Optomechanical Hamiltonian and Quantum Langevin Equations

Cavity Quantum Optomechanics

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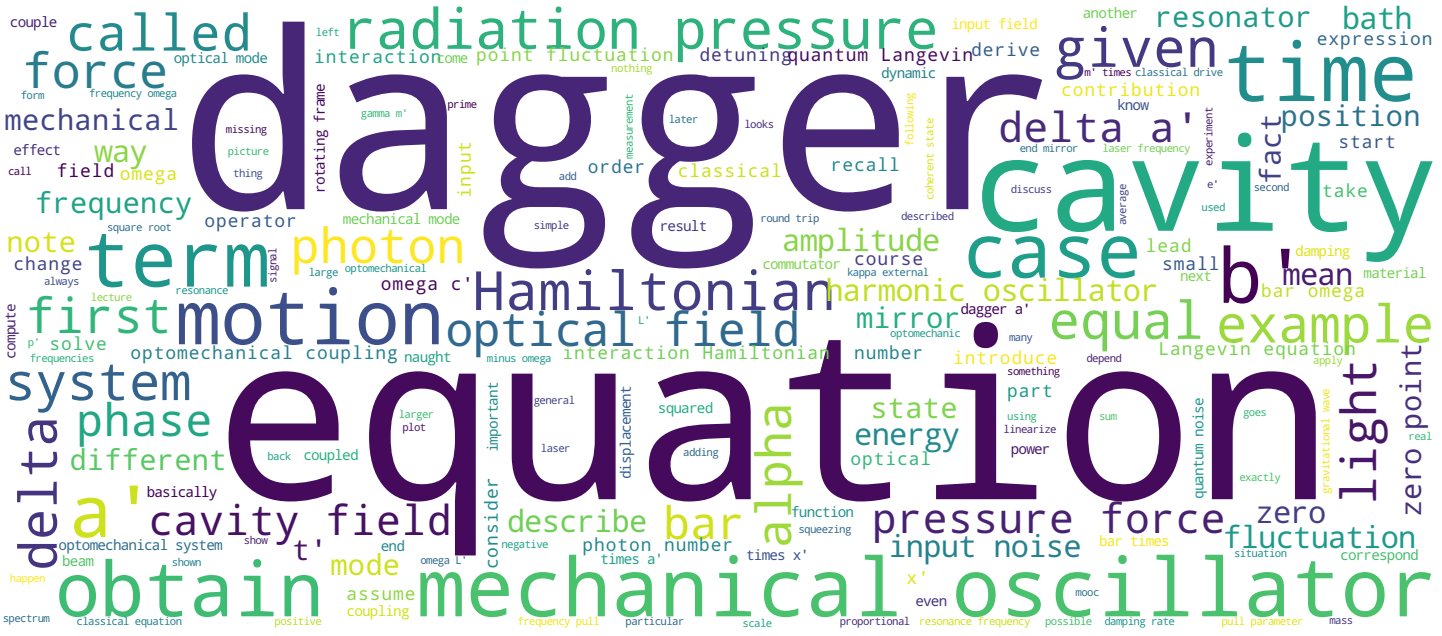
EPFL



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OUTLINE

In this lecture, you will learn how to quantum mechanically describe the dynamics of an optomechanical system:

- **Hamiltonian description of the optomechanical coupling**
- **Quantum equations of motion (quantum Langevin equation)**

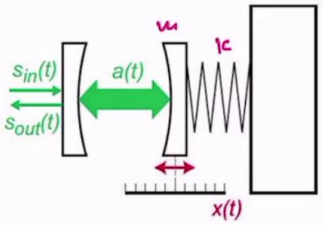
Hello. My name is Tobias Kippenberg. I work at EPFL, the Swiss Federal Institute of Technology in Lausanne, Switzerland and I'd like to talk today about the optomechanical coupling and how to describe it by Hamiltonian and the corresponding equations of motions that result from that. So in this mooc what you will learn are how to describe the dynamics of an optomechanical systems. I'll describe how the radiation pressure entails a Hamiltonian description of the optomechanical coupling and I'll describe how from this Hamiltonian, you can derive the equations of motion or in this case the so-called quantum equations of motion the quantum Langevin equations of the optomechanical system, that is, the mechanical oscillator and the optical field.

Notes

Summary

0m 00s

Hamiltonian description



$\hat{u} = \hat{a}^\dagger \hat{a}$ $\quad \hat{n}|u\rangle = u|u\rangle$

$\omega_m = \sqrt{\frac{k}{m}}$

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b}$$

optical mode — mechanical mode

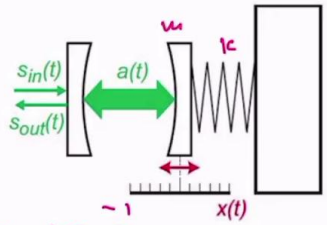
Okay, so I'd like to next discuss is the Hamiltonian description and so for this we consider at first the optical mode in the cavity and the optical mode is described by an operator a-dagger 'a' which is the photon number operator and so we know when the photon number operator acts on the number state we obtain number of photons inside the cavity. Then we also have a harmonic oscillator that is related to the mechanical mode. And this level spacing is h-bar omega 'm' and this describes the harmonic oscillator response of the end mirror. And so the frequency of omega 'm' is just square root of 'k' over 'm' if 'k' is our spring constant and 'm' is the mass of the mirror. So now we would like to find out and derive the so-called Hamiltonian that describes the interaction between the optical mode and the mechanical mode. And there is a simple way to do so. First of all we note that by coupling the mechanical end mirror, 2D optical mode, what we have done is we have made, we've entered the cavity mechanically compliant. What we can also state is that the photons in the cavity exert a radiation pressure force and we're going to use this first to derive the Hamiltonian.

Notes

Summary

0m 42s

Hamiltonian description



$\hat{u} = \hat{a}^\dagger \hat{a}$ $\quad \hat{n}|u\rangle = u|u\rangle$

$\omega_m = \sqrt{\frac{k}{m}}$

$T_{rt} = \left(\frac{c}{2L}\right)$

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b}$$

optical mode — mechanical mode

Radiation pressure force $\hat{F}_{RP} = 2\hbar k \left(\frac{\hat{a}^\dagger a}{T_{rt}}\right) = \hbar \frac{\hat{a}^\dagger \hat{a}}{L} \cdot \omega = \hbar \frac{\omega}{L} \hat{a}^\dagger \hat{a}$

Photon Flux

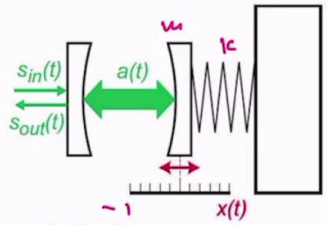
So the radiation pressure force that we have already described classically so the radiation pressure force is just given by F-rad is given by two h-bar ‘k’ times the number of photons per second impinging on the mirror. So and this can be expressed now if a-dagger 'a' is the photon number. We can express the radiation pressure force here as a-dagger ‘a’ so this number of photons intake in the cavity and now we have to divide here by the round trip time 'rt'. And why is that because if we divide by the round trip time, we obtain again the power in the cavity and again bear in mind now that a-dagger 'a' is the right number of photons so this is a flux, a photon flux. And this is nothing else than a-dagger ‘a’. And if you do the math and we also for the round trip time now so for 't' round trip, we have 'c' over '2L'. That's our round trip time then what we have here we obtain just a-dagger ‘a’ over 'L'. This is 'r' times omega, this is our radiation pressure force times h-bar and that is equal to h-bar times omega over 'L' a-dagger ‘a’. And this here is our frequency pull parameter 'G' so we recovered the result that we have already seen before which is that h-bar 'G'. This is our photon so the per photon radiation pressure force and a-dagger 'a' is number of photons in the cavity.

