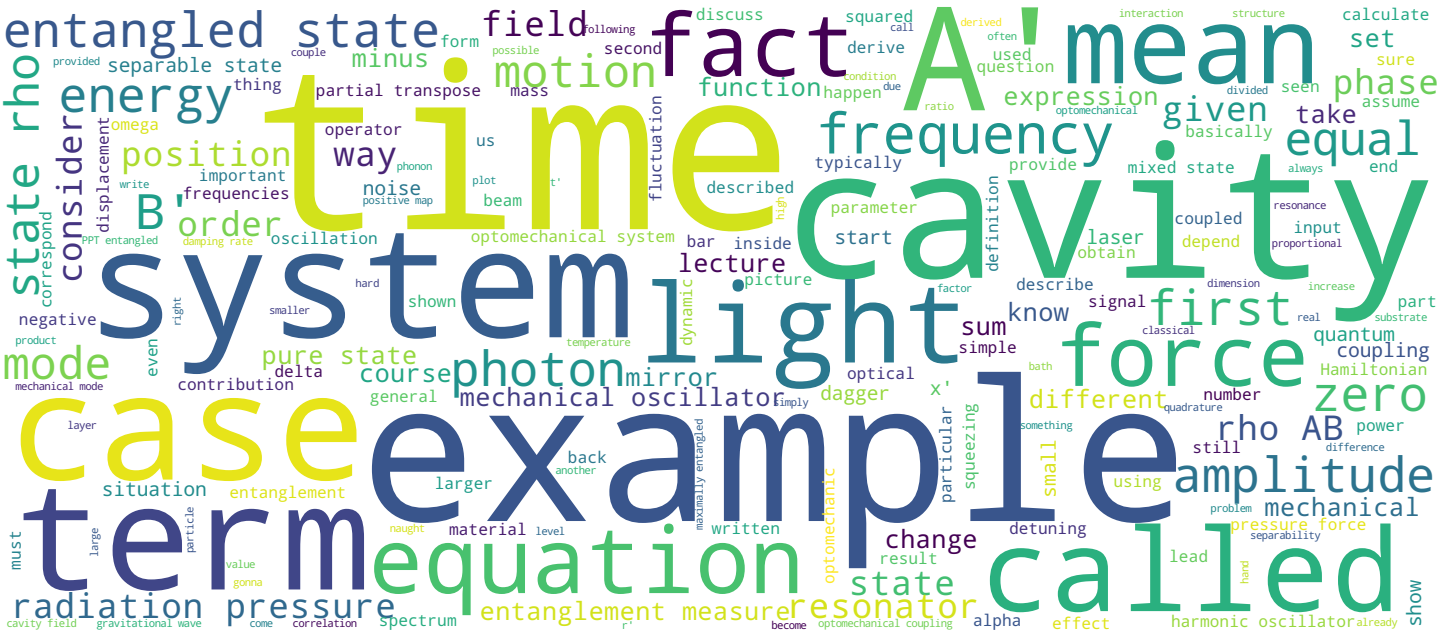
ENTANGLEMENT I

Cavity Quantum Optomechanics

David Vitali

Physics Division, School of Science and Technology
University of Camerino, Camerino, Italy



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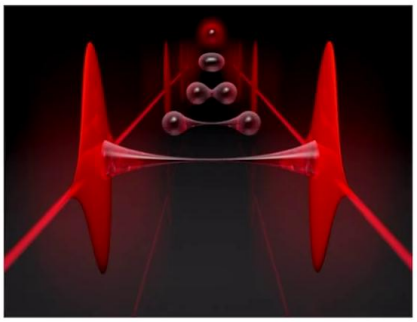
WELCOME & OUTLINE

- This lecture on the concept of entanglement will provide the basic mathematical definitions and the main properties of entanglement between two quantum systems (we restrict to *bipartite systems* only here)

The lecture is divided into two parts:

In the first part we will deal with:

- Basic definitions
- Entanglement criteria
- Entanglement measures



Welcome, I'm David Vitali and we'll talk about entanglement in cavity optomechanical systems. This lecture will provide the the basic definition of the concept of entanglement and also discuss the main properties and the particular case of bipartite system which means the systems formed by two parties which we'll call 'A' and 'B'. The lecture itself is divided into two parts. In this first part now we will deal with the basic definitions. The entanglement criteria which means the conditions under which we are able to establish if a state is entangled or not and finally, also the entanglement measures which allows us to quantify how much entanglement is contained in a state.

