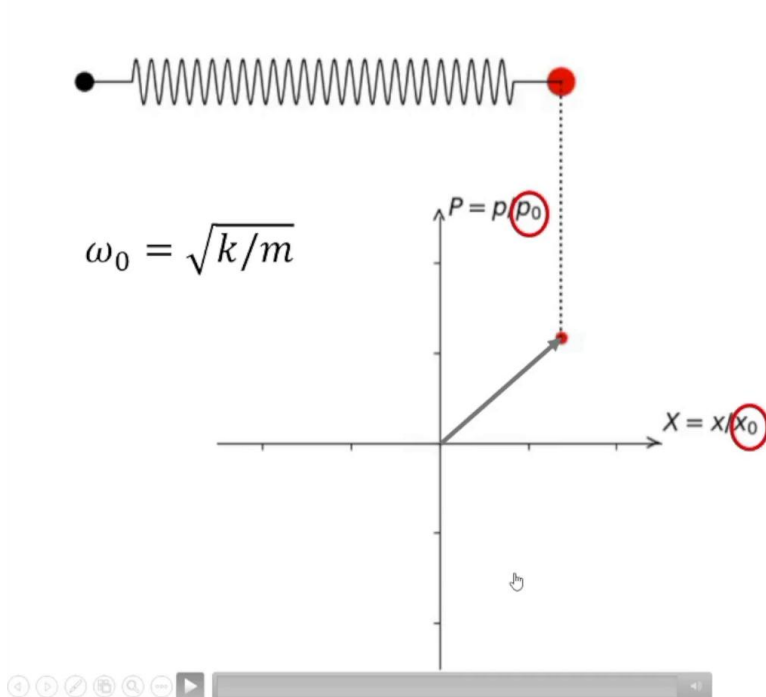
So classically, the state of a harmonic oscillator is encoded by two quantities, the position 'X' and the momentum 'P'. And a very useful way of looking at these quantities is to think about them as the real and imaginary part of a complex number called the complex amplitude alpha. The complex plane where this amplitude leaves is called the phase space and if we choose the right scaling between the real and imaginary axis, we can show that the complex amplitude will evolve according to simply a time varying phase such that it will draw a circle in the complex plane.

Notes

Summary



The classical harmonic oscillator



Complex amplitude
 $\alpha = X + iP$

Evolution:
 $\alpha = \alpha_0 e^{-i\omega_0 t}$

Energy

$E \propto \underbrace{X^2}_{\text{potential energy}} + \underbrace{P^2}_{\text{kinetic energy}} = |\alpha|^2$

Finally, the total energy is the sum of two terms, the potential energy which is quadratic with position as we said earlier and the kinetic energy which is quadratic with the momentum. And from Pythagoras theorem we see that this quantity is proportional to the modulus square of alpha.

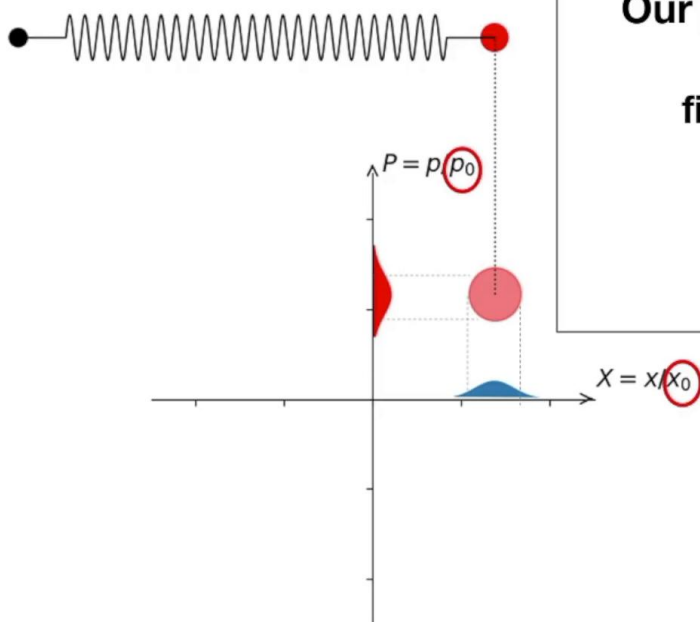
Notes

Summary



2m 04s

The quantum harmonic oscillator



Our goal:

find $\Psi_n(x), E_n$ such that

$$\hat{H}\Psi_n = E_n \Psi_n$$

$$\hat{p}[\Psi(x)] = -i\hbar \frac{\partial \Psi}{\partial x}$$

Hamiltonian

$$\hat{H} = \frac{1}{2}k\hat{x}^2 + \frac{\hat{p}^2}{2m}$$

Now quantum mechanically, the state of the harmonic oscillator is described by a wave function ψ of 'x' and position and momentum are now defined by observable that is operators acting on ψ . Position of the operator takes ψ and multiplies it by 'x' and the momentum operator takes ψ and returns a quantity that is proportional to derivative of ψ with respect to 'x'. An immediate consequence is that the position and momentum are not any more well-defined quantities. In a given state ψ we can associate to each of them a probability distribution that is represented in blue and red on the axis. These two quantities are called conjugate variables as they obey a particular commutation relation that we will study shortly. Finally, as in the classical case the Hamiltonian is the sum of a potential energy proportional to x-square and a kinetic energy proportional to p-square. What we want to achieve in this video is to diagonalize this Hamiltonian, that is, to find a set of wave function ψ_n of 'x' and corresponding energies ' E_n ' such that the following eigenvalue equations are fulfilled. And the first thing we are going to do is to find the most natural choice for this normalization constants x-naught and p-naught. And for this we will perform a simple dimensional analysis.

Notes

Summary



Dimensional analysis

Relevant parameters:

$$[k] = [FL^{-1}]$$

$$[m] = [M]$$

$$[\hbar] = [ET]$$



$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$x_0 = \sqrt{\frac{2\hbar}{m\omega_0}}$$

$$p_0 = \sqrt{2\hbar m\omega_0}$$

frequency
position
momentum

Dimensionless variables:

$$\hat{X} = \frac{\hat{x}}{x_0}, \quad \hat{P} = \frac{\hat{p}}{p_0}$$

In the classical harmonic oscillator, the only physical parameters that play a role are the spring constant 'k' that has units of newton per meter and the mass 'm' which is expressed in kilograms. From these units from these quantities we can form a frequency square root 'k' over 'm' which we immediately recognize as the natural oscillation frequency of the system. However, it is not obvious how to define x-naught and p-naught since there is no way to build a quantity homogeneous to a position or a momentum with these relevant parameters. On the other hand, in the quantum harmonic oscillator we have this extra parameter Planck's constant which appears in the definition of the momentum operator and this constant is homogeneous to an energy in times of time. And with these parameters we can form a position and a momentum which are basically the natural units in which to express the problem. We thus introduce the dimensionless operators capital 'X' equal 'x' over x-naught and capital 'P' equal 'p' over p-naught. I leave the following exercises for you at home. First, you should check that with those units the Hamiltonian takes this very pleasant form and second, you should calculate the commutator between 'X' and 'P' and show that it is equal to 'i' over two.

Notes

Summary



3m 57s

Dimensional analysis

Relevant parameters:

$$[k] = [FL^{-1}]$$

$$[m] = [M]$$

$$[\hbar] = [ET]$$



$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$x_0 = \sqrt{\frac{2\hbar}{m\omega_0}}$$

$$p_0 = \sqrt{2\hbar m\omega_0}$$

frequency
position
momentum

Dimensionless variables:

$$\hat{X} = \frac{\hat{x}}{x_0}, \quad \hat{P} = \frac{\hat{p}}{p_0}$$

Hamiltonian:

$$\hat{H} = \hbar\omega_0 \left(\hat{X}^2 + \hat{P}^2 \right)$$

Commutation relations:

$$[\hat{X}, \hat{P}] = \frac{i}{2}$$

This commutation relation is really fundamental essentially. Everything that we will do in the rest of this video is to deduce all the consequences of this commutation relation.

