

EPFL

# WELCOME & OUTLINE

In this lesson, we will learn about strong coupling regime between coupled resonators

- A toy-system : 2 masses and a spring
- Features of strong coupling :  
    Avoided crossing & energy transfer
- Quantum description : Jaynes-Cummings Hamiltonian
- Strong coupling in optomechanics

Hello everyone. I'm Rémy Braive from the Université de Paris and the Center of Nanoscience and Nanotechnologies and we will discuss about strong coupling regime in optomechanics. In this lesson, we will learn about strong coupling regime between couple resonators. In order to discuss this concept we will first achieve strong coupling with a toy model; namely, two masses coupled with a spring. With this toy model, we will be able to go through the two main feature of strong coupling, the avoided crossing and the energy transfer between the two resonators. We will then introduce Quantum model of strong coupling with the Jaynes-Cummings Hamiltonian. Finally, we will explain strong coupling in the frame of optomechanics.

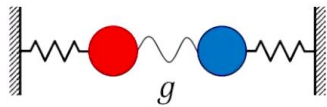
Notes

Summary



# CLASSICAL MODEL

## A TOY MODEL



With the ansatz solution

$$x_i = X_i e^{i\Omega t}$$

Amplitude of driven resonators

$$X_B = \frac{\overset{\circ}{G}}{[i\Omega\Gamma_B + (\Omega_B^2 - \Omega^2)]} X_A$$

$$X_A = \frac{[i\Omega\Gamma_B + (\Omega_B^2 - \Omega^2)]}{[i\Omega\Gamma_A + (\Omega_A^2 - \Omega^2)][i\Omega\Gamma_B + (\Omega_B^2 - \Omega^2)] - \overset{\circ}{G}^2}$$

## SOLVING EQUATION

- Coupled structures

$$\begin{cases} \ddot{x}_A + \Gamma_A \dot{x}_A + \Omega_A^2 x_A - G x_B = F_A \cos \Omega t \\ \ddot{x}_B + \Gamma_B \dot{x}_B + \Omega_B^2 x_B - G x_A = 0 \end{cases}$$

$$\begin{cases} [i\Omega\Gamma_A + (\Omega_A^2 - \Omega^2)] X_A - G X_B = F_A \\ [i\Omega\Gamma_B + (\Omega_B^2 - \Omega^2)] X_B - G X_A = 0 \end{cases}$$