Notes

Summary

0m 00s

BASIC DEFINITIONS

Very different situation between **pure states** (vectors $|\Psi\rangle$ in a Hilbert space) and **mixed states** (density operators $\rho > 0$) of a bipartite system AB. **For pure states it is much easier.**

For **pure states** one can always make the **Schmidt decomposition** over joint orthonormal basis, **which is unique**

$$|\Psi\rangle_{AB} = \sum_{k=1}^{r} c_k |\psi_k\rangle_A |\phi_k\rangle_B$$

$|\Psi\rangle_{AB}$ ***is entangled*** if and only if (iff) the number of terms in the sum ("Schmidt rank") ***r > 1*** .

If *r = 1*, $|\Psi\rangle_{AB}$ is a product (factorized, or separable) state
If ***r = d*** = dimension of A and B Hilbert state, and $|c_k| = 1/\sqrt{d}$, $\forall$k, $|\Psi\rangle_{AB}$ is **maximally entangled**

For mixed states, ρ_{AB} is separable if it can be written as a **mixture (convex sum) of product states,** with probabilities p_j

$$\rho_{AB} = \sum_j p_j \rho_A^{(j)} \rho_B^{(j)} \qquad \sum_j p_j = 1$$

ρ_{AB} entangled = not separable

Let us start with the basic definitions. In fact, there is a very different situation between pure states which are described by vectors psi in a Hilbert space and mixed states, which are described in quantum mechanics by positive density operators which means operators whose eigenvalues are always equal to zero or positive. In fact, for pure states the situation is much easier because one can always make the so-called Schmidt decomposition which is shown here and it is a sum over a unique index 'k' over a joint orthonormal basis, psi for the system 'A' and phi for the system 'B'. The Schmidt decomposition always exists and is unique and, therefore, it allows us to arrive at the simple definition. In fact, the state psi-AB is entangled if and only if the number of terms in the sum, which is called the so-called the Schmidt rank, is strictly larger than one. If on the contrary 'r' is equal to one, the state is a product or factorized or separable state. On the other hand, 'r' can be up to equal to 'd', the dimension of 'A' and 'B' which is the maximum number of terms that one can have. And, moreover, if the moduli of the Fourier coefficients 'c' sub 'k' are all identical, in this case, the state psi is called maximally entangled.

Notes

Summary

0m 53s

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The definition is different instead for mixed state. In fact, for the mixed state, rho-AB is separable if it can be written as a mixture of convex sum of product states with probabilities 'pj'. This decomposition is written here and it shows classical correlation between 'A' and 'B' so we have a mixture of different situation in each of which we have a correlation between 'A' and 'B' which are classical. If on the contrary, the state cannot be written in this way in this sum, the state is not separable and, therefore, it is entangled.

Notes

Summary

2m 34s

ENTANGLEMENT CRITERIA: PPT CRITERION

The definition of separability for mixed states is not operational (hard to be sure that no convex sum representation exists). One needs **simple criteria** to establish if a state is entangled or not.