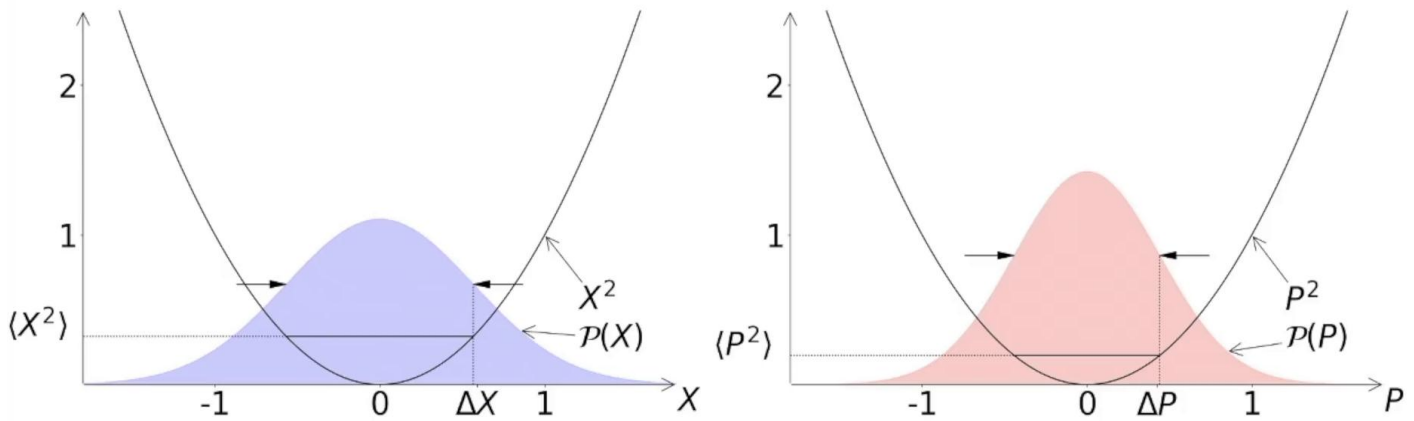
Notes

Summary



Hamiltonian in reduced variables

$$\hat{H} = \hbar \omega_0 (\hat{X}^2 + \hat{P}^2)$$



So the first very important consequence of this commutation relation is that we know that two non-commuting observables fulfill a Heisenberg inequality. In this particular instance, this inequality is $\Delta X \Delta P$ larger than one quarter. In other words, the width of the probability distribution in position times the width of the distribution in momentum is larger than one over four. So this value, of course, depends on the particular choice of units we have made over here and if we look in terms of energy now this means also that even if we take a probability distribution that is perfectly centered around zero, it will somehow extend to region where the potential is strictly positive such that the expectation value for the potential energy that is given by expectation of X^2 will be strictly positive. The same applies, of course, also for the kinetic energy. So if the position is very well localized around zero, the potential energy will be minimized. However, the distribution in momentum will be very large due to the Heisenberg relations such that kinetic energy will be larger and vice versa. So we already see at that point that if the goal is to minimize the total energy of the system, we will have to find a trade-off between kinetic and potential energy.

Notes

Summary



Factorizing the Hamiltonian (Operator method)

$$\Rightarrow \hat{H} = \hbar \omega_0 \underbrace{(\hat{X}^2 + \hat{P}^2)}_{\substack{(\hat{X} - i\hat{P})(\hat{X} + i\hat{P}) + \dots \\ \hat{a}^+ \quad \hat{a}}}$$

Definition: adjoint (or Hermitian conjugate) operator:

$$\forall |\varphi\rangle, |\psi\rangle \quad \langle \varphi | \hat{a}^+ | \psi \rangle = (\langle \psi | \hat{a} | \varphi \rangle)^*$$



So let's try to be more quantitative and to actually find the minimum energy of the quantum harmonic oscillator. So for this, we take some inspiration from the classical harmonic oscillator and let's imagine for a minute that 'X' and 'P' are just real numbers then this quantity this sum could be interpreted as the square magnitude of a complex amplitude alpha. So here in reality 'X' and 'P' are operators so that this operator 'X' plus 'iP' is in fact an operator which we denote by 'a' with a hat and the equivalent of the complex conjugate is the adjoint operator and we remind here the definition of the adjoint operator is an operator which fulfills this equality for any pairs of wave function phi and psi. I leave as an exercise to check that indeed 'X' minus 'iP' here is the adjoint operator of 'X' plus 'iP'. Another problem with the fact that we are dealing with operator is that if we develop this expression so okay, we are going to find X-squared here from the product between these two terms, P-squared from the product between these two terms but contrary to what would happen with complex numbers, the products 'X' and 'P' would not cancel out and we are left here with the commutator 'X' by 'P'.