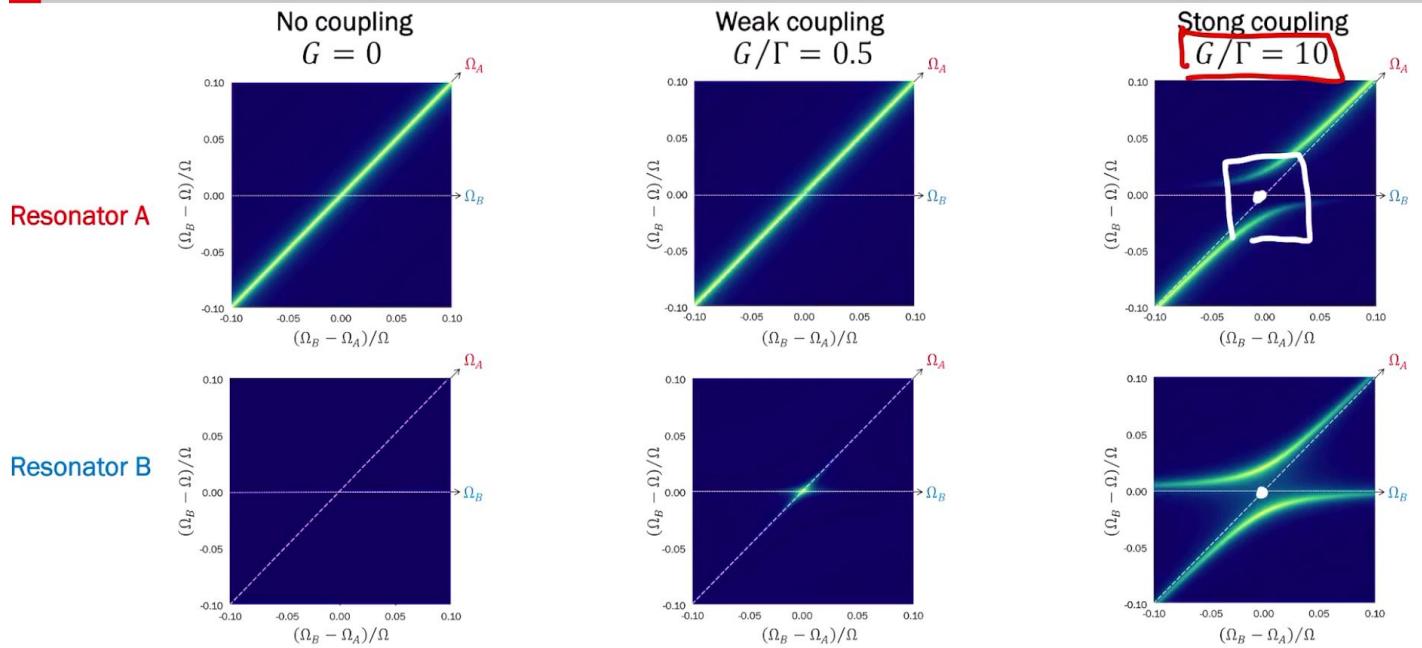
Let's introduce our toy model. Two uncoupled masses with a resting force due to a spring. With us we have two resonators. Their equation of motion are given by their acceleration, the mechanical damping and their mechanical frequencies. Only resonator 'A' is driven by a periodic forces here. If we now couple the two resonator by a spring with a coupling strength 'g' here, equation are modified as follow. An extra term appear directly related to the coupling strength here and here. At the same time, the mechanical frequency is modified and is now named a capital omega. Capital omega 'A' and capital omega 'B'. These new frequencies are the natural mechanical frequencies of the resonator shifted by the coupling here. We can solve these coupled equation by using this well-known ansatz solution here where capital 'Xi' is amplitude of the displacement. By injecting this expression in the coupled equation, we get frequency-dependent equation for amplitude. Thus we get the amplitude equation for the two resonators, 'XB' and 'XA'. This expression are resonant at the mechanical frequency of each resonators and depend on the coupling capital 'G' here and here. Now we can look at different values of coupling and see the behavior of the resonators.