Equivalent (still **unpractical**) definition of separability

$$\rho_{AB}\ separable \quad \Leftrightarrow I_A \otimes \Lambda_B(\rho_{AB}) > 0 \quad \forall \quad positive \quad map\ \Lambda_B$$

Transposition of a matrix is an example of positive map ⇒ **positive partial transpose (PPT) necessary criterion of separability (Peres-Horodecki)**

$$\rho_{AB}\ separable \quad \Rightarrow \rho_{AB}^{T_B} > 0$$

NPT entanglement criterion (sufficient condition)

$$\rho_{AB}^{T_B} < 0 \Rightarrow \rho_{AB} \quad inseparable$$

Successful because it detects many entangled states. **But it is only sufficient** and therefore **there are PPT entangled states** [at least dim($\mathcal{H}_{AB}$) > 6, e.g. two qutrits]

It is quite evident that the definition of separability for mixed state is not operational at all because it's very hard to be sure that no convex sum representation exist of the given state. Instead we need a simple criterion to establish if a given state is entangled or not. In order to arrive this criteria, we start from an equivalent but still unpractical definition of separability which we shown here. State of 'A' and 'B' is separable if and only if the application of every positive map lambda 'B' on a party 'B', so on one of the two parties, maintain the positivity of the state. This definition of separability is still unpractical because you have to verify this positivity preservation on all of the possible positive map and actually, it is still not yet known so this system is not yet fully characterized, we don't know all the possible positive maps. On the other hand, we can derive a simple necessary criterion from this because by simply considering one appropriate positive map. An example of this is the transposition of a matrix. So if this is separable for sure the partial transpose of rho-AB is certainly positive and this means so that positive partial transpose is a necessary condition for separability which is derived by Peres-Horodecki.

Notes

Summary

3m 12s

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From this we can derive a sufficient condition for entanglement, the so-called NPT entanglement criterion because by negating, so if negating this condition, so if the partial transpose is negative, it means that for sure they still will be is inseparable and, therefore, it is entangled. The Peres-Horodecki criterion is be very successful because it detects many entangled states. For example, it is necessary sufficient all the entangled states because it's necessary sufficient for two qubit and one qubit and one qutrit. However, in general it is always sufficient and, therefore, for sure we know that there would be the so-called PPT entangled states, the states which are entangled but with positive partial transpose. This occurs at the sufficiently large dimensions, for example, two qutrits so we need a dimension of the joint Hilbert space which is strictly larger than six.

Notes

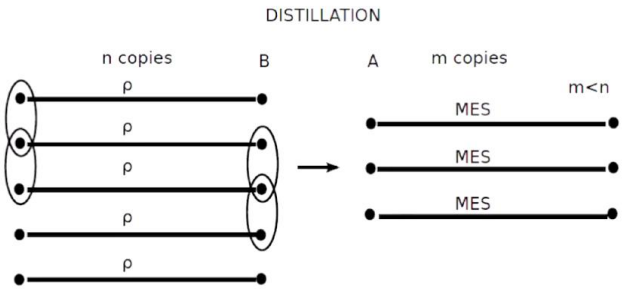
Summary

4m 50s

PPT ENTANGLED STATES

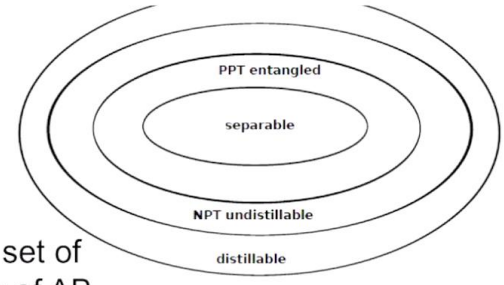
PPT entangled states = "bound" entangled states, because cannot be exploited for some tasks. Example: **distillation protocol**

Entanglement Distillation (purification) protocol: Starting from *n* copies of ρ_{AB}, "distill" from them $m < n$ copies of **maximally entangled pure states**, using only local operations and classical communication **(LOCC)**



• **PPT entangled states cannot be distilled (entanglement is "locked" and unusable)**

