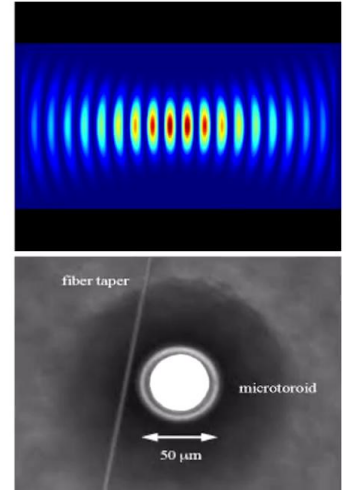


WELCOME & OUTLINE

- In this lecture you will learn about the quantum optics description of a cavity and how to define quantum input and outputs

The topics we will cover will include:

- Cavity mode as an harmonic oscillator
- Heisenberg-Langevin equation for a cavity mode
- Input-output relation and causality
- One-sided cavity
- Two-sided cavity



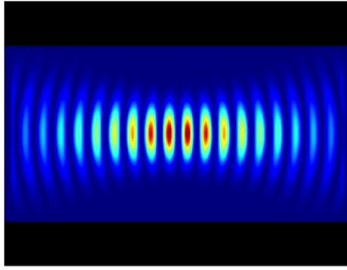
Welcome. In this lecture, you will learn how to derive in a quantum mechanically consistent way the dynamics of an optical cavity and how the input and output field the incoming outcoming in the cavity can be still described in terms of quantum mechanics. So that a proper dynamical description of the cavity can be provided. More in detail, you will learn how it's possible to define in a consistent quantum mechanical way the input and output field of an optical cavity and we will start from the second quantization description of a cavity mode as an harmonic oscillator. We will then see how to derive the Heisenberg-Langevin equation for a cavity mode which will describe in terms of input and output field. We will derive the relation between the input and the output field and how this relation is perfectly consistent with the demands of the principle of causality. At the end, we will provide two example. The case of one-sided cavity with only one port and the case of a two-sided cavity with two ports which is very common in practical situation.

Notes

Summary



QUANTIZED OPTICAL CAVITY MODE

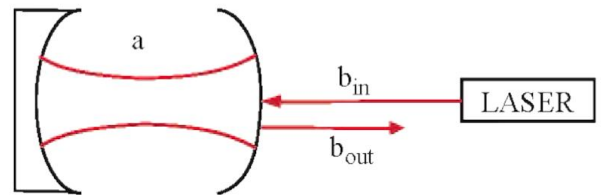


Second quantization of the e.m. field:
optical mode \leftrightarrow harmonic oscillator with the same frequency

$$H_{sys} = \sum_k \hbar \omega_k a_k^\dagger a_k$$

a_k^\dagger creates a photon in the k-th cavity mode, with commutation rules $[a_k, a_j^\dagger] = \delta_{jk}$

We drive and detect a single cavity mode with annihilation operator a (frequency ω_a)



$$H_{sys} = \hbar \omega_a a^\dagger a$$

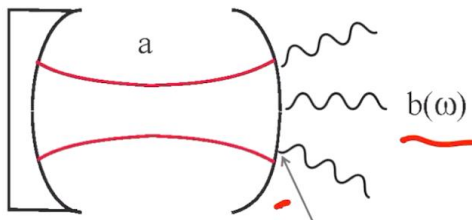
In second quantization of the electromagnetic field, each optical mode of the cavity which we remember or recall, confine light the field within a finite special region. Each optical mode is described in terms of an harmonic oscillator with the same frequency of the optical mode. At the end, the total Hamiltonian so the total energy of the cavity is the sum of many independent terms each associated to each cavity mode. Each mode is described in terms of annihilation creation operator so bosonic modes, which satisfies the usual commutation rules for Boson's field and the meaning is that ' a_k^\dagger ' creates a photon in a k-Th cavity mode. However, in general we focus on only one cavity mode only so we drive and detect a single cavity mode with a given frequency polarization which in our case it would be described by annihilation operator ' a ' and frequency ω_a . so that the Hamiltonian of the system is simply given by a single term.