Notes

Summary



# STRONG COUPLING REGIME SIGNATURE



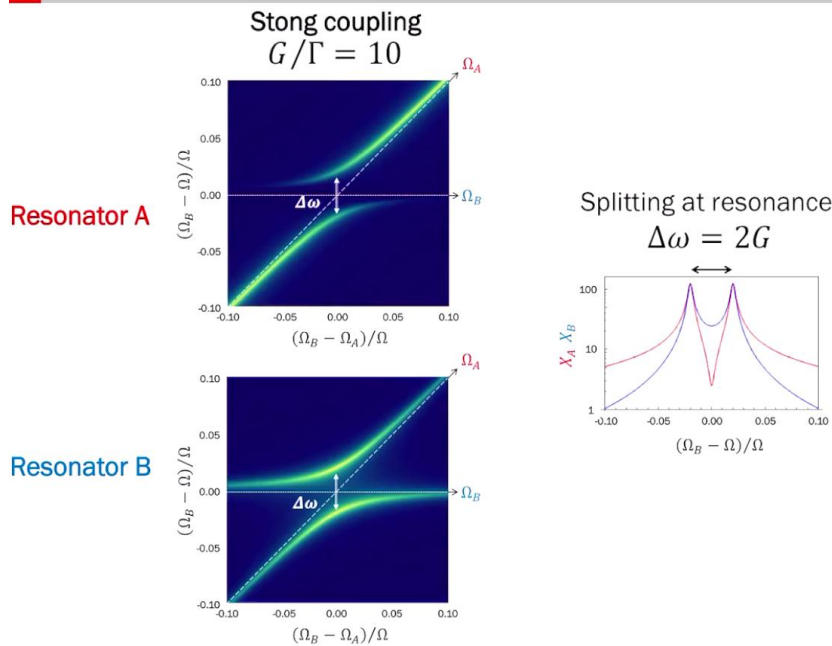
Let's start with the simplest case where 'G', the coupling, is equal to zero. On top, we have the mechanical response of resonator 'A' and on the bottom, the one of resonator 'B'. Since we are not driving resonator 'B', it's not excited and there is no response. Thus nothing happened to resonator 'B' and mechanical frequency of resonator 'A' is unchanged here. When resonator 'A' and 'B' are identical such that capital omega 'A' and 'B' are equal in the spectrum here, we only see the mechanical response of resonator 'A'. By slightly increasing the coupling here 'G' over gamma is equal to 0.5, we start seeing mechanical response in resonator 'B' here. But only close to omega 'A' is equal to omega 'B'. By looking at the spectrum close to this specific frequency, we are able to see two responses. With the one of 'B' much weaker than 'A'. By reaching much higher coupling here 'G' over gamma is equal to ten. We can clearly see the first feature of strong coupling; namely, the Avoided Crossing here. There is no more mechanical response when capital omega 'A' equals capital omega 'B' at this point here and this point here.

Notes

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# STRONG COUPLING REGIME SIGNATURE



What about time domain ?  
Possible transfer of energy  
between the two resonators ?

By looking at the spectrum, the splitting between the two mode is equal to two time the coupling here. The maximum of amplitude of the two our responses are similar here and here. And at the center of the spectrum, we see a large reduction of the amplitude here with a funnel-like resonance. We can now look at the time response of the resonator for this different domain and see the transfer of energy between resonators.

Notes

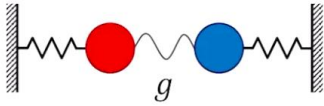
Summary



# ENERGY TRANSFERT IN STRONG COUPLING REGIME

No driving ( $F_A = 0$ ) : Damped oscillations

Solutions :  $X_i(t) \propto e^{\lambda_{1,2}^\pm t}$



$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\Omega_A^2 & G & -\Gamma_A & 0 \\ G & -\Omega_B^2 & 0 & -\Gamma_B \end{pmatrix}$$

Eigenvalues of Jacobian :

$$\lambda_{1,2}^\pm = -\mu_{1,2} \pm i\sigma_{1,2}$$

Exponential decay of the amplitude :  $\mu_{1,2} > 0$

Oscillation frequency :  $\sigma_{1,2} > 0$

Weak coupling :

$$\lambda_{1,2}^\pm = -\frac{\Gamma_A}{2} \pm i\sqrt{\Omega_A^2 - \frac{\Gamma_A^2}{4}}$$

Strong coupling :

$$\lambda_{1,2}^\pm = -\frac{\Gamma_A + \Gamma_B}{2} \pm i\sqrt{\Omega_A^2 \pm G}$$

In order to do so, we stop the driving 'FA' is equal to zero in order to see the damping of this resonators. The amplitude can be written as an exponential expression with a real part  $\mu$  which is the exponential decay of the amplitude and an imaginary part  $\sigma$  which gives the oscillation frequency. Now coupled equation can be written with this Jacobian here. By solving it we can extract the eigenvalues in the case of weak and strong coupling. In the case of weak coupling for identical resonator, coupling does not appear in the expression. In the case of strong coupling, damping is given by the mean value of the dampings of the two resonators and the oscillation involve the coupling strength here. From here we can show the time response of the resonators.