• It is conjectured that there are also NPT undistillable states



Partition of the set of quantum states of AB

Let us spend few more words on PPT entangled states. They are also called a bound entangled states because in these states entanglement cannot be easily exploited for some useful tasks. Example is provided by the so-called distillation protocol also called purification. In this protocol, one starts from 'n' copies of a given state rho-AB and wants to distill from them a smaller number 'm' of copies of maximally entangled pure states. Using, however, only local operation on 'A' and 'B' sides and at the same time eventually classical communication which means performing measurements either on 'A' and 'B' and is changing the information on the results of the measurement. Many distillation protocol has been derived. It has been proved that a PPT entangled states cannot be distilled which means that entanglement in this state is locked and unusable for distillation at least. The situation is a bit involved because it has been conjectured that there are also some NPT states which are undistillable not yet found I have to say. And so, in general, one proposes the following partition of set of quantum state of *[inaudible]* system. The center we have separable states.

Notes

Summary

5m 49s

PPT ENTANGLED STATES

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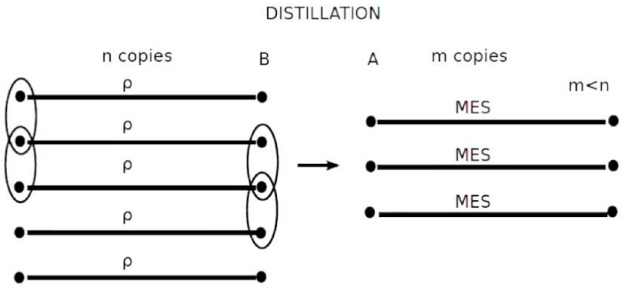

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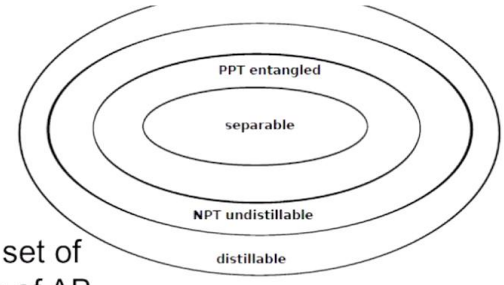


Partition of the set of quantum states of AB

Outside of it there are all the entangled states which are in turn respectively are divided into subsets. There will be PPT entangled state, possibly NPT undistillable states and finally distillable entangled states, which are the most useful one because one can use them for useful quantum information protocols.

Notes

Summary

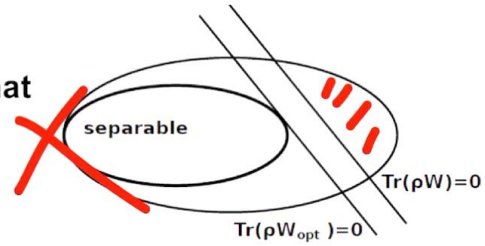
7m 13s

ENTANGLEMENT WITNESSES

A different sufficient condition for entanglement

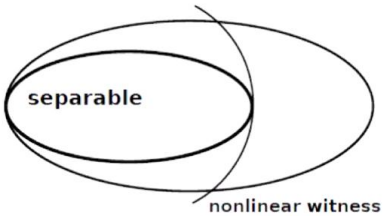
ρ_{AB} is entangled if an Hermitian operator *W* exists such that

$$\langle W \rangle = Tr[\rho_{AB} W] < 0 \quad and \quad Tr[\rho_{sep} W] > 0 \quad \forall \rho_{sep}$$



Both the whole set of states and **the set of separable states are convex**, and therefore one needs in general **infinite** operators *W* to fully characterize the set of entangled states.

One can better "approximate" the set of separable states **with "nonlinear witnesses"**, for example **using variances** (we shall see some examples later), which have a quadratic term in ρ.