Notes

Summary

2m 02s

Hamiltonian description



$\hat{u} = \hat{a}^\dagger \hat{a}$ $\quad \hat{n}|u\rangle = u|u\rangle$

$\omega_m = \sqrt{\frac{k}{m}}$

$T_{rt} = \left(\frac{c}{2L}\right)$

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b}$$

optical mode — mechanical mode

Per Photon radiation force

Radiation force $\hat{F}_{rp} = 2\hbar k \left(\frac{\hat{a}^\dagger \hat{a}}{T_{rt}}\right) = \hbar \frac{\hat{a}^\dagger \hat{a} \cdot \omega}{L} = \hbar \frac{G}{\omega} \hat{a}^\dagger \hat{a} = \hbar G \hat{a}^\dagger \hat{a}$

Photon Flux

zero point fluctuation $x_{zpf} = \sqrt{\frac{\hbar}{m\Omega_m}}$

Hamiltonian: $\hat{H} = \hat{F}_{rp} \cdot \hat{x} = \hbar G \hat{a}^\dagger \hat{a} \cdot x_{zpf} (\hat{b} + \hat{b}^\dagger)$

$\hat{x}$

$\hat{H}_{int} = \hbar \cdot G \, x_{zpf} (\hat{a}^\dagger \hat{a})(\hat{b} + \hat{b}^\dagger)$

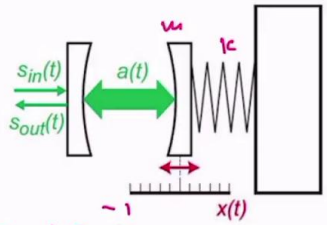
So now we can obtain the Hamiltonian. The way we obtain the Hamiltonian is now very simple. What we do is we recall that the energy of interaction is just the radiation pressure force of the mirror times the position coordinate. Okay. That is just for the classical of the energy done on the system so it's force times distance. In our case then we obtain h-bar 'G' times a-dagger ‘a’ times the mechanical oscillator coordinate which is the zero point fluctuation multiplied by 'b' plus b-dagger. So this is just the oscillator coordinate 'x' of the harmonic oscillator and the zero point fluctuation so normalization of position operator is just h-bar omega times 'm' omega ‘m’. That is the normalization of the so-called zero point fluctuation. So if we now put this together what we have for the interaction Hamiltonian, we obtain h-bar times ‘G’ times zero point fluctuations times a-dagger ‘a’ 'b' plus b-dagger. And now, I would like to recall and make you aware of a new interpretation of the product of ‘G’ and zero point fluctuations. So ‘G’ was our frequency pull parameter so ‘G’ had the units of frequency per displacement and if we multiply this quantity now with a zero point fluctuation, what we obtain is the so-called G-naught and G-naught this is the 'x' times 'zpf'.

Notes

Summary

3m 44s

Hamiltonian description



$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \hbar g_0 \hat{a}^\dagger \hat{a}(\hat{b} + \hat{b}^\dagger)$$

optical mode — mechanical mode

$$\hat{F}_{rp} = 2\hbar k \left(\frac{\hat{a}^\dagger \hat{a}}{T_{rt}}\right) = \hbar \frac{\hat{a}^\dagger \hat{a}}{L} \cdot \omega = \hbar \frac{G}{L} \hat{a}^\dagger \hat{a} = \hbar G \hat{a}^\dagger \hat{a}$$

$$\hat{H} = \hat{F}_{rp} \cdot \hat{x} = \hbar G \hat{a}^\dagger \hat{a} \cdot x_{zpf}(\hat{b} + \hat{b}^\dagger)$$

$$\hat{H}_{int} = \hbar \cdot G\, x_{zpf} (\hat{a}^\dagger \hat{a})(\hat{b} + \hat{b}^\dagger) \qquad g_0 = G \cdot x_{zpf}$$

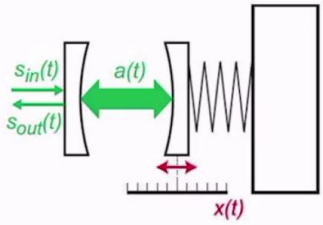
This is now our optomechanical vacuum rate and it tells us how much frequency variation we have for our cavity so how much our cavity shifts due to the zero point fluctuations. So it's a very natural way to quantify the interaction of the mechanical oscillator and the optical field at the quantum level and this is the so-called G-naught of the interaction. So now we have an optomechanical Hamiltonian. We can just derive it here. We have our Hamiltonian which is h-bar G-naught times a-dagger ‘a’ 'b' plus b-dagger. This is our interaction Hamiltonian that we obtain that describes the interaction of the mirror with the light field. Now a couple of comments are in place. First of all, note that this interaction Hamiltonian commutes with the original optical mode so therefore, you can use in principle the the mechanical response to measure the photon number something also that later in this mooc will be covered and secondarily note that this Hamiltonian is a three-wave mixing Hamiltonian. That is to say it is nonlinear. It has a nonlinearity associated with it given that it has three operators and leads to dynamic instabilities as we'll see and generally also in the context of the quantum treatment needs to be linearized.

Notes

Summary

5m 13s

Hamiltonian description



$$\hat{F} = \hbar G\hat{n}$$

measurement leads to a radiation pressure backaction

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar\Omega_{\rm m}\hat{b}^\dagger\hat{b} + \hbar G\hat{a}^\dagger\hat{a}\hat{x}$$

Radiation pressure Force: $F_{\rm RP} = \frac{P}{\hbar\omega}2\hbar\kappa = \bar{n}_{\rm p}\hbar G;\ \hat{F}_{\rm RP} = \hbar G\hat{n}$

Force can be derived from a Hamiltonian: $\hat{H}_{\rm int} = (\hbar G\hat{n})\hat{x} = \hbar g_0\hat{a}^\dagger\hat{a}(\hat{b}^\dagger + \hat{b})$

$$\hat{H}_{\rm int} = \hbar G x_{\rm zpf}\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger) \qquad \hat{x} = x_{\rm zpf}(\hat{b} + \hat{b}^\dagger) \qquad x_{\rm zpf} = \sqrt{\frac{\hbar}{2m\Omega_{\rm m}}}$$

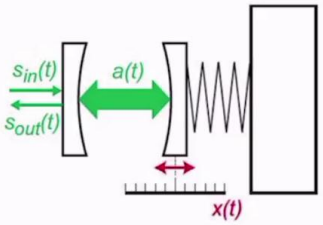
So here is one more time the result written on a separate slide derived in a previous slide. The radiation pressure force is h-bar ‘G’ times 'n'. ‘n’ is a number operator associated with the cavity field and we have now the radiation pressure interaction term h-bar ‘G’ a-dagger ‘a’ times 'x' derived from this radiation pressure force analysis outlined in the previous slide.

Notes

Summary

6m 27s

Hamiltonian description



$\omega_0(x) = \omega_c + G \cdot x(t)$ $\quad G = -\frac{\omega}{L}$

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b}$$

$= \hbar\omega_0(\hat{x})\hat{a}^\dagger\hat{a} + \hbar\Omega_m \hat{b}^\dagger\hat{b}$ $\quad \hat{x} = \sqrt{\frac{\hbar}{2m\Omega_m}}(\hat{b} + \hat{b}^\dagger)$

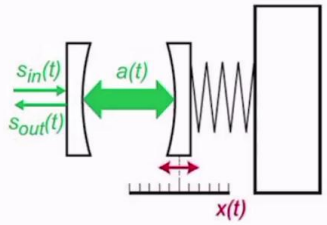
So I'd like to return one more time to Hamiltonian description of another way to derive the optical Hamiltonian by reconsidering the previous result which was that if the resonance frequency becomes dependent on the oscillator position, we can express the resonance frequency of the cavity as omega 'c' plus 'G' the frequency pull parameter times 'x of t'. And here 'G' was minus omega over 'L', the minus sign is for the fact that as the cavity becomes elongation is positive, there's a red shift in the cavity frequency. So what we can do next is we can use this result to derive the interaction Hamiltonian. The way we do this is now we replace the frequency omega by omega of 'x' and now 'x' becomes an operator a-dagger 'a' plus h-bar omega times b-dagger 'b' and here 'x' is the position operator which is the zero point fluctuations of the mechanical oscillator and 'b' plus b-dagger. Now what we can do next is simply insert the expression and so what we obtain here is we obtain the bare cavity frequency omega 'c' plus 'G' times zero point fluctuation 'b' plus b-dagger so this is the mechanical oscillator position coordinate times a-dagger 'a' plus h-bar omega 'm' b-dagger 'b'.

Notes

Summary

6m 49s

Hamiltonian description



$\omega_0(x) = \omega_c + G \cdot x(L) \qquad G = -\frac{\omega}{L}$

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b}$$

$$= \hbar\omega_0(\hat{x})\hat{a}^\dagger\hat{a} + \hbar\Omega_m \hat{b}^\dagger\hat{b} \qquad \hat{x} = x_{zpf}(\hat{b} + \hat{b}^\dagger)$$

$$= \hbar(\omega_c + G x_{zpf}(\hat{b} + \hat{b}^\dagger))\hat{a}^\dagger\hat{a} + \hbar\Omega_m \hat{b}^\dagger\hat{b}$$

$$= \hbar\omega_c \hat{a}^\dagger\hat{a} + \hbar\Omega_m \hat{b}^\dagger\hat{b} + \underbrace{\hbar G x_{zpf}}_{g_0} \hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger)$$

And now we can recollect terms and we recover first of all the harmonic oscillator of the cavity mode. We have the harmonic oscillator associated with the mechanical oscillator and now we have interaction Hamiltonian which is h-bar ‘G’ times 'zpf' so this is our G-naught optomechanical coupling rate and then we have here a-dagger ‘a’ times 'b' plus b-dagger so we again recover the optomechanical interaction Hamiltonian that describes interaction of our moving mirror with the cavity field as derived before.