Notes

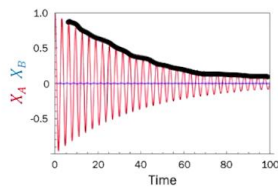
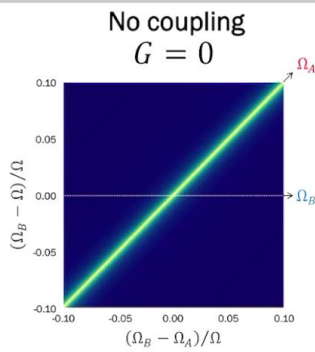
Summary



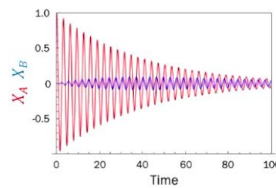
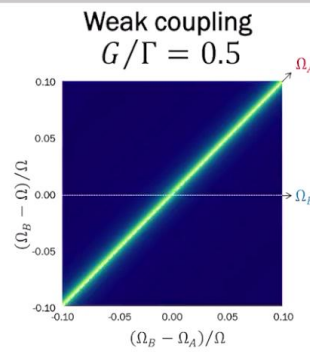
4m 54s

# ENERGY TRANSFER

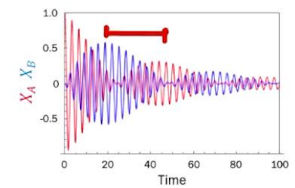
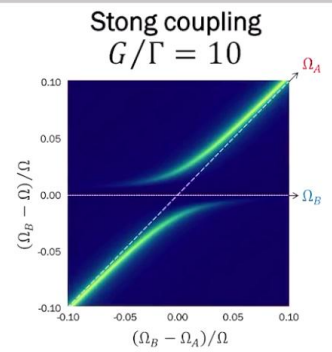
Resonator A



No excitation of  
resonator B



Weak excitation  
of resonator B



Energy transfer  
between A & B

On the top, there are the same curve as before for different couplings and on the bottom without coupling here, we see the exponential decay of resonator 'A' here with time and no answer from 'B'. In the weak coupling regime, the damping of resonator 'A' is weakly modified by the coupling and we start seeing a small excitation of resonator 'B'. For strong coupling decays are strongly modified and we see a beating between the two resonator here. And thus we can show an energy transfer between the two resonator 'A' and 'B'.

Notes

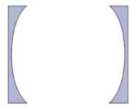
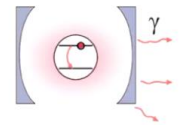
Summary





# JAYNES-CUMMINGS MODEL

Quantum optic model = interaction { Single quantized mode of an optical cavity  
+  
Two-level atom



Quantizing the electromagnetic field

$$\hat{H}_{field} = \hbar\omega_c(\hat{a}(t)\hat{a}^\dagger(t) + \frac{1}{2}) \approx \hbar\omega_c\hat{a}(t)\hat{a}^\dagger(t)$$

Lot of field quanta so that  $\hbar\omega_c/2$  is negligible



Two-level atom Hamiltonian  $\hat{H}_{atom} = E_g|g\rangle\langle g| + E_e|e\rangle\langle e|$

$$\hat{H}_{atom} = \begin{pmatrix} E_e & 0 \\ 0 & E_g \end{pmatrix} = \frac{1}{2}(E_g + E_e)\mathbb{I} + \frac{1}{2}(E_e - E_g)\hat{\sigma}_z$$

$$\hat{H} = \hat{H}_{atom} = \frac{1}{2}\hbar\omega_a\hat{\sigma}_z$$

Shift zero of energy, only energy difference interesting

We can now go to the quantum description of strong coupling by describing Jaynes-Cummings Hamiltonian. In this case, the system is made of an optical cavity and a two-level atoms here. We will describe these two elements separately and then the interaction between them. The Hamiltonian for the field inside the cavity is given by this expression with the frequency of the cavity  $\omega_c$  and the product of the creation and annihilation operators. In the following, the term one-half here is neglected due to the large number of quanta inside the cavity. The Hamiltonian of the two-level atom is given by this matrix here where  $E_e$  is the energy of the excited level and  $E_g$  is the energy of ground state. This Hamiltonian can be rewritten in term of the identity matrix here and the sigma 'Z' operator, a Pauli matrix. In the following, we will only consider the second term here this one. The first one can be seen as a shift in energy and only the difference is interesting. Thus we have this expression for the atom.