Notes

Summary



7m 25s

Factorizing the Hamiltonian (Operator method)

$$\Rightarrow \hat{H} = \hbar \omega_0 (\hat{X}^2 + \hat{P}^2) = \hbar \omega_0 (\underbrace{\hat{a}^\dagger \hat{a}}_{\substack{(\hat{X} - i\hat{P})(\hat{X} + i\hat{P}) \\ \hat{a}^\dagger \quad \hat{a}}} + \underbrace{\frac{1}{2}}_{-i[\hat{X}, \hat{P}] = \frac{1}{2}})$$

Number operator

Definition: adjoint (or Hermitian conjugate) operator:

$$\forall |\varphi\rangle, |\psi\rangle \quad \langle \varphi | \hat{a}^\dagger | \psi \rangle = (\langle \psi | \hat{a} | \varphi \rangle)^*$$



Since this quantity equals 'i' over two, we are left at the end of the day with a-dagger 'a' plus one-half. So we have here the sum of an operator which we call the number operator and which is reminiscent of a square magnitude of a complex amplitude and one-half times identity. What we will show next is that this number operator can only have positive eigenvalues such that the energy cannot be smaller than one-half of h-bar omega naught.

Notes

Summary



The quantum ground state

Let's take $|\Psi\rangle$ a (normalized) eigenvector of $\hat{a}^\dagger \hat{a}$ with eigenvalue λ :

$$\lambda = \|\varphi\|^2 \quad \text{with} \quad |\varphi\rangle = \hat{a}|\Psi\rangle$$

So for this let's take an eigenvector ψ of the number operator with an eigenvalue λ and for simplicity we'll suppose that ψ is normalized to one. So we have the following eigenvalue equation. $\hat{a}^\dagger \hat{a} \psi = \lambda \psi$ and we will first multiply this equation on the left by ψ . So let's take care of the right-hand side first. We see that we have a scalar value λ inside the product so we can take it out and we end up with this expression and since ψ is normalized, this bracket equals to one such that the right-hand side of the equation is λ . For the left-hand side, we will just introduce a new vector φ which is the action the result of the action of \hat{a} on ψ . And you can show using the properties of the adjoint operator that the ψ multiplied by \hat{a}^\dagger is exactly the φ . So the left-hand side of the equation is the square norm of the vector φ . So at the end of the day we have $\lambda = \|\varphi\|^2$ where φ is the result of the action of \hat{a} on ψ . The first consequence of this is that the eigenvalue λ is positive because it's the norm of a vector, a square norm of vector and the second consequence is that this value can only be zero if and only if $\hat{a} \psi = 0$.

Notes

Summary



9m 45s

The quantum ground state

Let's take $|\Psi\rangle$ a (normalized) eigenvector of $\hat{a}^\dagger \hat{a}$ with eigenvalue λ :

$$\lambda = \|\varphi\|^2 \quad \text{with } |\varphi\rangle = \hat{a}|\Psi\rangle$$

Consequence 1: $\lambda \geq 0$

Consequence 2: $\lambda = 0 \quad \Leftrightarrow \quad \hat{a}|\Psi\rangle = 0$

$$\hat{X} + i\hat{P}$$

Can this lower bound be reached ?

$$\left(X + \frac{1}{2} \frac{\partial}{\partial X}\right) \Psi(x) = 0 \quad \Leftrightarrow \quad \Psi_0(x) \propto e^{-x^2}$$



So can we actually reach this lower bound? To see this, we can remember the expression of the operator 'a' which is 'X' plus 'iP' and we can just introduce the expression of 'X' and 'P' and we see that this operator 'X' plus 'iP' is just the multiplication by 'X' plus one-half derivative with respect to the variable capital 'X'. So this operator operating on psi of 'X' should be equal to zero and we have here a first order differential equation that can be solved with standard techniques and we find that the solution is psi naught of 'X' proportional to Gaussian exponential minus X-squared.