Notes

Summary

8m 07s

Hamiltonian description

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \hbar G \hat{a}^\dagger \hat{a}\hat{x}$$

Note: Hamiltonian is singly QND Hamiltonian for photon number

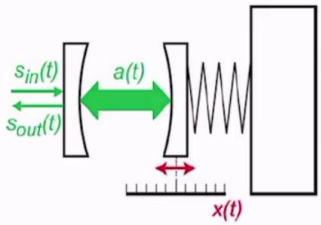
One can derive the equations of motion for the operators (optics and mechanics)

$[\hat{a}, \hat{a}^\dagger] = 1$

Optical frequency shift $\hbar \frac{d}{dt}\hat{a} = [\hat{H}, \hat{a}]$

Radiation pressure force

From Hamiltonian formulation one recovers the classical equation of motion (without the respective damping terms)



g0 is the vacuum optomechanical coupling rate

Okay. I'd like to return now to the Hamiltonian description and I want to comment on the fact that the Hamiltonian now, of course, allows also to reconstruct again our equations of motion for 'a', the annihilation operator of the optical field and 'b', the annihilation operator of the mechanical oscillator. And if we treat these the averages of the field operator 'a' and 'b' then we recover also the classical equations of motion that we derived earlier in this mooc. And so let's consider how we can obtain the classical equations of motion from this? So if we want or in this case the so-called quantum equations of motion the quantum Langevin equations, we also introduce damping. So the equation of motion for the optical field is, let's first recall that 'a' and a-dagger the commutator is equal to one. So now if we compute just from the Heisenberg equation of motion we obtain that 'i' h-bar 'd dt' of 'a' is given by 'H' and 'a'. So we have to compute the commutator of the Hamiltonian with 'a' the operator of the optical field. Now let's note that 'a' commutes, of course with 'b' because this acts on the mechanical oscillator 'a' on the optical field so they commute with each other, however, 'a' does not commute with a-dagger.

Notes

Summary

8m 41s

Hamiltonian description

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \hbar G \hat{a}^\dagger \hat{a}\hat{x}$$

Note: Hamiltonian is singly QND Hamiltonian for photon number

One can derive the equations of motion for the operators (optics and mechanics)

damping: +

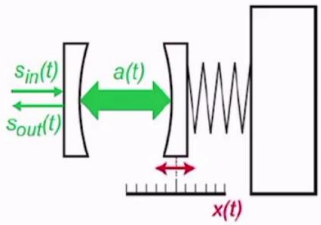
$[\hat{a}, \hat{a}^\dagger] = 1$

Optical frequency shift $i\hbar \frac{d}{dt}\hat{a} = [\hat{H}, \hat{a}] = \hbar\omega_0 \cdot \hat{a} + \hbar G \cdot \hat{x} \cdot \hat{a} \Rightarrow \frac{d}{dt}\hat{a} = i(\omega_0 + G\hat{x})\hat{a}$

missing: $-\kappa/2\,\hat{a}$ $\sqrt{\kappa_{ex}}s_{in}$

Radiation pressure force

From Hamiltonian formulation one recovers the classical equation of motion (without the respective damping terms)



g0 is the vacuum optomechanical coupling rate

And so we obtain as a standard basic quantum mechanics, we obtain here then just the expression omega naught times for the first term times 'a' and if you compute the commutator in this case of 'a' with a-dagger ‘a’ times 'x', we obtain a term that has the form h-bar times ‘G’ multiplied by 'x' times 'a'. And so now what we can observe from this is that we have an equation that says that ‘d dt’ of ‘a’ is nothing else than minus ‘i’ omega cavity plus ‘G’ times 'x' times 'a'. So this is just the same as our classical equation of motion except one aspect that this equation is missing the damping. And so the damping the damping term that we had before and also missing the driving term which was root kappa external times S-in. So these two terms need to be added separately for the damping and the driving one needs to add Hamiltonians in the case of damping. This is obtained by adding a dissipative dynamics. So by adding a so-called bath Hamiltonian that corresponds to the outgoing modes and incoming modes and the treatment of this bath Hamiltonian is done typically in the context of the so-called open quantum systems and the dynamics of this bath can then be and this was first described by a famous paper by Gardiner and Collett and this leads then to the appearance of this additional term minus kappa over two times 'a', a loss term.

Notes

Summary

10m 00s

Hamiltonian description

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \hbar G \hat{a}^\dagger \hat{a}\hat{x}$$

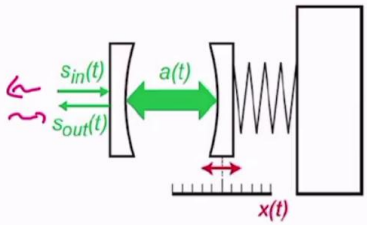
Note: Hamiltonian is singly QND Hamiltonian for photon number

One can derive the equations of motion for the operators (optics and mechanics)

$[\hat{a}, \hat{a}^\dagger] = 1$ Optical frequency shift $i\hbar \frac{d}{dt}\hat{a} = [\hat{H}, \hat{a}] = \hbar\omega_0 \hat{a} + \hbar G \hat{x}\hat{a} \Rightarrow \frac{d}{dt}\hat{a} = i(\omega_0 + G\hat{x})\hat{a}$

$[\hat{x}, \hat{p}] = i\hbar$ Radiation pressure force $i\hbar\partial_t \hat{p} =$

From Hamiltonian formulation one recovers the classical equation of motion (without the respective damping terms)



g0 is the vacuum optomechanical coupling rate

Also we have here missing the input noise term the input term of the driving and this root kappa external times S-in and this can be added to the Hamiltonian by adding a so-called drive term that I'll omit for the moment here and just add by hand. Nevertheless, if one takes a look we see that the Hamiltonian obviously and should, of course, reproduce the classical equations of motion and does so also in this current case. We can also look at the mechanical oscillator so note that 'x' and 'p' the commutators 'i' h-bar so we compute here as 'd dt' in this case of 'p' so change momentum. That's given by 'H' and 'p' so 'p' commutes with a-dagger 'a' so no contribution there but we get a contribution in this case from first of all, from we get a contribution here from the second term which is h-bar 'G' a-dagger 'a' times 'x' so we obtain here is just a contribution h-bar times 'G' a-dagger 'a' and then we have the commutator 'i' h-bar that comes from 'x' and 'p' which is vanishing then for a harmonic oscillator we can also express the b-dagger 'b' in terms of p-squared and x-squared and we get a term here that corresponds to a mechanical eigenfrequency that is 'i' omega 'm' times 'p' and so what we obtain here is times h-bar so we obtain here now is an equation that in the end just looks like our classical counterpart.

Notes

Summary

11m 47s

Hamiltonian description

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \hbar G \hat{a}^\dagger \hat{a} \hat{x}$$

Note: Hamiltonian is singly QND Hamiltonian for photon number

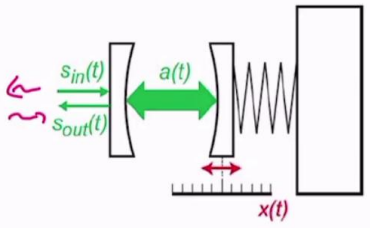
One can derive the equations of motion for the operators (optics and mechanics)

$[\hat{a}, \hat{a}^\dagger] = 1$ Optical frequency shift $i\hbar \frac{d}{dt}\hat{a} = [\hat{H}, \hat{a}] = \hbar\omega_0 \hat{a} + \hbar G \hat{x}\hat{a} \Rightarrow \frac{d}{dt}\hat{a} = i(\omega_0 + G\hat{x})\hat{a}$

$[\hat{x}, \hat{p}] = i\hbar$ Radiation pressure force $i\hbar\partial_t \hat{p} = [\hat{H}, \hat{p}] = \hbar G \hat{a}^\dagger \hat{a} (i\hbar) + i\Omega_m \hat{p}\hbar$

$\Rightarrow \frac{\partial}{\partial t}\hat{p} = i\Omega_m \hat{p} + \hbar G \hat{a}^\dagger \hat{a}$

From Hamiltonian formulation one recovers the classical equation of motion (without the respective damping terms)



g0 is the vacuum optomechanical coupling rate

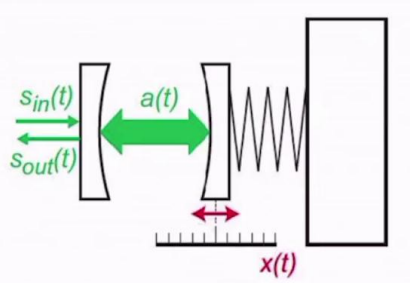
We get ‘d dt’ of ‘p’ so change in momentum is given by 'i' omega ‘m’ times 'p' so this is just the harmonic oscillator frequency plus a term h-bar times ‘G’ times a-dagger ‘a’. And this term here is precisely again the force the radiation pressure force that acts on the mechanical oscillator. And so we conclude that the Hamiltonian treatment reproduces again the two classical equations of motion, mind that also an equation for the mechanical oscillator, there's one term missing and this term that is missing is an input noise term from the environment so this is the Langevin force that drives the mechanical oscillator to the Brownian motion so ‘FL of t’ is so-called Langevin force. The second term that is missing here is the term minus gamma 'm' times 'p' which is a damping term and again this damping term is not contained in the Hamiltonian dynamics but can only be recovered if one introduces a bath to which a mechanical oscillator is coupled and then there will be additional term here that describes the damping. Nevertheless, the two equations of motion as we see them here are the coherent dynamics of the optomechanical equations of motion that we derived classically before.