Notes

Summary





# JAYNES-CUMMINGS MODEL

Interaction Hamiltonian  $\hat{H}_{int} = -\hat{d} \cdot \hat{E} = -\hat{d} \cdot E_0(\hat{a} + \hat{a}^\dagger) \sin(kz) = \lambda \hat{d} \cdot (\hat{a} + \hat{a}^\dagger)$

The dipole operator  $\hat{d} = d|g\rangle\langle e| + d^*|e\rangle\langle g|$

Due to parity :  
 $\langle e|\hat{d}|e\rangle = 0 \quad \langle g|\hat{d}|g\rangle = 0$

With the Pauli matrices

$$\left[ \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$\hat{\sigma}_+ = \hat{\sigma}_1 + i\hat{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \hat{\sigma}_- = \hat{\sigma}_1 - i\hat{\sigma}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\rightarrow \hat{d} = d(\hat{\sigma}_+ + \hat{\sigma}_-)$$

$$\langle g|\hat{d}|e\rangle = d \text{ is real}$$

$$\hat{H}_{int} = \hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-) \cdot (\hat{a} + \hat{a}^\dagger)$$

Now let's discuss about the interaction between the dipole and the field. The field can be expressed as a constant E-naught with multiplying the creation and annihilation operators times the spatial distribution of the field. These two constant E-naught and sinus 'kz' are put in the constant alpha here. Then the dipole operator can be expressed as follow where the first term expresses the excitation from the ground state to the excited state and the second term here express the de-excitation of the atom. The other element 'L' the different... the other matrix element are equal to zero here and here. Thanks to Pauli matrix recalled here we can rewrite the interaction Hamiltonian as a linear combination of sigma plus and sigma minus here. The interaction Hamiltonian is thus the scalar product between the sigma operators here and the creation annihilation operators.

Notes

Summary



8m 30s

# JAYNES-CUMMINGS MODEL

Operators evolve like

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega_c t} \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega_c t}$$

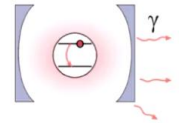
$$\hat{\sigma}_\pm(t) = \hat{\sigma}_\pm(0)e^{\pm i\omega_a t}$$

In the *rotation wave approximation*,  $\hat{\sigma}_+ \hat{a}^\dagger$  and  $\hat{\sigma}_- \hat{a}$  terms vary more rapidly than the others.

$$\hat{H}_{int} = \hbar\Omega(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger)$$

The Jaynes-Cummings Hamiltonian is then :

$$\hat{H}_{JC} = \hbar\omega_c \hat{a}(t)\hat{a}^\dagger(t) + \frac{1}{2}\hbar\omega_a \hat{\sigma}_z + \hbar\Omega(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger)$$



In the rotating wave approximation, operators evolved following this expression with omega 'C', the cavity frequency, and omega 'a' here the one of the two-level system. By neglecting the rapidly varying terms, the interaction Hamiltonian has only two terms sigma plus 'a' and sigma minus 'a' dagger. The two orders are too fast. Finally, we achieve the Jaynes-Cummings Hamiltonian with the cavity term. The two-level system Hamiltonian and the interaction between the two elements.