Notes

Summary



CAVITY MODE AS AN OPEN SYSTEM



Interaction (1D photon scattering) at the output mirror

$$H_{\text{int}} = i\hbar \int_0^{\infty} d\omega \kappa(\omega) [b(\omega) + b^+(\omega)] [c - c^+]$$

Cavity mode = open system, interacting with the outside optical modes

$$H_B = \int_0^{\infty} d\omega \hbar \omega b^+(\omega) b(\omega)$$

$$[b(\omega), b^+(\omega')] = \delta(\omega - \omega')$$

Large (\sim optical) frequency $\omega_a \Rightarrow$ rotating wave approximation (RWA) (only quasi-resonant terms are kept, and extension to negative frequencies)

$$H_{\text{int}} \approx i\hbar \int_{-\infty}^{\infty} d\omega \kappa(\omega) [b^+(\omega)c - b(\omega)c^+]$$

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{int}} + H_B$$

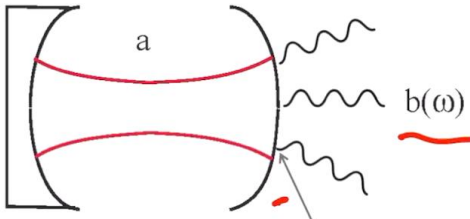
However, a cavity mode is an open system which means that it interacts with the outside world in the set environment which is in this case is composed by the all electromagnetic modes of the outside space so the space outside the cavity. In particular, since it is an infinite volume, it will be described by a continuum of bosonic mode, 'b' of omega with the usual commutation rules and this Hamiltonian is just the integral over all the possible modes. The interaction between the cavity mode and the reservoir of this electromagnetic field operator occurs just at the input-output mirror here. And we are using a single 1D photon scattering model in which the interaction comes from a typical dipole interaction which is linear in terms of the field of operator of the reservoir operator and moreover, is depends upon a generic cavity field operator 'c' which provides the coupling in order to be as general as possible. However, here we are considering optical frequency which means that omega sub 'a' is typically much larger than the typical frequency we are detecting. So it is reasonable to assume a rotating wave approximation which means keeping in interaction only the quasi-resonant terms and so this card being the so-called counter-rotating terms which are fast oscillating.

Notes

Summary



CAVITY MODE AS AN OPEN SYSTEM



Interaction (1D photon scattering) at the output mirror

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$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{int}} + H_B$$

So at the end one ends up with the simplified expression for the interaction. And moreover, we also extend it to negative frequency all the integral so we include also negative frequency components which are not so physical but on the other end allows us to get in the future more simplified expression. However, this approximation is reasonable because all the dynamics occurs around the frequency ω_a which is much larger than zero. At the end we will end up with the total Hamiltonian of this system which will provide the full dynamics we're interested in as a sum of three terms. The system Hamiltonian of the cavity, the interaction Hamiltonian in the rotating wave approximation and the Hamiltonian of the reservoir.

Notes

Summary



QUANTUM LANGEVIN EQUATION

Heisenberg equations

$$(i) \quad \dot{b}(\omega) = -i\omega b(\omega) + \kappa(\omega)c$$

Reservoir field operator

$$(ii) \quad \dot{a} = \frac{i}{\hbar} [H_{sys}, a] + \int_{-\infty}^{\infty} d\omega \kappa(\omega) \{b^+(\omega)[a, c] - [a, c^+]b(\omega)\}$$

Cavity field operator

Solution of (i)

$$b(\omega) = e^{-i\omega(t-t_0)}b(\omega, t_0) + \kappa(\omega) \int_{t_0}^t dt' e^{-i\omega(t-t')}c(t')$$

$$b(\omega, t_0) \quad t > t_0$$

boundary condition in the past

• Inserting the solution of (i) within (ii)

• Flat spectral density (first Markov approximation) $\kappa(\omega) = \sqrt{\kappa/2\pi}$

κ = cavity linewidth = photon decay rate through the input-output mirror

Now we derive the Quantum Langevin equation for the dynamics of the cavity field and we start from the Heisenberg equation. So we write the Heisenberg equation for the reservoir field operator, the Heisenberg equation for the cavity field operator then we solve the first equation for the reservoir field operator assuming a boundary condition in the past where by choosing a given initial condition 'b' of omega t-naught at time t-zero which is very far in the past. And then we will set the solution one into two in order to get an effective equation for the system we are interested in so the cavity field dynamics. Moreover, we also make the so-called first Markov approximation which means assuming a flat spectral density in the interaction between the cavity and the reservoir of electromagnetic field modes. In this way, we introduce a coupling constant 'K' which is the second important parameter for the cavity provides the cavity linewidth so 'K' is just the cavity linewidth which physically means is just the photon decay rate through the input-output mirror.