Notes

Summary

13m 22s

Hamiltonian description



$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \underbrace{\hbar G \hat{a}^\dagger \hat{a} \hat{x}}_{\hat{H}_{int}}$$

$\alpha, \alpha^* \in \mathbb{C}$

Linearization around the driven cavity

$$[\hat{a}, \hat{a}^\dagger] = [\alpha + \delta\hat{a}, \alpha^* + \delta\hat{a}^\dagger] \qquad [\delta a, \delta\hat{a}^\dagger] = 1$$

$$\hat{a} = \frac{\alpha}{c} + \underbrace{\delta\hat{a}}_{\text{operator}}$$

Quantum theory of optomechanical cooling:
I. Wilson-Rae, Nooshi, Zwerger, Kippenberg, PRL 99, 093901 (2007)
J. Dobrindt, Wilson-Rae, Kippenberg, PRL, 101, 263602 (2008)
F. Marquardt, Chen, Clerk, Girvin, PRL 99, 093902 (2007)

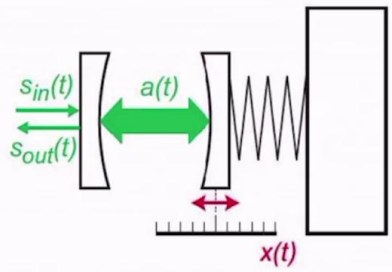
Now in order to solve the equations of motion, it is typically permissible in many settings to linearize the Hamiltonian. And again the reason we want to linearize the Hamiltonian is that we would like to solve the equations and this is more easily done if the equations break down to a linear set of differential equations that we can solve and for this, what we need to do is look at this nonlinear term a-dagger ‘a’ times 'x' and one way to linearize it is to note that the cavity field 'a' that we have is typically composed of a mean plus a fluctuation. So the idea here is to replace our commutator a-dagger 'a' by commutator of alpha plus 'da' and alpha conjugate plus delta a-dagger. Here alpha and alpha star are just complex numbers. They describe the classically driven cavity fields and the fluctuations due to the vacuum are now contained just in the in the fluctuations delta 'a' delta a-dagger so delta 'a' delta a-dagger satisfy the commutator equal to one. So if you do this, this linearization so this again is a classical number ‘c’ number, this is an operator. Then we can linearize the Hamiltonian. We can do the same for the mechanical oscillator coordinate.

Notes

Summary

14m 42s

Hamiltonian description



$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \underbrace{\hbar G \hat{a}^\dagger \hat{a} \hat{x}}_{\hat{H}_{int}}$$

Linearization around the driven cavity

$\alpha, \alpha^* \in \mathbb{C}$

$[\hat{a}, \hat{a}^\dagger] = [\alpha + \delta\hat{a}, \alpha^* + \delta\hat{a}^\dagger] \quad [\delta a, \delta\hat{a}^\dagger] = 1$

$\hat{a} = \frac{\bar{\alpha}}{\mathbb{C}} + \delta\hat{a}$ operator

$\hat{x} = \bar{x} + x_{zpf}(\hat{b} + \hat{b}^\dagger)$ fluctuations

$\hat{H}_{int} = \hbar G \cdot \hat{a}^\dagger \hat{a} \hat{x}$

$= \hbar G x_{zpf} (\alpha^* + \delta a^\dagger)(\alpha + \delta\hat{a})(\hat{b} + \hat{b}^\dagger)$

$= \hbar g_0 \alpha \cdot (\delta\hat{a} + \delta a^\dagger)(\hat{b} + \hat{b}^\dagger)$

Quantum theory of optomechanical cooling:
I. Wilson-Rae, Nooshi, Zwerger, Kippenberg, PRL 99, 093901 (2007)
J. Dobrindt, Wilson-Rae, Kippenberg, PRL, 101, 263602 (2008)
F. Marquardt, Chen, Clerk, Girvin, PRL 99, 093902 (2007)

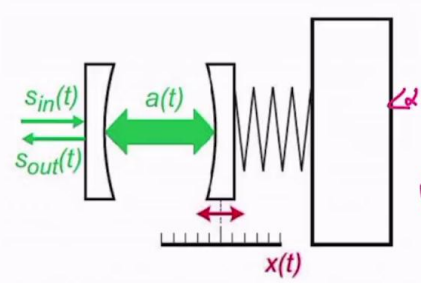
We assume that it is a static shift x-bar so it's just a static displacement from the radiation pressure force in the cavity and there are fluctuations around the mean and this is again given by 'b' plus b-dagger so now these are the fluctuations around this mean. And what we can do next is we can linearize this Hamiltonian and to do that we consider H-int as h-bar times ‘G’ and times now a-dagger ‘a’ times 'x' and if we now introduce our linearized operators here we would have a term alpha plus delta a-dagger alpha plus delta 'a' and then we have here our 'b' plus b-dagger times the zero point fluctuation x-zpf. So and this is nothing else then h-bar G-naught and now we can retain several terms if we assume for simplicity that alpha is real, and we can do that by a suitable choice of input field then all we have left here is alpha times delta 'a' plus delta a-dagger times 'b' plus b-dagger. So now we have obtained the Hamiltonian which is bi-linear. So it's a bi-linear Hamiltonian, only contains term in delta 'a' and 'b' and we have here alpha and alpha is the amplitude of the coherent state and this is given nothing else than the square root of the photon number inside the cavity.

Notes

Summary

16m 02s

Hamiltonian description



$\langle\alpha|a^\dagger a|\alpha\rangle = \alpha^2 = n_c$

$\sqrt{n_c} = \alpha$

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} + \underbrace{\hbar G\hat{a}^\dagger\hat{a}\hat{x}}_{\hat{H}_{int}}$$

Linearization around the driven cavity

$\alpha, \alpha^* \in \mathbb{C}$

$[\hat{a}, \hat{a}^\dagger] = [\alpha + \delta\hat{a}, \alpha^* + \delta\hat{a}^\dagger]$ $[\delta a, \delta\hat{a}^\dagger] = 1$

$\hat{a} = \frac{\bar{\alpha}}{\mathbb{C}} + \underbrace{\delta\hat{a}}_{\text{operator}}$

$\hat{x} = \bar{x} + x_{zpm}(\hat{b} + \hat{b}^\dagger)$ fluctuations

$\hat{H}_{int} = \hbar G \cdot \hat{a}^\dagger\hat{a}\hat{x}$

$= \hbar G(\alpha^* + \delta a^\dagger)(\alpha + \delta\hat{a})(\hat{b} + \hat{b}^\dagger)$

$H_{int} = \hbar g_0 \alpha \cdot (\delta\hat{a} + \delta\hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger)$

$\downarrow$

$\sqrt{n_c}$

Quantum theory of optomechanical cooling:
I. Wilson-Rae, Nooshi, Zwerger, Kippenberg, PRL 99, 093901 (2007)
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So recall just basics from quantum optics that alpha here is the strength of the coherent state that we have in the cavity. And we have that a-dagger ‘a’ for coherent state is nothing else than alpha squared and this is also equal to the cavity photon number so we have 'nc' is just square root of 'nc' is given by alpha. So now we have our linearized optomechanical Hamiltonian from which we can now derive equations of motion and importantly, these equations of motion will now just give a linear set of equations for delta 'a' and 'b' because of this quadratic nature of the Hamiltonian so we've removed the non-linearity that initially was present due to this three-wave mixing term.