Notes

Summary



9m 52s

# DRESSED STATES

- Only transition of the type  $|e\rangle|n\rangle \leftrightarrow |g\rangle|n+1\rangle$  (the bare states of the Jaynes-Cummings model)
- For fixed  $n$ , dynamics confined to the 2-D space of product states  $\{|e, n\rangle, |g, n+1\rangle\}$
- In this basis :

$$\langle e, n | g, n+1 \rangle = 0 \text{ and } \hat{H}^{(n)} = \begin{pmatrix} n\hbar\omega_c + \frac{1}{2}\hbar\omega_a & \hbar\Omega\sqrt{n+1} \\ \hbar\Omega\sqrt{n+1} & (n+1)\hbar\omega_c - \frac{1}{2}\hbar\omega_a \end{pmatrix}$$

Energy eigenvalues are

$$E_{\pm}(n) = \left(n + \frac{1}{2}\right)\hbar\omega_c \pm \hbar\sqrt{(\omega_a - \omega_c)^2 + 4\Omega^2(n+1)}$$

On resonance ( $\omega_a - \omega_c = 0$ ):  $E_{\pm}(n) = \left(n + \frac{1}{2}\right)\hbar\omega_c \pm \hbar\Omega\sqrt{n+1}$

The Rabi oscillations with frequency  $\omega_n = 2\Omega\sqrt{n+1}$

Atom spontaneously emitting a photon and absorbing it then re-emitting it, etc

We only take into account the bare state of the Jaynes-Cummings Hamiltonian; namely, the transition between an excited state with 'n' photon and the ground state with 'n' plus one photon. In this basis Hamiltonian can be written by a two by two matrix written here. The eigenfrequency are given by this expression. At resonance when the frequency of the cavity and the one of the two-level system are equal, We can evidence the splitting in energy with a frequency omega 'n' directly linked to the coupling here also called the Rabi frequency.

Notes

Summary



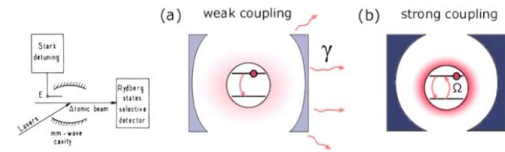
# STRONG COUPLING : EXPERIMENTS

With Jaynes-Cummings model, experiments have been carried out with different systems

## Cavity – 2 level systems (atoms, QD ...)

Brune et al, Phys. Rev. Lett. **76**, 1800 (1996)

Yoshie et al, *Nature* volume **432**, 200 (2004)

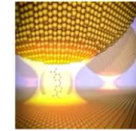


## Plasmonic cavity – Single molecule

Chikkaraddy et al, *Nature* **535**, 127 (2016)

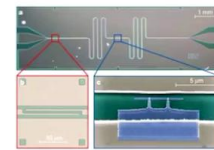
## Plasmonic cavity – 2D materials

Kleeman et al, *Nature Communications* volume **8**, 1296 (2017)



## Single photon – Superconducting qubit

Walraff et al, *Nature* **431**, 162 (2004)



## Single photon – Magnetic Vortex

Martinez-Perez et al, *ACS Photonics* 2019, 6, 2, 360-367

**ALSO IN OPTOMECHANICS !**

This frequency is the frequency at which atoms emit a photon in the cavity and reabsorb this photon inside Jaynes-Cummings model have been intensively used and strong coupling has been evidence in many system such as cavity with two-level atoms, or Quantum Dots, plasmonic cavities with single molecule or even single photons with superconducting qubit or magnetic vortex. But it has also been shown in optomechanics.

Notes

Summary



11m 33s

# OPTOMECHANICAL NORMAL-MODE SPLITTING

Non-dissipative part of the Hamiltonian  
dominate all decay channels :

$$g_{OM} \gg \kappa, \Gamma_m$$

$$\hat{H} = \underbrace{-\hbar\Delta\delta\hat{a}^\dagger\delta\hat{a}}_{\text{Optical oscillator}} + \underbrace{\hbar\Omega_m\hat{b}^\dagger\hat{b}}_{\text{Mechanical oscillator}} - \underbrace{\hbar g_{OM}(\delta\hat{a}^\dagger + \delta\hat{a})(\hat{b}^\dagger + \hat{b})}_{\text{Coupling}}$$

In optomechanics when the non-dissipative part of the Hamiltonian dominate all the decay channel; namely, when the optomechanical coupling is larger than the linewidth of the cavity and the linewidth of the mechanical modes. The Hamiltonian can be written as follow with the optical part here with the optical detuning between the laser and the mode of the cavity. The mechanical term here with the mechanical frequency omega 'm' and the interaction term here.