Notes

Summary



4m 45s

QUANTUM LANGEVIN EQUATION II

Defining the **input radiation field**
(reservoir operator)

$$b_{in}(t) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)} b(\omega, t_0)$$

$$[b_{in}(t), b_{in}^+(t')] = \delta(t-t')$$

Quantum Langevin equation
for the intracavity field operator
 $a(t)$, $\forall t > t_0$

$$\dot{a} = \frac{i}{\hbar} [H_{sys}, a] + \left[\frac{\kappa}{2} c^+ - \sqrt{\kappa} b_{in}^+(t) \right] [a, c] - \left[\frac{\kappa}{2} c - \sqrt{\kappa} b_{in}(t) \right] [a, c^+]$$

b_{in} = classical laser amplitude
driving + vacuum quantum
noise operator

$$b_{in}(t) = \beta_L + \delta b_{in}(t) \quad |\beta_L| = \sqrt{\frac{P_L}{\hbar \omega_L}} \quad P_L = \text{laser power}$$

Quantum noise properties specified by the correlations $\langle \delta b_{in}^+(t) \delta b_{in}(t') \rangle$, $\langle \delta b_{in}(t) \delta b_{in}(t') \rangle$,
fixed by the reservoir quantum state at time t_0 in the past

We then go on in order to derive the final formula Quantum Langevin equation. For that we need also to define the input radiation field which is a reservoir operator which is defined here and is simple reservoir operator so it's just the combination of the operator field fixed at the initial condition at time t_0 due to the definition and also because we have a standard integral over all the real axis so also to minus, so to negative frequencies. Again, also the input field as that are correlated commutation rules similar to the bosonic commutation rules. With this definition of b_{in} , we end up with a very simple form of the Quantum Langevin equation for the cavity field operator forward in time so for all times larger than t_0 , which should be expressed only in terms of the input operator and the input coupling operator ' c '. It's also interesting to notice that the input field can be always decomposed, in general, into two terms. A classical laser amplitude so it's a number which is we call β_L and which is proportional to the square root of the laser power plus a purely quantum noise operator which has zero mean and describe the vacuum fluctuation of the optical field outside the cavity.

Notes

Summary



TIME-REVERSED QUANTUM LANGEVIN EQUATION

Solution of (i)

$$b(\omega, t) = e^{-i\omega(t-t_1)} b(\omega, t_1) - \kappa(\omega) \int_t^{t_1} dt' e^{-i\omega(t-t')} c(t')$$

$$b(\omega, t_1) \quad t < t_1$$

boundary condition in the **future**

Defining the **output radiation field**
(reservoir operator)

$$b_{out}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_1)} b(\omega, t_1)$$

$$[b_{out}(t), b_{out}^+(t')] = \delta(t-t')$$

Time-reversed Quantum Langevin equation for the intracavity field operator $a(t)$,
 $\forall t < t_1$

$$\dot{a} = \frac{i}{\hbar} [H_{sys}, a] - \left[\frac{\kappa}{2} c^+ - \sqrt{\kappa} b_{out}^+(t) \right] [a, c] + \left[\frac{\kappa}{2} c - \sqrt{\kappa} b_{out}(t) \right] [a, c^+]$$

What are the properties which are interesting here for the fluctuation of the vacuum field? Just the quantum noise properties which are characterized by the correlations which we are describing here. In general, they are fixed by the quantum state of the reservoir at the time t_1 we have fixed in the past. It's also interesting to look at the time-reversed Quantum Langevin equation which looks at the dynamics backward in time so in this case, we fix the boundary condition in the future here at time ' t_1 ' which is far in the future and then we look at the dynamics backward in time so for all times ' t ' smaller than ' t_1 '. In this case, we define an output radiation field which is still a radiation reservoir operator and which depends upon this bosonic operator fixed at the future time ' t_1 '. Just because the definition is analogous to the input operator, the commutation rules are the same, still delta correlated. Again, we are considering all the positive and negative frequencies. And we end up in a very time-reversed Quantum Langevin equation which is very similar to the standard one, to the forward Langevin equation which is expressed now in terms of the output field in the future and the coupling operator ' c '.