Notes

Summary

17m 37s

Quantum Langevin equations

Recall: Coupling between system and bath

$$H_{\mathrm{B}} = \int d\omega \hbar\omega \hat{b}_\omega^\dagger \hat{b}_\omega \quad \text{and} \quad H_{\mathrm{SB}} = \int d\omega g(\omega)(\hat{a}\hat{b}_\omega^\dagger + \hat{a}^\dagger \hat{b}_\omega).$$

Time-evolution of the operators in the Heisenberg picture gives a dissipation term and a fluctuation term:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{a}, \hat{H}_{\mathrm{sys}}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\mathrm{in}}.$$

For the optomechanical Hamiltonian, $\hat{H} = \hbar\omega_{\mathrm{c}}(1 - \frac{\hat{x}}{L})\hat{a}^\dagger\hat{a} + \hbar\Omega_{\mathrm{m}}\hat{b}^\dagger\hat{b}$, we get the following equations of motion:

$$\frac{d\hat{a}}{dt} = -i\omega_{\mathrm{c}}\hat{a} + i\frac{\omega_{\mathrm{c}}}{L}x_{\mathrm{zpf}}(\hat{b}^\dagger + \hat{b})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}(\hat{a}_{\mathrm{in}} + \alpha e^{i\omega_{\mathrm{L}}t}),$$

$$\frac{d\hat{b}}{dt} = -i\Omega_{\mathrm{m}}\hat{b} + i\frac{\omega_{\mathrm{c}}}{L}\hat{a}^\dagger\hat{a} - \frac{\Gamma_{\mathrm{m}}}{2}\hat{b} + \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}.$$

I would like to return one more time to the quantum Langevin equations and recall one more time that we started out with a system Hamiltonian that we linearized that was h-bar times G-naught a-dagger ‘a’ 'b' plus b-dagger and this could be linearized to h-bar G-naught and then we add a term that in this case looked like alpha. This is the mean field then we have delta 'a' plus delta a-dagger plus 'b' and plus b-dagger and here we had chosen the phase the input the alpha to be real such that we don't differentiate between the alpha and alpha conjugate and we could do that by an appropriate choice of the input phase 'S e i phi' that is a classical drive field that is subjected to the cavity. Now, in this slide here what I’d like to recall is again how to describe the coupling to a bath. So now let's recall how the bath is introduced and this is something that is covered by Gardiner and Collett in their famous PRA paper on describing the dissipative quantum dynamics of an open system. It's also covered in quantum noise by Peter Zoller and Gardiner and what we need to do is if we have a cavity, for instance, in our case with an end mirror then what we have to do is we have to account for the fact that the end mirror both lets fluctuations inside the cavity also coupled to the outside.

Notes

Summary

18m 23s

Quantum Langevin equations

Recall: Coupling between system and bath

$$H_{\mathrm{B}} = \int d\omega \hbar\omega \hat{b}_\omega^\dagger \hat{b}_\omega \quad \text{and} \quad H_{\mathrm{SB}} = \int d\omega g(\omega)(\hat{a}\hat{b}_\omega^\dagger + \hat{a}^\dagger \hat{b}_\omega).$$

Time-evolution of the operators in the Heisenberg picture gives a dissipation term and a fluctuation term:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{a}, \hat{H}_{\mathrm{sys}}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\mathrm{in}}.$$

For the optomechanical Hamiltonian, $\hat{H} = \hbar\omega_{\mathrm{c}}(1 - \frac{\hat{x}}{L})\hat{a}^\dagger\hat{a} + \hbar\Omega_{\mathrm{m}}\hat{b}^\dagger\hat{b}$, we get the following equations of motion:

$$\frac{d\hat{a}}{dt} = -i\omega_{\mathrm{c}}\hat{a} + i\frac{\omega_{\mathrm{c}}}{L}x_{\mathrm{zpf}}(\hat{b}^\dagger + \hat{b})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}(\hat{a}_{\mathrm{in}} + \alpha e^{i\omega_{\mathrm{L}}t}),$$

$$\frac{d\hat{b}}{dt} = -i\Omega_{\mathrm{m}}\hat{b} + i\frac{\omega_{\mathrm{c}}}{L}\hat{a}^\dagger\hat{a} - \frac{\Gamma_{\mathrm{m}}}{2}\hat{b} + \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}.$$

And for this, we introduce a bath. These are the bath modes. These are all the modes of the Universe that so b-dagger 'b' here is our these are two operators associated with the creation and annihilation of the bath so this is a generally it's not written as a sum but rather as integral we assume many densely spaced bath modes. We also need to have a system bath interaction. The interaction here is bi-linear that is to say we can annihilate a mode from the cavity 'a' and create a quanta in the bath. This would correspond to output coupling. Or we can annihilate a photon from the bath and create one in the cavity and this would correspond to input noise, an input noise process. And without going into the calculation but I'd like to mention that one of the seminal results which is true under certain approximation, the approximations being that the bath modes are very densely spaced, is that one can then eliminate the dynamics of the bath and obtain a so-called quantum Langevin equation. And this so-called QLE has some more important features. There is first of all the interaction of the operator 'a' which is our system operator which is our cavity mode and so this in the absence of any optomechanical coupling, this would just be given by h-bar omega times 'a', so just the normal dynamics of the mechanical oscillator.

Notes

Summary

19m 58s

Quantum Langevin equations

Recall: Coupling between system and bath

$$H_{\mathrm{B}} = \int d\omega \hbar\omega \hat{b}_\omega^\dagger \hat{b}_\omega \quad \text{and} \quad H_{\mathrm{SB}} = \int d\omega g(\omega)(\hat{a}\hat{b}_\omega^\dagger + \hat{a}^\dagger \hat{b}_\omega).$$

Time-evolution of the operators in the Heisenberg picture gives a dissipation term and a fluctuation term:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{a}, \hat{H}_{\mathrm{sys}}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\mathrm{in}}.$$

For the optomechanical Hamiltonian, $\hat{H} = \hbar\omega_{\mathrm{c}}(1 - \frac{\hat{x}}{L})\hat{a}^\dagger\hat{a} + \hbar\Omega_{\mathrm{m}}\hat{b}^\dagger\hat{b}$, we get the following equations of motion:

$$\frac{d\hat{a}}{dt} = -i\omega_{\mathrm{c}}\hat{a} + i\frac{\omega_{\mathrm{c}}}{L}x_{\mathrm{zpf}}(\hat{b}^\dagger + \hat{b})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}(\hat{a}_{\mathrm{in}} + \alpha e^{i\omega_{\mathrm{L}}t}),$$

$$\frac{d\hat{b}}{dt} = -i\Omega_{\mathrm{m}}\hat{b} + i\frac{\omega_{\mathrm{c}}}{L}\hat{a}^\dagger\hat{a} - \frac{\Gamma_{\mathrm{m}}}{2}\hat{b} + \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}.$$

Then we have a new term appearing here which is the loss. This loss comes from the coupling to the bath and then we also have input noises or input drive fields depending which part we are considering of a-in. a-in contains typically a classical part. This could be our input field with some arbitrary phase 'e i omega L t' and also the input noises and the input noises here in this case are the vacuum noises for an ideal laser field. So these are the quantum fluctuations that enter. And so these two are linked so we can also view this equation as a manifestation of the fluctuation dissipation theorem. We have kappa over two being the dissipation of the system but the dissipation also entails that there is a fluctuating force 'a' incoming from the outside so this can be viewed as coming from the fluctuation dissipation theorem. So the same analogous considerations also apply to the mechanical oscillator. Also here there is a bath that a mechanical oscillator is coupled to. So if we make our mechanical omega compliant, there's a bath of mechanical modes that we couple to that gives rise to both the dissipative term and also a fluctuation. So this is the same analysis that can be repeated.

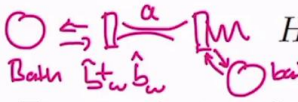
Notes

Summary

21m 25s

Quantum Langevin equations

Recall: Coupling between system and bath



$$H_{\mathrm{B}} = \int d\omega \hbar\omega \hat{b}_\omega^\dagger \hat{b}_\omega \quad \text{and} \quad H_{\mathrm{SB}} = \int d\omega g(\omega)(\hat{a}\hat{b}_\omega^\dagger + \hat{a}^\dagger \hat{b}_\omega).$$

Time-evolution of the operators in the Heisenberg picture gives a dissipation term and a fluctuation term:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{a}, \hat{H}_{\mathrm{sys}}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\mathrm{in}}.$$

For the optomechanical Hamiltonian, $\hat{H} = \hbar\omega_{\mathrm{c}}(1 - \frac{\hat{x}}{L})\hat{a}^\dagger\hat{a} + \hbar\Omega_{\mathrm{m}}\hat{b}^\dagger\hat{b}$, we get the following equations of motion:

$$\frac{d\hat{a}}{dt} = -i\omega_{\mathrm{c}}\hat{a} + i\frac{\omega_{\mathrm{c}}}{L}x_{\mathrm{zpf}}(\hat{b}^\dagger + \hat{b})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}(\hat{a}_{\mathrm{in}} + \alpha e^{i\omega_{\mathrm{L}}t}),$$

$$\frac{d\hat{b}}{dt} = -i\Omega_{\mathrm{m}}\hat{b} + i\frac{\omega_{\mathrm{c}}}{L}\hat{a}^\dagger\hat{a} - \frac{\Gamma_{\mathrm{m}}}{2}\hat{b} + \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}.$$

And so for the optomechanical Hamiltonian now, we get the following equations of motion. We obtain equation of motion for the optical fields or the cavity fields that has now a loss rate appearing. We have the optomechanical coupling that we described before and we have here the cavity eigenfrequency. So note that the cavity eigenfrequency omega ‘c’ generally can be different or different from the laser frequency and this was exactly what in our case was the laser detuning delta so this is varied in the experiment. We also have here now our drive field so our input field and this is what we called here a-in. So here a-in has two contributions. One is the quantum noise that I called vacuum noise above. One can also and it's often conventional to make a delta here to a designated fluctuation and we have here a part that is classical and that is just a classical drive. The classical drive field which now at this nomenclature I’ve called not 'S' bar but rather alpha times 'e i omega L t'. And note and this is very important. There is a time dependence in this equation. So we drive this optical field now not its resonant frequency which is omega 'c' but rather detuned.