Notes

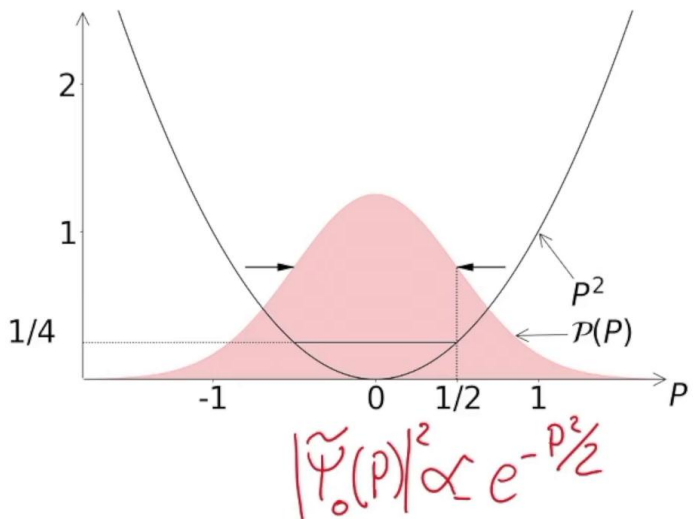
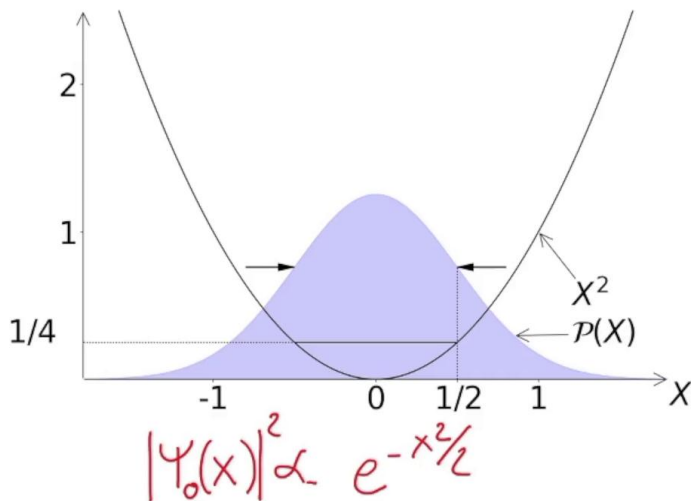
Summary



11m 40s

The quantum ground state

$$\hat{H}|\psi_0\rangle = E_0|\psi_0\rangle \text{ with } E_0 = \hbar\omega_0/2$$



So the situation is the following. We have a first eigenvector ψ_0 of the Hamiltonian with the energy E_0 equals one-half $\hbar\omega_0$ and moreover, we have shown that there are no other states with an energy below this value. So we call this state the quantum ground state of the harmonic oscillator. Moreover, we have found that the wave function at this quantum state is a Gaussian centered around zero. And since the wave function in the momentum space is the Fourier transform of that in the position space, it is also a Gaussian centered around zero. And as anticipated, the width of the probability distribution in position and in momentum are equal to one-half such that the potential energy and the kinetic energy contribute equally to one-quarter $\hbar\omega_0$ into the total energy of the state.

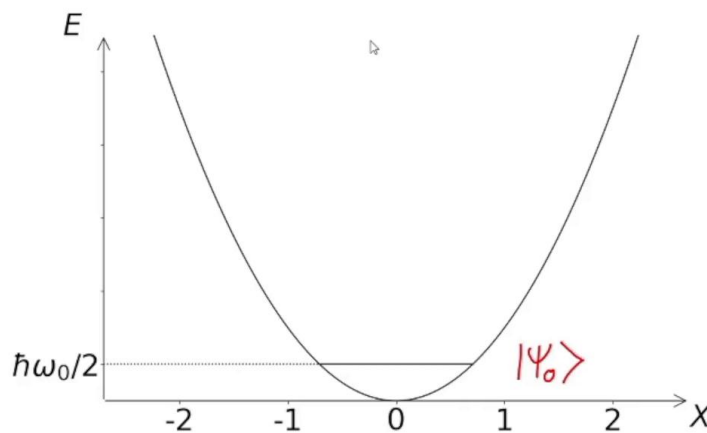
Notes

Summary



12m 33s

The quantum ground state



What about the other eigenstates ?



So this is the current situation. We have the lower energy eigenstate of the harmonic oscillator ψ_0 . We know that there are no energy eigenstate with an energy below this value and we would like now to find the other energy eigenstate of the problem.

Notes

Summary

13m 40s



Building the Fock state ladder...