Notes

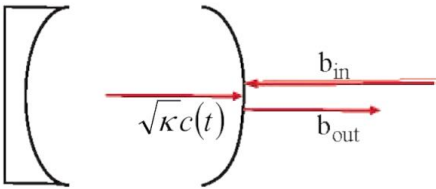
Summary



7m 34s

INPUT-OUTPUT RELATION AND CAUSALITY

Since $\iint \frac{1}{\sqrt{2\pi}} \int d\omega b(\omega, t) = b_{out}(t) - \frac{\sqrt{\kappa}}{2} c(t) = -b_{in}(t) + \frac{\sqrt{\kappa}}{2} c(t)$ 

Input-output relation $b_{out}(t) = -b_{in}(t) + \sqrt{\kappa} c(t)$ 

Quantum causality: system operators are not influenced by the input at later times and by the output at earlier times

$$\begin{aligned} [a(t), b_{in}(t')] &= \theta(t - t') \sqrt{\kappa} [a(t), c(t')] \\ [a(t), b_{out}(t')] &= \theta(t' - t) \sqrt{\kappa} [a(t), c(t')] \end{aligned}$$

$$\theta(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases} \quad \text{Heaviside step function}$$

Now, we are able to derive the input-output relation and also how it is consistent with causality. In fact, this quantity here which is a combination of the old input operator of the reservoir operator can be expressed in two equivalent forms, one depending upon the output and one depending upon the input. The two expressions must be the same and so we end up with the very simple input-output relation which we write here which has been derived only exploiting quantum mechanics so the Heisenberg equation, the Hamiltonian and the laws of the time evolution in quantum mechanics according to the Heisenberg picture. On the other hand, this is a very intuitive expression and describes another course at the input mirror because as we can see here, the output field is just the superposition of the reflected input field, the minus sign describes the reflection, plus the transmitted field into the cavity and square root of 'K' just provided is related to the transmission coefficient.

Notes

Summary



INPUT-OUTPUT RELATION AND CAUSALITY

Since

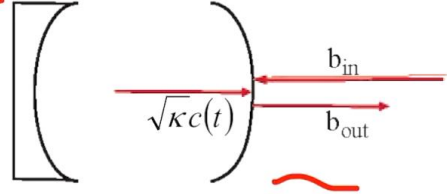
||

$$\frac{1}{\sqrt{2\pi}} \int d\omega b(\omega, t) = b_{out}(t) - \frac{\sqrt{\kappa}}{2} c(t) = -b_{in}(t) + \frac{\sqrt{\kappa}}{2} c(t)$$



Input-output relation

$$b_{out}(t) = -b_{in}(t) + \sqrt{\kappa} c(t)$$



Quantum causality: system operators are not influenced by the input at later times and by the output at earlier times

||

$$\begin{aligned} [a(t), b_{in}(t')] &= \theta(t-t') \sqrt{\kappa} [a(t), c(t')] \\ [a(t), b_{out}(t')] &= \theta(t'-t) \sqrt{\kappa} [a(t), c(t')] \end{aligned}$$

$$\theta(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

Heaviside
step
function

Moreover, it's interesting to notice that the definition of the input and output field is perfectly consistent with the causality which imposes the fact that the system operator cannot be influenced by the input at later times and by the output at earlier times and therefore, the commutation rules here between the input operator at later times with the cavity field must be zero and again, for the same argument the commutation between the commutator between the output field at earlier times and the cavity field must be zero as expressed here by this Heaviside step function which I've defined here. And therefore, the fact that this commutator is zero at the relevant times this time interval is just a manifestation of causality.

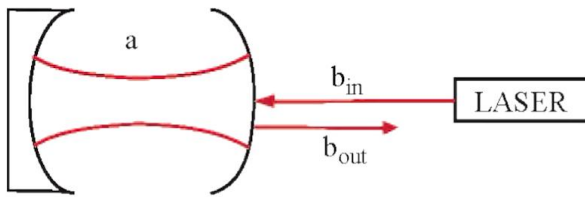
Notes

Summary



10m 20s

EXAMPLE I: EMPTY ONE-SIDED CAVITY



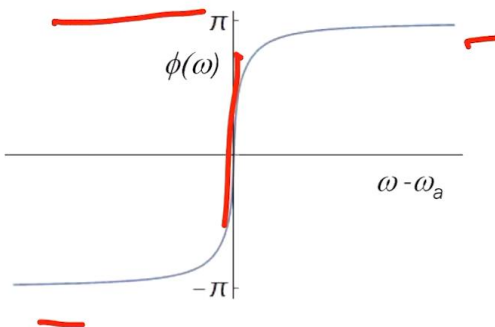
$$H_{sys} = \hbar \omega_a a^\dagger a \quad c = a$$