Notes

Summary



12m 16s

# OPTOMECHANICAL NORMAL-MODE SPLITTING

Non-dissipative part of the Hamiltonian  
dominate all decay channels :

$$g_{OM} \gg \kappa, \Gamma_m$$

$$\hat{H} = \underbrace{-\hbar\Delta\delta\hat{a}^\dagger\delta\hat{a}}_{\text{Optical oscillator}} + \underbrace{\hbar\Omega_m\hat{b}^\dagger\hat{b}}_{\text{Mechanical oscillator}} - \underbrace{\hbar g_{OM}(\delta\hat{a}^\dagger + \delta\hat{a})(\hat{b}^\dagger + \hat{b})}_{\text{Coupling}}$$

In the case of red-detuned regime :

$$\Delta \approx -\Omega_m$$

$$\hat{H} = -\hbar\Delta\delta\hat{a}^\dagger\delta\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \hbar g_{OM}(\delta\hat{a}^\dagger\hat{b} + \delta\hat{a}\hat{b}^\dagger)$$

beam-splitter  
Hamiltonian

Eigenmodes = hybridize  $\left\{ \begin{array}{l} \text{mechanical oscillations } \hat{b} \\ + \\ \text{fluctuations of the driven cavity mode } \delta\hat{a} \end{array} \right.$

In the case of the red detuned regime, the detuning is set at minus omega 'm'. The a Hamiltonian can be rewritten with only two terms in the interaction terms. Thus looking like the beam-splitter Hamiltonian. In this configuration, like in the Jaynes-Cummings Hamiltonian the eigenmodes are hybridize here between the mechanical oscillator and the fluctuation of the driven cavity mode.

Notes

Summary



12m 56s

# OPTOMECHANICAL NORMAL-MODE SPLITTING

Eigenfrequencies :

$$\omega_{\pm} = \frac{\Omega_m - \Delta}{2} \pm \sqrt{g_{OM}^2 + \left(\frac{\Omega_m + \Delta}{2}\right)^2}$$

In the case of red-detuned regime :

$$\Delta \approx -\Omega_m$$

$$\omega_+ - \omega_- = 2g_{OM}$$

The eigenfrequency can be written similarly to what have been done for Jaynes-Cummings Hamiltonian and in the red detuned regime the difference of frequency is directly equals to two time the optomechanical coupling.

Notes

Summary



13m 31s



# OPTOMECHANICAL NORMAL-MODE SPLITTING

Eigenfrequencies :

$$\omega_{\pm} = \frac{\Omega_m - \Delta}{2} \pm \sqrt{g_{OM}^2 + \left(\frac{\Omega_m + \Delta}{2}\right)^2}$$

In the case of red-detuned regime :

$$\Delta \approx -\Omega_m \quad \omega_+ - \omega_- = 2g_{OM}$$

At resonance : Symmetric / antisymmetric superpositions of light and mechanics

$$(\delta \hat{a}^\dagger \pm \hat{b})/\sqrt{2}$$

Far from resonance :

Purely optical  $-\Delta$  or mechanical  $\Omega_m$  frequencies

At resonance we are dealing with the superposition of light and mechanics and far from this resonance, we have purely optical and mechanical frequencies.