...from the following commutation relations:

$$[\hat{H}, \hat{a}] = -\hbar\omega_0 \hat{a}$$

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega_0 \hat{a}^\dagger$$

Let's take an eigenvector of \hat{H} :

$$\hat{H} |\psi\rangle = E |\psi\rangle$$

$$\hat{H}(\hat{a} |\psi\rangle) =$$



And for this, we are going to use a beautiful method that only exploits one single thing, commutation relation between the Hamiltonian and the operator 'a' and a-dagger. So I leave as an exercise at home to derive this commutation relation from the expression of 'H' a' and a-dagger. It should take you just a minute. You can pause the video and if you've done the calculation, you should have found something very particular which is that you take your initial operator 'a' or a-dagger, you calculate the commutator with 'H' and at the end, you get an operator that's proportional to the initial operator, minus $\hbar\omega_0$ a' in the first case and plus $\hbar\omega_0$ a-dagger. And what we will see in a minute is that from this very particular structure of the commutation relation we can deduce many many things. So let's start from an eigenvector of 'H'. 'H' psi equals 'E' psi so here the energy 'E' can be whatever. We don't suppose anything at the moment. But let's just try to calculate the action of 'H' on the vector 'a' psi. So we don't know the action of 'a' on psi. We don't know nothing about that but what we know is the way 'H' acts on psi so what we're going to do is we are going to exploit the commutation relation between 'a' and 'H'.

Notes

Summary



13m 58s

Building the Fock state ladder...

...from the following commutation relations:

$$[\hat{H}, \hat{a}] = -\hbar\omega_0 \hat{a} \quad [\hat{H}, \hat{a}^\dagger] = \hbar\omega_0 \hat{a}^\dagger$$

Let's take an eigenvector of \hat{H} : $\hat{H} |\Psi\rangle = E |\Psi\rangle$

$$\hat{H}(\hat{a} |\Psi\rangle) = (E - \hbar\omega_0)(\hat{a} |\Psi\rangle)$$

$$\hat{H}(\hat{a}^\dagger |\Psi\rangle) = (E + \hbar\omega_0)(\hat{a}^\dagger |\Psi\rangle)$$

→ $\hat{a} |\Psi\rangle$ is an eigenvector with energy $E - \hbar\omega_0$

→ $\hat{a}^\dagger |\Psi\rangle$ is an eigenvector with energy $E + \hbar\omega_0$



So we can always write the things like that. We commute 'a' and 'H' and for this equality to hold we need to add here the commutator and now 'H' acting on psi simply gives the scalar 'E' times psi and here commutator 'H' with 'a' is minus h-bar omega naught 'a'. So we can now since this is a scalar which commutes with 'a' and we can group the terms and we find 'E' minus h-bar omega naught 'a' psi. In other words, we have now found a new eigenvector of the Hamiltonian with energy 'E' minus h-bar omega naught. This vector 'a' psi is another solution of our eigenvalue problem. You can also make the same calculation with a-dagger and you will find that you have an eigenvector with an energy 'E' plus h-bar omega naught. So what these operators are doing is something very handy. Let's say you already have an eigenvector of the problem and you let 'a' act on it, and what you get is a new eigenvector with an energy decreased by h-bar omega naught. Let a-dagger act on it and you get a new eigenvector with an energy increased by h-bar omega naught.