Standard single-photon loss through the mirror

Fourier-transforming

$$\dot{a} = -i\omega_a a - \frac{\kappa}{2}a + \sqrt{\kappa}b_{in}(t)$$

$$a(\omega) = \frac{\sqrt{\kappa}}{\kappa/2 - i(\omega - \omega_a)} b_{in}(\omega)$$



$$b_{out}(\omega) = \frac{\kappa/2 + i(\omega - \omega_a)}{\kappa/2 - i(\omega - \omega_a)} b_{in}(\omega) = e^{i\phi(\omega)} b_{in}(\omega)$$

Now let us look at some example. First of all, I'll consider a very simple example of a empty cavity so we have the standard system Hamiltonian and moreover, we can lose photon from the input-output mirror, which means that we can choose here as coupling operator 'c' is exactly the cavity field operator 'a' so we describe one photon loss through the input-output mirror. The corresponding Langevin equation is very simple, describes the 'K' through the input mirror and driving by the input field. By Fourier transforming, one can have a full picture in this case actually, and also derive the output as a function of the input. In this case, the amplitude of the output must be the same of the input. What happens that we have only a phase shift described by a phase shift phi whose behavior is shown here in the picture here. As we can see, it goes from minus Pi to Pi and it has a linear dependence around resonance.

Notes

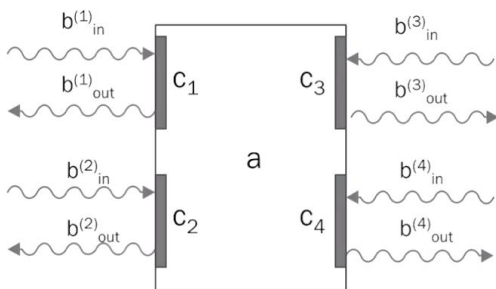
Summary



11m 17s

SEVERAL INPUT-OUTPUT PORTS

Each port is coupled to its own independent reservoir, via a system operator c_l in general



Quantum Langevin equation
(for the intracavity field operator)

κ_l = photon decay rate through the l -th input-output port

$$\dot{a} = \frac{i}{\hbar} [H_{\text{sys}}, a] + \sum_l \left[\frac{\kappa_l}{2} c_l^+ - \sqrt{\kappa_l} b_{\text{in}}^{l,+}(t) \right] [a, c_l] - \sum_l [a, c_l^+] \left[\frac{\kappa_l}{2} c_l - \sqrt{\kappa_l} b_{\text{in}}^l(t) \right]$$

Each port has its own input-output relation

$$b_{\text{out}}^l(t) = -b_{\text{in}}^l(t) + \sqrt{\kappa_l} c_l(t) \quad \forall l$$

In order to look at the second example, we have to generalize our Quantum Langevin equation to the K's of several input-output ports, as we have shown, for example, in the picture here. Each Port is characterized by its own input its own output field because it's characterized by its own independent reservoir and moreover, by in general, also by different coupling operator ' c_l ' in order to be as general as possible. One can repeat all the same steps and arrive at this Langevin equation which has the same form of the one we've already seen except that now we sum over all the possible dissipative channels so over all the possible ' l ' ports. In this way, we will have now many terms. Each term is characterized by its own input field ' b_{in}^l ', it's own coupling parameter a coupling operator ' c_l ' of the cavity and its own photon decay rate through the l -Th input port ' κ_l '. Each port is characterized by its own input-output relation relating as we can see here the output and the input of the same port with the corresponding transmitted component coming from the cavity.

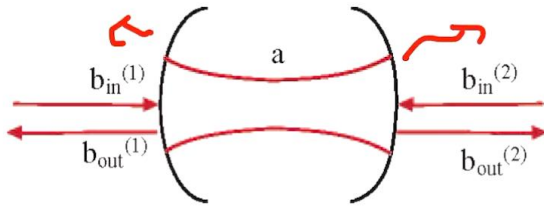
Notes

Summary



12m 26s

EXAMPLE II: TWO-SIDED CAVITY



$$H_{sys} = \hbar \omega_a a^\dagger a \quad c_l = a \quad \forall l$$

$$\dot{a}(t) = -i\omega_a a(t) - \frac{\kappa_1 + \kappa_2}{2} a(t) + \sqrt{\kappa_1} b_{in}^{(1)}(t) + \sqrt{\kappa_2} b_{in}^{(2)}(t)$$

$$\begin{pmatrix} b_{out}^{(1)}(\omega) \\ b_{out}^{(2)}(\omega) \end{pmatrix} = \begin{pmatrix} s_{11}(\omega) & s_{12}(\omega) \\ s_{21}(\omega) & s_{22}(\omega) \end{pmatrix} \begin{pmatrix} b_{in}^{(1)}(\omega) \\ b_{in}^{(2)}(\omega) \end{pmatrix}$$

Scattering matrix

Fourier transforming
and using input-output
relations



We can apply this general theory to the case of a two-sided cavity in which again, we choose for the coupling operator 'cl' again 'a' so we describe single photon laws through each mirror here and here. The resulting Langevin equation is still simple. We have a total decay rate which is the sum of the two decay rates through the each port, two input noise term. And in general, we can relate the input and the output by using the scattering matrix formalism, as you can see here and the expression of the scattering matrix can be evaluated by looking at the Fourier transform and using input-output relations.