A different sufficient condition for entanglement is that its provide if we consider the so-called entanglement witnesses. In fact, state rho-AB is entangled if an Hermitian operator 'W' exists such that the average value of 'W' on the state rho-AB is negative and at the same time this mean value is positive for all the possible separable states. This means that actually we essentially divide the operator 'W' is going to any sort of hyperplane which divide the set of states into two sets. One in which we are sure to have entangled states and the other where we are not sure to entangled states. However, the set of separable states is convex. And, therefore, it is evident that we can have tangent witnesses with tangentially separable states. But it indicate we'll need an infinite number of tangent witnesses in order to fully characterize the set of separable states. This is why in some cases one consider the so-called nonlinear witnesses which are functional, which are nonlinear functional of the density matrix rule. For example, variances which have a quadratic dependence upon the density state rho. They provide a better approximation to the set of separable states. We should see this later on in the second part of the lecture.

Notes

Summary

7m 33s

ENTANGLEMENT MEASURES

They address the questions: i) **how much** is ρ_{AB} entangled ? ii) is state $\rho_{AB}^{(1)}$ **more entangled than** $\rho_{AB}^{(2)}$?

Required properties of an entanglement measure $E(\rho)$:

- ρ separable $\Rightarrow E(\rho) = 0$
- no increase under LOCC $\Leftrightarrow E\left(\wedge_{LOCC}(\rho)\right) \leq E(\rho)$
- local unitary invariance $\Leftrightarrow E(U(\rho)) = E(\rho)$ when $U = U_A \otimes U_B$= local unitary operation
- Continuity: $||\rho_1 - \rho_2|| \to 0 \Rightarrow |E(\rho_1) - E(\rho_2)| \to 0$

Still unsatisfactory situation: very many proposals, only few possess all the above properties and those who do are very hard to calculate.
Answering the above questions is not so simple !

Let us now consider the problem of entanglement measures. They address two quite intuitive questions. First of all, how much a state rho-AB is entangled? Another question is that if a state rho-AB one is more entangled than another state rho-AB two. In order to address this question, we need an entanglement measure for which we require some quite reasonable property, which we listed here. First of all, it must be zero for every separable states. Another condition is that it must not increase under local operation in classical communication. The third condition is that the so-called local unitary invariance, which means that if we apply unitary operation, local unitary operation on 'A' and 'B' separately then we expect that entanglement measure must not change so it must be remain the same. Finally, we require continuity which means that if a state rho one approach another state rho two on the same distance, at the same time the corresponding entanglement measure 'E' of rho one should approach the entanglement measure 'E' of rho two.

Notes

Summary

9m 12s

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Still unsatisfactory situation: very many proposals, only few possess all the above properties and those who do are very hard to calculate.
Answering the above questions is not so simple !

From a theoretical point of view, the situation is still unsatisfactory because there have been many many proposal in the literature but only few have been proved to possess the above required properties and, moreover, the few which we have been proven to possess the necessary properties are often very hard to calculate and therefore in general, original state. Answering the two above questions on how much the state is entangled is typically, for example, is not, typically not so simple at all.

Notes

Summary

10m 30s

ENTANGLEMENT MEASURE FOR PURE STATES

Again the situation **is solved and easy for pure states**, i.e., $\rho_{AB} = (\rho_{AB})^2$

Reduced states of the two parties,
$\rho_A = Tr_B[\rho_{AB}]$, $\rho_B = Tr_A[\rho_{AB}]$

von Neumann entropy of ρ

$$S(\rho) = -\langle \log \rho \rangle = -Tr[\rho \log \rho]$$

The unique entanglement measure with all the desired properties, is the von Neumann entropy of the reduced state

Entanglement entropy

$$E(\rho_{AB}) = S(\rho_A) = S(\rho_B)$$

From the Schmidt decomposition $|\Psi\rangle_{AB} = \sum_{k=1}^{r} c_k |\psi_k\rangle_A |\phi_k\rangle_B$

$$E(\rho_{AB}) = E(|\psi\rangle_{AB}\langle\psi|) = -\sum_{j=1}^{r} |c_i|^2 \log |c_i|^2$$