Notes

Summary

22m 36s

Quantum Langevin equations

Recall: Coupling between system and bath

$$H_{\mathrm{B}} = \int d\omega \hbar\omega \hat{b}_\omega^\dagger \hat{b}_\omega \quad \text{and} \quad H_{\mathrm{SB}} = \int d\omega g(\omega)(\hat{a}\hat{b}_\omega^\dagger + \hat{a}^\dagger \hat{b}_\omega).$$

Time-evolution of the operators in the Heisenberg picture gives a dissipation term and a fluctuation term:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{a}, \hat{H}_{\mathrm{sys}}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\mathrm{in}}.$$

For the optomechanical Hamiltonian, $\hat{H} = \hbar\omega_{\mathrm{c}}(1 - \frac{\hat{x}}{L})\hat{a}^\dagger\hat{a} + \hbar\Omega_{\mathrm{m}}\hat{b}^\dagger\hat{b}$, we get the following equations of motion:

$$\frac{d\hat{a}}{dt} = -i\omega_{\mathrm{c}}\hat{a} + i\frac{\omega_{\mathrm{c}}}{L}x_{\mathrm{zpf}}(\hat{b}^\dagger + \hat{b})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}(\hat{a}_{\mathrm{in}} + \alpha e^{i\omega_{\mathrm{L}}t}),$$

$$\frac{d\hat{b}}{dt} = -i\Omega_{\mathrm{m}}\hat{b} + i\frac{\omega_{\mathrm{c}}}{L}\hat{a}^\dagger\hat{a} - \frac{\Gamma_{\mathrm{m}}}{2}\hat{b} + \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}.$$

And this leads when we transform into rotating frame to the appearance of the laser detuning in the equation of motion. Now also let's recall that a mechanical oscillator if we go through this treatment of an open quantum system, picks up two new terms. Again we have the fluctuation dissipation theorem. We have a dissipation from the environment. This is the energy decay rate. So gamma 'm' is the energy relaxation rate. And now we pick up a root gamma 'm' times b-in and this is the noise and this noise term here that are the fluctuations that are entering due to the dissipation. And in the case of harmonic oscillator because we're at high temperature, the properties of delta b-in of 't' and b-in dagger of t-prime, this is just delta function 't' minus t-prime times 'nth' where 'nth' is the number of thermal quantity in the oscillator. Now the key point is that we don't have to worry about these classical these quantum noises because if we take the averages of these equations on both sides, the important point to note is that both b-in and also a-in average equal to zero so what we obtain are just classical equations of motion that we can solve if we are attentive and pay attention to the detunings because these equations are driven with the frequency omega 'L' difference and the cavity frequency and generally 'a' and 'b' will rotate with two different frequencies.

Notes

Summary

24m 03s

Quantum Langevin equations

Recall: Coupling between system and bath

$$H_{\mathrm{B}} = \int d\omega \hbar\omega \hat{b}_\omega^\dagger \hat{b}_\omega \quad \text{and} \quad H_{\mathrm{SB}} = \int d\omega g(\omega)(\hat{a}\hat{b}_\omega^\dagger + \hat{a}^\dagger \hat{b}_\omega).$$

Time-evolution of the operators in the Heisenberg picture gives a dissipation term and a fluctuation term:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{a}, \hat{H}_{\mathrm{sys}}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\mathrm{in}}.$$

For the optomechanical Hamiltonian, $\hat{H} = \hbar\omega_{\mathrm{c}}(1 - \frac{\hat{x}}{L})\hat{a}^\dagger\hat{a} + \hbar\Omega_{\mathrm{m}}\hat{b}^\dagger\hat{b}$, we get the following equations of motion:

$$\left\langle \frac{d\hat{a}}{dt} \right\rangle = \left\langle i\omega_{\mathrm{c}}\hat{a} + i\frac{\omega_{\mathrm{c}}}{L}x_{\mathrm{zpf}}(\hat{b}^\dagger + \hat{b})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}(\hat{a}_{\mathrm{in}} + \alpha e^{i\omega_{\mathrm{L}}t}), \right\rangle$$

$$\left\langle \frac{d\hat{b}}{dt} \right\rangle = \left\langle -i\Omega_{\mathrm{m}}\hat{b} + i\frac{\omega_{\mathrm{c}}}{L}\hat{a}^\dagger\hat{a} - \frac{\Gamma_{\mathrm{m}}}{2}\hat{b} + \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}} \right\rangle$$

The cavity field would rotate with omega 'L' and the mechanical oscillator with omega ‘m’ and so if we properly introduce rotating frames for both the cavity field and a mechanical oscillator then not all terms in the interaction Hamiltonian will be resonance and this leads precisely to the described dynamics before that depending on which detuning we choose, we either obtain terms that are bi-linear so that scale with 'a' and 'b' dagger or we have terms that scale with 'a' and 'b' which are, therefore, the parametric pair generation processes. And this leads to the dynamics I've described in the previous slide.

Notes

Summary

25m 35s

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:

$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

The interaction Hamiltonian $\hat{H}_{\rm int} = \hbar\frac{\omega_c}{L}x_{\rm zpf}(\hat{b}^\dagger + \hat{b})\hat{a}^\dagger\hat{a}$ is thus *linearized*: $\hat{H} = \hat{H}_{\rm int} + \hat{H}_{\rm sys}$

$\frac{d}{dt}\hat{a} = \frac{1}{i\hbar}[\hat{H}, \hat{a}] = -i\omega_c\hat{a}$

Okay. Next let's just discuss the quantum Langevin equations and how to derive them for a very specific case, that's the case of resonant laser excitation where the cavity field exactly is equal to the laser frequency. And this allows us one more time to go through the various various transformations we are making which in particular is the transfer to the rotating frame and also the linearization. These are two key approximations one needs to bear in mind and one uses actually when deriving the Langevin equations. So we start out by the bare interaction Hamiltonian and we start by writing down the cavity field so the cavity field so is given by minus 'i' h-bar 'H' and 'a' so this is the commutator of the system Hamiltonian which is the interaction Hamiltonian plus the free Hamiltonian of the system so the two harmonic oscillators and if we compute this, this is just given by minus omega 'c' times 'a'. We also have the cavity ‘k’ that comes from the fact that we have an open quantum systems from this dissipative dynamics that's not contained in Hamiltonian but can be recovered by adding a bath and then we have the optomechanical dynamics 'iG’ so ‘G’ is here omega over 'L' x-zpf 'b' plus b-dagger times 'a' and then we have the coupling term which contains two contributions and we have first of all the laser frequency and then we have a term a-in so it's a mean field.

Notes

Summary

26m 12s

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:

$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

$\langle \hat{a} \rangle = \alpha$: real

The interaction Hamiltonian $\hat{H}_{\text{int}} = \hbar\frac{\omega_c}{L}x_{\text{zpf}}(\hat{b}^\dagger + \hat{b})\hat{a}^\dagger\hat{a}$ is thus *linearized*:

$\hat{H} = \hat{H}_{\text{int}} + \hat{H}_{\text{sys}}$

$\frac{d}{dt}\hat{a} = \frac{1}{i\hbar}[\hat{H}, \hat{a}] = -i\omega_c\hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\text{zpf}}(\hat{b} + \hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}}\, e^{-i\omega_L t}(\langle a_{in}\rangle e^{i\varphi} + \delta\hat{a}_{in})$

assume $\omega_c = \omega_L$

classical drive — quantum noise input

Okay. We'll introduce a phase that I will say something about in a second and we have the fluctuations. So this is the classical drive. Why do we introduce the phase here 'e i phi'? We introduce this phase 'e i phi' to later on make the field inside the cavity so that is the expectation value of 'a'. We make that in the end real. And the way we can make this real is by applying arbitrary phase to the input and this will be quite handy for a lot of the manipulations. And this is the input quantum noise or classical noise into the cavity. So the approximations now in these equations are several ones and they can be quite confusing. That's why we'll go to them here step by step. So first of all note that we are in a frame that is not yet rotating. We are driving here with a laser field that at a laser frequency and we assume now the situation where omega is equal to omega 'L' is equal to omega 'c'. So the approximation we're going to make first is we're gonna transform to a rotating frame and what we obtain in this case so what we're doing is we're gonna set 'a' and replace it by 'a i omega L t' and if we do that our equation of motion simply becomes delta 'ta' is given by detuning minus delta.