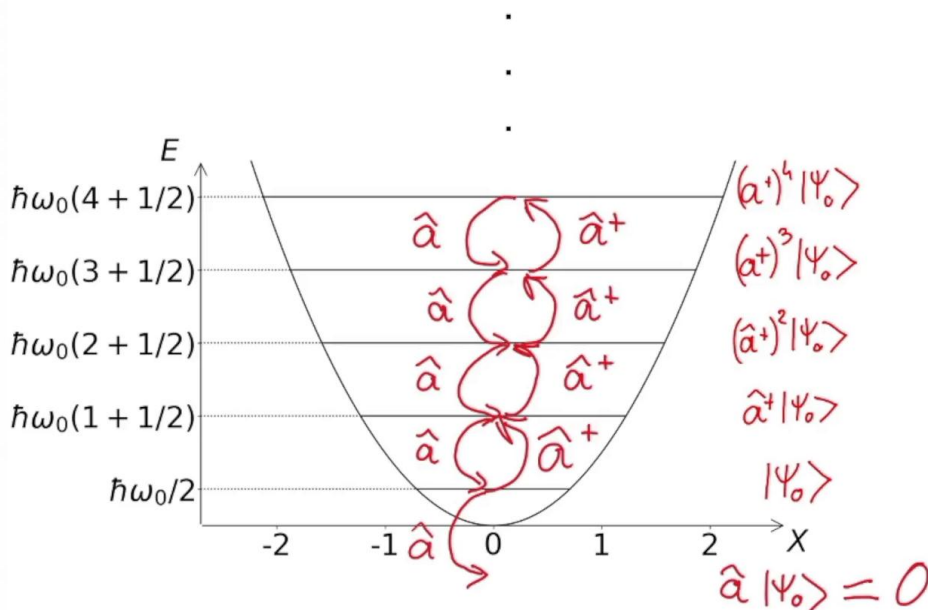
Notes

Summary



15m 33s

Building the Fock state ladder



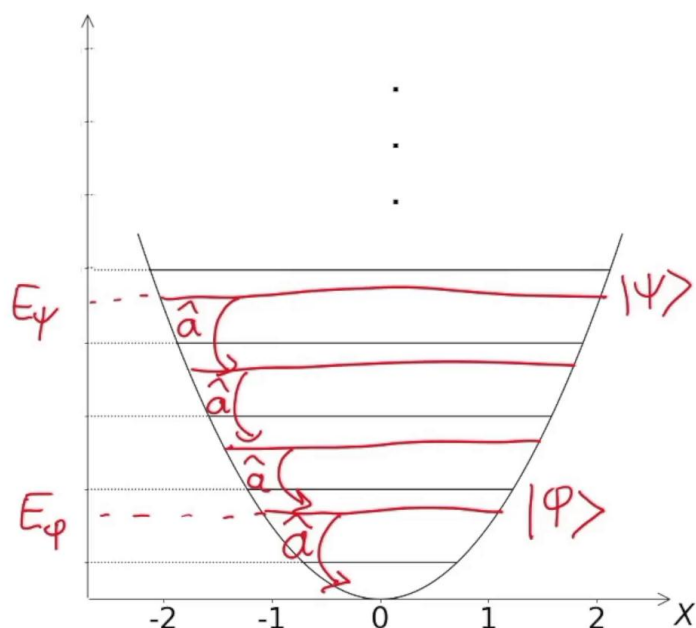
So let's take the only eigenvector that we have derived so far which is the quantum ground state and let \hat{a}^\dagger act on it and we get a second eigenvector. We repeat the operation to create a third, a fourth and so on. And we see that we can build an infinite ladder of eigenvectors equally spaced by an energy interval $\hbar\omega_0$. We can also apply the operator ' \hat{a} ' to climb this ladder downwards. And we can repeat this as many times until we reach the quantum ground state. So what about acting one more time with ' \hat{a} ' on this vector? We should get a new eigenvector with energy decreased by $\hbar\omega_0$. Wait. No. We have seen that ' \hat{a} ' acting on the quantum ground state $|\psi_0\rangle$ gives the scalar zero and not a vector so the ground state has just the right property such that it can stop this infinite ladder from propagating downwards.

Notes

Summary



Are there other eigenstates ?



Let's take an arbitrary eigenstate $|\Psi\rangle$ of energy E_Ψ

... then, for n large enough, $|\phi\rangle = \hat{a}^n |\Psi\rangle$ has an energy $E_\phi < 3/2 \hbar \omega_0$...

... and $\hat{a}|\phi\rangle$ is an eigenstate of energy $E_\phi - \hbar \omega_0 < \hbar \omega_0/2 \rightarrow$ **Impossible**

... unless $\hat{a}|\phi\rangle = 0$, then $|\phi\rangle$ is the ground state !

That's actually how we can prove that the ladder we have just constructed contains all the eigenvectors of the problem. Let's take an eigenvector ψ with an energy that would lie outside of the ladder so we can apply 'a' as many times as needed such that we create an eigenvector ψ with an energy between one-half and three-half $\hbar \omega_0$. Then if we apply 'a' one more time, we will create an eigenstate of energy $\hbar \omega_0$ smaller than $\hbar \omega_0/2$. So this is absurd unless ϕ is actually the ground state but since the red ladder is equally spaced with an energy $\hbar \omega_0$, then this is in contradiction with the fact that our initial state was outside of the black ladder. So we have constructed this way all the states of the quantum harmonic oscillator.