For maximally entangled states in dimension d $E(\rho_{AB}) = \log d$

Again, if we consider pure states, the situation is quite easy and has been actually fully solved. In fact, in this case for pure states, one, first of all, could see that the reduced states of two parties obtained by tracing over the other parties so rho-A and rho-B are the two reduced states of the two parties. And then one consider the Von Neumann entropy of rho, which is minus the average value of logarithm of rho. The unique entanglement measure with all desired properties for pure states is the just the Von Neumann entropy of the reduced state which is also called the entanglement entropy we show here. It is actually the same thing if we either use rho-A or rho-B is just the same just because the initial state is pure. And using the Schmidt decomposition which we know it always exist, one can even arrive at the simple analytical expression of rho of the entanglement entropy which is minus the sum over the corresponding using as a function of the square modulus of the Fourier coefficient 'c'. Fine. For maximally entangled states in dimension 'd', the maximum value of the entanglement to be is just the logarithm of the dimension 'd'. So maximally entangled states corresponds to the maximal entanglement entropy, of course.

Notes

Summary

11m 04s

SOME ENTANGLEMENT MEASURES (MIXED STATES)

We show **only two examples** out of the very many proposals in the literature

Entanglement of formation: $E_F(\rho_{AB}) = \inf \sum_j p_j S(|\psi_j\rangle\langle\psi_j|)$

Where the infimum is taken **over all possible pure state decompositions** of ρ_{AB}; it has the desired properties, but it is difficult to evaluate in general

Logarithmic negativity: $E_N(\rho_{AB}) = \log[1 + 2N(\rho_{AB})] \qquad N(\rho_{AB}) = \max\left\{0, -\sum_i \lambda_i^-\right\}$

$N(\rho)$ negativity, λ^-_i all the negative eigenvalues of the partial transpose ρ^{T_B}

Pros: it is intuitive (**it quantifies how much the PPT criterion is violated**), and it is the easiest to calculate

Cons: it is not a real measure (it is only an entanglement monotone): **it is not continuous** $\Rightarrow$ it does not always reduce to the unique measure for pure states, $S(\rho)$

Let us now finally look what happens in the case of mixed states. In this case, we consider only two example out of the very many proposal in the literature. The entanglement of formation, which is obtained by taking the infimum over all possible pure state decomposition of rho-AB as we can show here. It has all the desired properties. However, as you can imagine it is difficult to evaluate in general because one has to take the infimum over an infinite number of possible situation because it's very hard to be consider all the possible pure state decomposition of a given mixed state rho-AB. And now the proposed entanglement measure is the so-called logarithmic negativity which has a simple formula just the logarithm of one times one plus two times the negativity which is just the maximum between zero and the sum over all the possible negative eigenvalues of the partial transpose. So actually corresponds to the intuitive idea of quantifying how much the PPT criterion is violated so by looking directly at the negative eigenvalues of the partial transpose density matrix. So it is computable and so this is a nice property. And is also an intuitive an intuitive meaning.

Notes

Summary

12m 39s

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Cons: it is not a real measure (it is only an entanglement monotone): **it is not continuous** $\Rightarrow$ it does not always reduce to the unique measure for pure states, $S(\rho)$

On the other hand, it has a very bad expert because it's not a real measure. In fact, it is called as an entanglement monotone because it's not continuous and, therefore, it does not always reduce to the unique measure for pure states.

Notes

Summary

14m 01s

SUMMARY

- We have seen how one can establish if a state is entangled or not (entanglement criteria)
- We have seen how one can mathematically quantify the amount of entanglement (entanglement measures)
- In the second part of the lecture we shall apply these definitions to cavity optomechanical systems and discuss some physical implications

So in conclusion we'll see how one can establish if a state is entangled or not so we have provided some entanglement criteria the PPT, the NPT criterion, the entanglement witnesses. And we've seen that how one can mathematically quantify the amount of entanglement by looking at some entanglement measure. In the second part of the lecture, we shall apply these definitions to cavity optomechanical systems and discuss some physical implication of entanglement.

Notes

Summary

14m 15s