Notes

Summary

27m 45s

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:

$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

$\langle \hat{a} \rangle = \alpha$: real

The interaction Hamiltonian $\hat{H}_{\text{int}} = \hbar \frac{\omega_c}{L} x_{\text{zpf}} (\hat{b}^\dagger + \hat{b}) \hat{a}^\dagger \hat{a}$ is thus *linearized*:

$\hat{H} = \hat{H}_{\text{int}} + \hat{H}_{\text{sys}}$

$$\frac{d}{dt}\hat{a} = \frac{1}{i\hbar}[\hat{H}, \hat{a}] = -i\omega_c \hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\text{zpf}} (\hat{b} + \hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}}\, e^{-i\omega_L t} (\langle a_{in} \rangle e^{i\varphi} + \delta\hat{a}_{in})$$

classical drive; quantum noise input

assume $\omega_c = \omega_L$ $\quad \hat{a} \to \hat{a} e^{i\omega_L t}$

$$\frac{d}{dt}\hat{a} = -i\underbrace{\Delta}_{=0}\hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\text{zpf}} (\hat{b} + \hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}} (\langle a_{in} \rangle e^{i\varphi} + \delta\hat{a}_{in}) + \sqrt{\kappa_0}\, \delta\hat{a}_{in}^{(\text{vac})}$$

Drive; input quantum noise

$$\hat{a} = \bar{\alpha} + \delta\hat{a}$$

That's equal to zero for the resonant case. We have the cavity loss rate plus 'iG' x-zpf. And then we have b-dagger plus b-dagger ‘a’ plus root kappa external. Now we don't have any rotation anymore here. So we have eliminated this and we just have left a-in with some arbitrary phase plus the input vacuum noise. Now the next approximation we're going to make is that this equation still is an equation that describes two parts. We're going to assume this equation 'a' has two contributions. First contribution of 'a' is a part that is a classical 'c' number or classical real number that comes from the drive term here from the classical drive and then we have also the fluctuations delta-in that are driven by the input quantum noise. Or by quantum noise inside the cavity which we can also add by a second vacuum port that we have omitted but in general, if there was internal losses to the cavity, there will be a term kappa naught delta a-in vacuum port here that would describe internal losses to the cavity. So if we do this, what we can do is we can solve the equations for alpha and so alpha can be solved simply by putting the left-hand side equal to zero so we do we compute this steady state value here and we obtain simply an equation of alpha that will depend and proportional to the input a-in and also to the arbitrary phase.

Notes

Summary

29m 00s

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:

$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

$\langle \hat{a} \rangle = \alpha$: real

The interaction Hamiltonian $\hat{H}_{\rm int} = \hbar \frac{\omega_c}{L} x_{\rm zpf} (\hat{b}^\dagger + \hat{b}) \hat{a}^\dagger \hat{a}$ is thus *linearized*:

$\hat{H} = \hat{H}_{\rm int} + \hat{H}_{\rm sys}$

$\frac{d}{dt}\hat{a} = \frac{1}{i\hbar}[\hat{H}, \hat{a}] = -i\omega_c \hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\rm zpf}(\hat{b}+\hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}}\, e^{-i\omega_L t}(\langle a_{in}\rangle e^{i\varphi} + \delta\hat{a}_{in})$

classical drive; quantum noise input

assume $\omega_c = \omega_L$ $\quad \hat{a} \to \hat{a} e^{i\omega_L t}$

$\frac{d}{dt}\hat{a} = -i\underbrace{\Delta}_{=0}\hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\rm zpf}(\hat{b}+\hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}}(\langle a_{in}\rangle e^{i\varphi} + \delta\hat{a}_{in}) + \sqrt{\kappa_0}\,\delta\hat{a}_{in}^{(\rm vac)}$

Drive; input quantum noise

$\hat{a} = \bar{\alpha} + \delta\hat{a}$ $\quad$ solve for $\bar{\alpha}$ $\quad \langle \frac{d}{dt}\hat{a} \rangle = 0$ $\quad \bar{\alpha} \propto \langle a_{in}\rangle e^{i\varphi}$ $\quad \varphi : \alpha \in \mathbb{R}$

$\frac{d}{dt}\delta\hat{a} = -\frac{\kappa}{2}\delta\hat{a} + iG x_{\rm zpf}(\hat{b}+\hat{b}^\dagger)\bar{\alpha} + \sqrt{\kappa_{ex}}\,\delta\hat{a}_{in}$

linearized QLE optical field ($\omega_L = \omega_c$ ($\Delta = 0$))

And we're gonna put the phase such a way that alpha becomes real. If we do that and then we have found what our steady state field in the cavity is and we can next linearize and for this we're going to do the following. We're gonna write now this cavity field for the fluctuation. Again delta is equal to zero because there's no detuning so we just have cavity loss. Now we have 'iG' at 'zpf', 'b' is also a fluctuations but I'll leave the original operators here and now 'a' is replaced by 'a' plus delta 'a' and here in the linearization the only term we keep is alpha. We neglect the term that goes like delta 'a' times 'b' because that's two fluctuations multiplied which we are going to neglect in this linearized treatment. And in the next term we have really now is our input noise kappa external times delta a-in. Okay, so this is the linearized quantum Langevin equation for the optical field. For this specific case of omega 'L' equals omega 'c' so where the detuning is equal to zero and again we've done two approximations. First go into rotating frame and then the second approximation is linearization. We can next also consider the mechanical oscillator and do exactly the same.

Notes

Summary

30m 33s

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:

$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

$\langle \hat{a} \rangle = \alpha$: real

The interaction Hamiltonian $\hat{H}_{\rm int} = \hbar \frac{\omega_c}{L} x_{\rm zpf} (\hat{b}^\dagger + \hat{b}) \hat{a}^\dagger \hat{a}$ is thus *linearized*:

$\hat{H} = \hat{H}_{\rm int} + \hat{H}_{\rm sys}$

$\frac{d}{dt}\hat{a} = \frac{1}{i\hbar}[\hat{H}, \hat{a}] = -i\omega_c \hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\rm zpf}(\hat{b}+\hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}}\, e^{-i\omega_L t}(\langle a_{in}\rangle e^{i\varphi} + \delta\hat{a}_{in})$

classical drive — quantum noise input

assume $\omega_c = \omega_L$ $\quad \hat{a} \to \hat{a}e^{i\omega_L t}$

$\frac{d}{dt}\hat{a} = -i\underbrace{\Delta}_{=0}\hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\rm zpf}(\hat{b}+\hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}}(\langle a_{in}\rangle e^{i\varphi} + \delta\hat{a}_{in}) + \sqrt{\kappa_0}\,\delta\hat{a}_{in}^{(\rm vac)}$

Drive — input quantum noise

$\hat{a} = \bar{\alpha} + \delta\hat{a}$ $\quad$ solve for $\bar{\alpha}$ $\langle \frac{d}{dt}\hat{a} \rangle = 0$ $\quad \bar{\alpha} \propto \langle a_{in}\rangle e^{i\varphi}$ $\quad \varphi : \alpha \in \mathbb{R}$

$\frac{d}{dt}\delta\hat{a} = -\frac{\kappa}{2}\delta\hat{a} + iG x_{\rm zpf}(\hat{b}+\hat{b}^\dagger)\bar{\alpha} + \sqrt{\kappa_{ex}}\,\delta\hat{a}_{in}$ $\quad$ linearized QLE optical field with $(\Delta = 0)$

$\frac{d}{dt}\hat{b} = \frac{1}{i\hbar}[\hat{H}, \hat{b}] = -i\Omega_m \hat{b} - \frac{\Gamma_m}{2}\hat{b} + i\frac{\omega_c}{L}(\hat{a}^\dagger\hat{a}) + \sqrt{\Gamma_m}\, b_{in} = -i$

$e^{i\omega_L t}$ $\quad e^{-i\omega_L t}$

So we have an equation for 'b' that is one over 'i' h-bar 'H' and 'b'. We have here an equation that obtains so if minus omega 'm' times b. We have a damping rate and energy relaxation rate and then we have an optomechanical coupling term and this optomechanical coupling term goes like 'i' omega 'c' over 'L' times a-dagger and 'a' and then we have the input noise plus root gamma 'm' times b-in. Now note first of all that if you could move the rotating frame, nothing changes for the laser field here because 'e' might 'L delta dagger and delta a-dagger go like 'e i omega L t' times and it's minus 'i omega L t' if you do the transformation, and so this implies that the two phases here are opposite and they cancel each other so the rotation the optical field being rotated doesn't change here the equation for 'b' and that also makes a lot of sense because a-dagger 'a' is a portion of the photon number. And that does not exhibit any, it doesn't exhibit a phase dependence, that term. Now this equation can be further simplified. We have now 'b' minus gamma over two 'b' and now what we're going to do is, we're going to simplify the term of the radiation pressure and this becomes just simply alpha which is now real as I mentioned delta 'a' plus delta a-dagger.