Notes

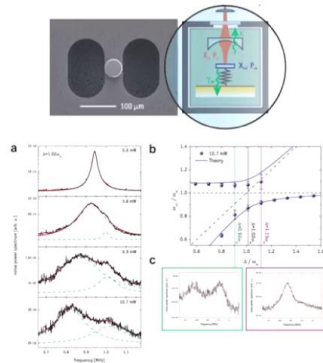
Summary



13m 48s

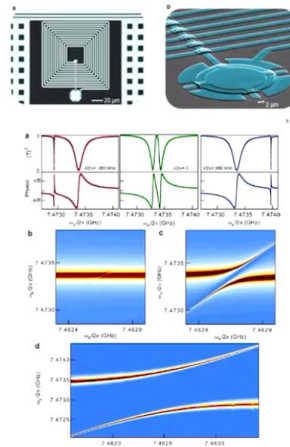
# OPTOMECHANICS : EXPERIMENTAL EVIDENCE

Micro-mirror



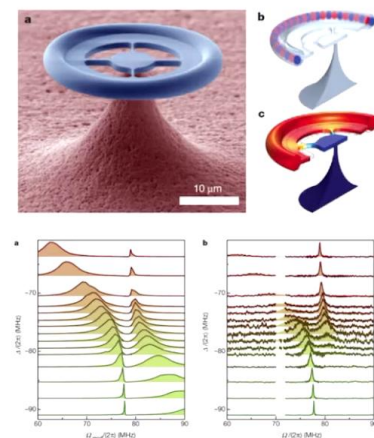
Groblacher et al,  
*Nature* **460**, 724 (2009)

Circuit cavity electromechanics



Teufel et al,  
*Nature* **471**, 204 (2011)

Micro-toroid



Verhagen et al,  
*Nature* **482**, 63 (2012)

In this frame, there have been several evidence of strong coupling with different optomechanical resonators. The first demonstration was made in 2009 with a micro-mirror on a mechanical resonator. Then in the microwave domain in 2011 with an electromechanical circuit. And finally, with the micro-toroid in 2012.

Notes

Summary



14m 01s

# STRONG COUPLING ... AND BEYOND

## Weak coupling

$$G \ll \gamma, \kappa, \Omega_0, \Omega$$

$\Omega$  : driving frequency

$\Omega_0$ : resonant frequency

## Strong coupling

$$\gamma, \kappa \ll G \ll \Omega_0, \Omega$$

Absorb / coherently re-emit a photon many time before it leaks from the cavity

- Rabi oscillations between 2 eigenstates possible

## Ultra Strong coupling

$$\gamma, \kappa \ll G \lesssim \Omega_0 \ll \Omega$$

interaction strength becomes of the order **OR** larger than the bare frequencies

- Photon blockade
- Non-classical state generation
- Superradiant phase transition
- Ultra efficient light emission
- ...

## Deep Strong coupling

$$\gamma, \kappa, \Omega_0 \ll G \lesssim \Omega$$

- Simulation of relativistic quantum phenomena
- Realisation of Dicke spin-boson model
- ...

Kockum et al, *Nature Reviews Physics* **1**, 19 (2019)  
 Forn-Diaz et al, *Rev. Mod. Phys.* **91** 025005 (2019)  
 Casanova et al, *Phys. Rev. Lett.* **105**, 263603 (2010)  
 De Liberato, *Phys. Rev. Lett.* **112**, 016401 (2014)  
 Ballester et al, *Phys. Rev. X* **2**, 021007 (2012)

In order to conclude we have seen the weak and strong coupling which depend on the coupling 'G' is smaller or larger than the optical and mechanical damping here or here. In the case of strong coupling, we discuss the emission and reabsorption of the coherent photon many times. And we can evidence the Rabi oscillation between the two eigenstate two possible eigenstate. We can even go beyond to coupling with ultra strong coupling and deep strong coupling where the coupling is of the order or even larger than the resonant frequency here or the driving frequency here. In the case of optomechanics, the resonant frequency could be the mechanical frequency and the driving frequency could be the optical frequency and for this two new coupling regimes, photon blockades or non-classical state generation can be achieved and even simulation of relativistic quantum phenomena can be achieved as well.

Notes

Summary



14m 29s