Notes

Summary

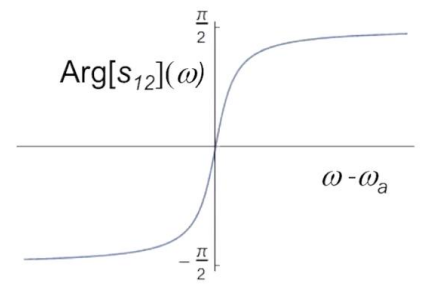
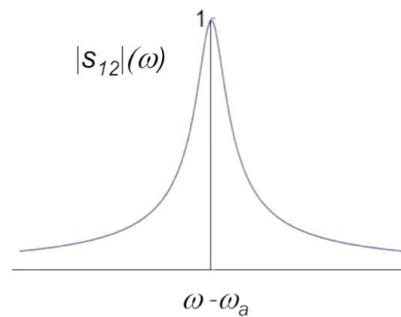


13m 55s

TWO-SIDED CAVITY II

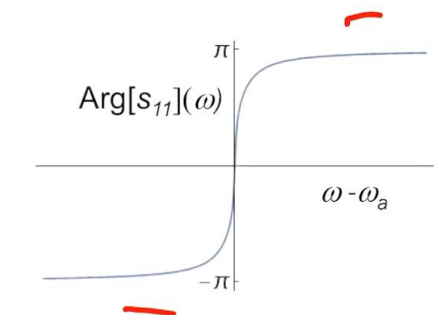
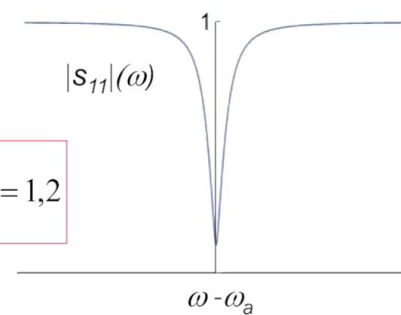
Amplitude transmission

$$s_{12}(\omega) = s_{21}(\omega) = \frac{\sqrt{\kappa_1 \kappa_2}}{(\kappa_1 + \kappa_2)/2 - i(\omega - \omega_a)}$$



Amplitude reflection

$$s_{jj}(\omega) = \frac{(-1)^j (\kappa_2 - \kappa_1)/2 + i(\omega - \omega_a)}{(\kappa_1 + \kappa_2)/2 - i(\omega - \omega_a)} \quad j = 1, 2$$



Applying this formalism, we end up with this explicit expression of the scattering matrix elements so the off-diagonal elements are identical and they provide the amplitude transmission. The amplitude, the modulus of the amplitude just shows as expected the usual Lorentzian form around the peak so at some peak around the cavity with the width given by the sum of the two decay rates here. The phase of the transmission instead goes from minus $\pi/2$ to $\pi/2$ and it is linear around resonance. The off-diagonal terms 'S11 S22' instead describe reflection at each port and the modulus shows the usual reflection dip whose properties depends upon the explicit value of ' κ_1 ' and ' κ_2 '. While the phase shift they're gaining in this case is similar to what occurs in the case of a one-sided cavity because it goes from minus π to π and again, it is linear around resonance.

Notes

Summary



14m 41s

SUMMARY

- We have provided a consistent quantum picture of the dynamics of an optical cavity field in terms of Quantum Langevin equations
- We have defined input and output fields of a generic optical cavity, characterized their quantum properties, and derived their relationships

In conclusion, we have provided two main results here in this lecture. We have provided a consistent quantum picture of the dynamics of an optical cavity field in terms of the Quantum Langevin equation and we have considered a forward and backward and also generalized to many input case, many input-output port cases. Moreover, we have defined and studied the properties of input and output fields of the cavity and we have seen how this is consistent with the general principle quantum mechanics. We have characterized their quantum properties and we have especially verified that they satisfy all the constraints imposed by the causality principle.

Notes

Summary



15m 56s