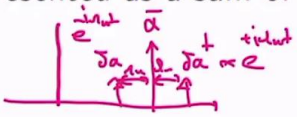
Notes

Summary

31m 52s

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:



$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

$\langle \hat{a} \rangle = \alpha$: real

The interaction Hamiltonian $\hat{H}_{\text{int}} = \hbar \frac{\omega_c}{L} x_{\text{zpf}} (\hat{b}^\dagger + \hat{b}) \hat{a}^\dagger \hat{a}$ is thus *linearized*:

$\hat{H} = \hat{H}_{\text{int}} + \hat{H}_{\text{sys}}$

$$\frac{d}{dt}\hat{a} = \frac{1}{i\hbar}[\hat{H}, \hat{a}] = -i\omega_c \hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\text{zpf}} (\hat{b} + \hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}}\, e^{-i\omega_L t} (\langle a_{in} \rangle e^{i\varphi} + \delta\hat{a}_{in})$$

classical drive; quantum noise input

assume $\omega_c = \omega_L$ $\quad \hat{a} \to \hat{a} e^{i\omega_L t}$

$$\frac{d}{dt}\hat{a} = -i\underbrace{\Delta}_{=0}\hat{a} - \frac{\kappa}{2}\hat{a} + iG x_{\text{zpf}} (\hat{b} + \hat{b}^\dagger)\hat{a} + \sqrt{\kappa_{ex}} (\langle a_{in} \rangle e^{i\varphi} + \delta\hat{a}_{in}) + \sqrt{\kappa_0}\, \delta\hat{a}_{in}^{(\text{vac})}$$

Drive; input quantum noise

$\hat{a} = \bar{\alpha} + \delta\hat{a}$ $\quad$ solve for $\bar{\alpha}$ $\langle \frac{d}{dt}\hat{a} \rangle = 0$ $\quad \bar{\alpha} \propto \langle a_{in} \rangle e^{i\varphi}$ $\quad \varphi : \alpha \in \mathbb{R}$

$$\frac{d}{dt}\delta\hat{a} = -\frac{\kappa}{2}\delta\hat{a} + iG x_{\text{zpf}} (\hat{b} + \hat{b}^\dagger)\bar{\alpha} + \sqrt{\kappa_{ex}}\, \delta\hat{a}_{in}$$

$\propto \bar{x}$

linearized QLE optical field ($\omega_L = \omega_c$ ($\Delta = 0$))

$$\frac{d}{dt}\hat{b} = \frac{1}{i\hbar}[\hat{H}, \hat{b}] = -i\Omega_m \hat{b} - \frac{\Gamma_m}{2}\hat{b} + i\omega_c (\hat{a}^\dagger \hat{a}) + \sqrt{\Gamma_m}\, b_{in} = -i\Omega_m \hat{b} - \frac{\Gamma_m}{2}\hat{b} - i\frac{\omega_c}{L}\alpha(\delta a + \delta a^\dagger) + \sqrt{\Gamma_m}\, \hat{b}_{in}$$

$e^{i\omega_L t}$ $\quad e^{-i\omega_L t}$

And then we have false input noise which goes like gamma 'm' times b-in. And so this now term here has a very intuitive kind of appeal and also interpretation so we see that the mechanical oscillator is driven in this case now by a term that looks like the original pump field in the cavity, alpha bar multiplied by delta 'a' plus delta a-dagger and these two fields here can be viewed as sidebands so if delta 'a' fluctuates with 'e' minus omega 'mt' and delta a-dagger with 'e' plus omega 'Lt' then, in fact, it's the sidebands beating with the carrier that drive the mechanics. So this one can corroborate also pictorially so if one has the optical carrier alpha and one plots here sidebands delta 'a' and delta a-dagger, and these rotate with 'e' plus omega ‘mt' and 'e' minus omega 'mt'. These beat with the carrier frequency and therefore, they can drive the mechanical oscillator at a difference frequency here which is chosen to be omega ‘m’ in this pictorial representation. So likewise if one looks at the equation for the mechanical oscillator appearing in the optical field, one sees that the optical field only couples to 'x' so to the position of the oscillator because only the position of the oscillator change the cavity length.

Notes

Summary

33m 22s

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:

$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

The interaction Hamiltonian $\hat{H}_{\rm int} = \hbar\frac{\omega_c}{L}x_{\rm zpf}(\hat{b}^\dagger + \hat{b})\hat{a}^\dagger\hat{a}$ is thus *linearized*:

$$\hat{a}^\dagger\hat{a} = (\bar{\alpha}^* + \delta\hat{a}^\dagger)(\bar{\alpha} + \delta\hat{a}) \to \bar{\alpha}(\delta\hat{a}^\dagger + \delta\hat{a})$$

Redefining $\delta\hat{a}$ as $\hat{a}$, we get *linearised quantum Langevin equations*:

Fluctuations
$\delta\hat{\alpha} = \hat{a}$

$$\frac{d\hat{a}}{dt} = i\frac{\omega_c}{L}x_{\rm zpf}(\hat{b}^\dagger + \hat{b})\bar{\alpha} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\rm in},$$

$\delta\hat{b} = \hat{b}$

$$\frac{d\hat{b}}{dt} = -i\Omega_m\hat{b} + i\frac{\omega_c}{L}\bar{\alpha}(a + a^\dagger) - \frac{\Gamma_m}{2}\hat{b} + \sqrt{\Gamma_m}\hat{b}_{\rm in}.$$

So summarizing we obtain the following equations. If we apply the application to a rotating frame, we are resonant with the cavity. We apply the linearization. Alpha-bar and beta-bar are the solutions of the means of the equations of motion for the optical field and the mechanical oscillator, we obtain the linear quantum Langevin equations and here I've set delta 'a' equal to a-bar and delta 'b' equal to 'b' and we obtain here again the two equations of motion where we have here alpha-bar the mean cavity field multiplied with b-dagger and 'b'. This is how the mechanical oscillator coordinate 'x' transduces to the optical field fluctuation and we have the equation for 'b'. Now note that in many of the literatures in cavity optomechanics, one doesn't distinguish between the fluctuation delta 'a' and 'a' and so also here I've chosen the notation where a-dagger designates the fluctuations of the cavity field and 'b' is the fluctuation of the mechanical oscillator and we have linearized alpha bar that is determined by the input field. Likewise for the input noises, this here is the input noise in the optical field and we have an input noise into the mechanical oscillator.

Notes

Summary

34m 49s

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:

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The interaction Hamiltonian $\hat{H}_{\rm int} = \hbar\frac{\omega_c}{L}x_{\rm zpf}(\hat{b}^\dagger + \hat{b})\hat{a}^\dagger\hat{a}$ is thus *linearized*:

$$\hat{a}^\dagger\hat{a} = (\bar{\alpha}^* + \delta\hat{a}^\dagger)(\bar{\alpha} + \delta\hat{a}) \to \bar{\alpha}(\delta\hat{a}^\dagger + \delta\hat{a})$$

Redefining $\delta\hat{a}$ as $\hat{a}$, we get *linearised quantum Langevin equations*:

Fluctuations $\delta\hat{a} = \hat{a}$

$$\frac{d\hat{a}}{dt} = i\frac{\omega_c}{L}x_{\rm zpf}(\hat{b}^\dagger + \hat{b})\bar{\alpha} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\rm in},$$

input noise

$\delta\hat{b} = \hat{b}$

$$\frac{d\hat{b}}{dt} = -i\Omega_m\hat{b} + i\frac{\omega_c}{L}\bar{\alpha}(a + a^\dagger) - \frac{\Gamma_m}{2}\hat{b} + \sqrt{\Gamma_m}\hat{b}_{\rm in}.$$

mechanical oscillator

$\langle a_{in}^\dagger(t)a_{in}(t')\rangle = \delta(t-t')$

$\langle b_{in}(t)b_{in}^\dagger(t')\rangle = \delta(t-t')(n_m+1)$

$\langle b_{in}^\dagger(t)b_{in}(t')\rangle = \delta(t-t')n_m$

The input field themselves obey several relations so the input field here delta a-in a-in of 't' and t-prime and the two time correlation function here would be delta 't' minus t-prime and for the mechanical oscillator we would have delta b-in of 't' b-in of t-prime obeying a delta function as well multiplied by 'nth' the occupancy of the mechanical oscillator. And if we compute delta b-in of 't' b-in of t-dagger of t-prime then here we obtain delta 't' minus t-prime times 'nth' plus one. So these are input noises that, in particular, play a role for computing so-called spectral densities or two time expectation values of operators.

Notes

Summary

36m 07s