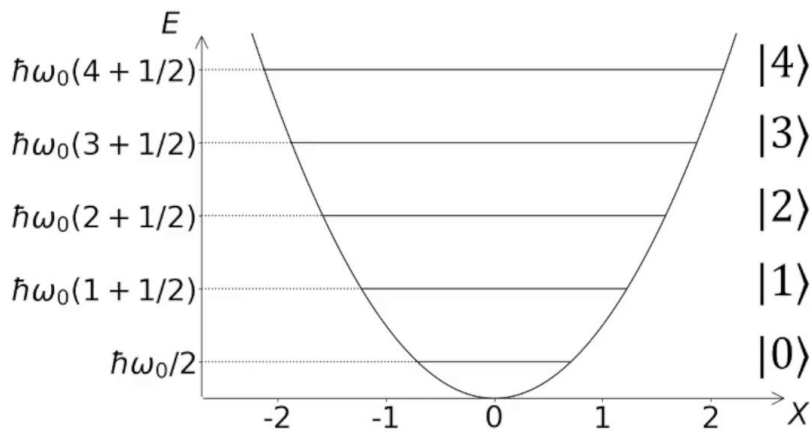
Notes

Summary



17m 46s

The phonons



\hat{a}^\dagger : creation operator

“Creates a quantum of excitation in the oscillator”

\hat{a} : annihilation operator

“annihilates a quantum of excitation in the oscillator”

... These elementary excitations are called *phonons*

So we have now derived the full excitation spectrum of the harmonic oscillator and we can see that it's totally analogous to that of an ensemble of identical particles each with an energy $\hbar\omega_0$. These particles are the elementary excitation of the oscillator. The action of \hat{a}^\dagger is to create an excitation in the system and the action of \hat{a} is to destroy an excitation. That's why these operators are called the creation and annihilation operators. For the material harmonic oscillator that we have discussed so far where the conjugate variables are the position and the momentum of a mechanical object, these elementary excitations are called phonons. However, there are other systems that can be described by a perfectly equivalent forms.

Notes

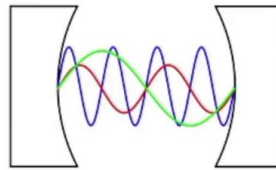
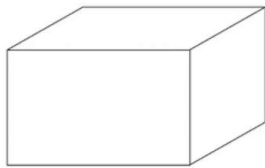
Summary

18m 47s



E. M. field quantization

Each mode of the electromagnetic field...



$\hat{a}_1, \hat{a}_3, \hat{a}_7$

... is associated with a quantum harmonic oscillator

$$\hat{H} = \sum \hbar \omega_k (\hat{a}_k^\dagger \hat{a}_k + 1/2)$$

Photons: elementary excitations of these quantum harmonic oscillators

A particularly important example are the individual modes of the field, of the electromagnetic field. A mode of the field is a solution of Maxwell's equation that fulfills some binary conditions. For instance, plane waves can be seen as the mode of a fictitious box with periodic binary condition. Another important example are the standing waves confined by the mirror of a Fabry–Pérot cavity. Each of the modes is associated with the harmonic oscillator at the mode frequency. All the notion that we have derived for the material harmonic oscillator equally applies to single modes of the field. The elementary excitation of these oscillators are called a photons. So up to this point we have only treated the case of perfectly isolated harmonic oscillator. You see that total Hamiltonian of the field here is just the sum of the Hamiltonian of the individual modes of the field. In the following lecture, you will see how to treat the more realistic scenario of a cavity that is coupled to various input and output fields using the quantum Langevin equations.

Notes

Summary



19m 40s

CONCLUSION / SUMMARY

- Position, momentum, and the phase space of the harmonic oscillator
- Commutation relations between conjugate variables
- Minimum energy eigenstate: the quantum ground state
- Spectrum of the Harmonic oscillator: the Fock state ladder
- Creation/annihilation operators
- Elementary excitations of the Harmonic oscillator: phonons/photons

To summarize this lecture we have reminded the definition of the phase space and its relation to the position and momentum for the classical and the quantum harmonic oscillator. We have discussed the commutation relation between these conjugate variables and we have shown that this commutation relation lead to the existence of the lowest energy state with strictly positive energy, the quantum ground state. We have then derived the full spectrum of the harmonic oscillator using Dirac's method and we have shown how to navigate within the spectrum using the creation and the annihilation operator. Finally, we have defined the elementary excitation of various kinds of harmonic oscillator, the phonons for mechanical systems and the photons for the electromagnetic field. These concepts are at the heart of quantum optomechanics which ultimately aims at creating an interaction between these two kinds of excitations.

Notes

Summary



20m 58s