MODAL ANALYSIS IN A FINITE ELEMENT SIMULATION

Cavity Quantum Optomechanics

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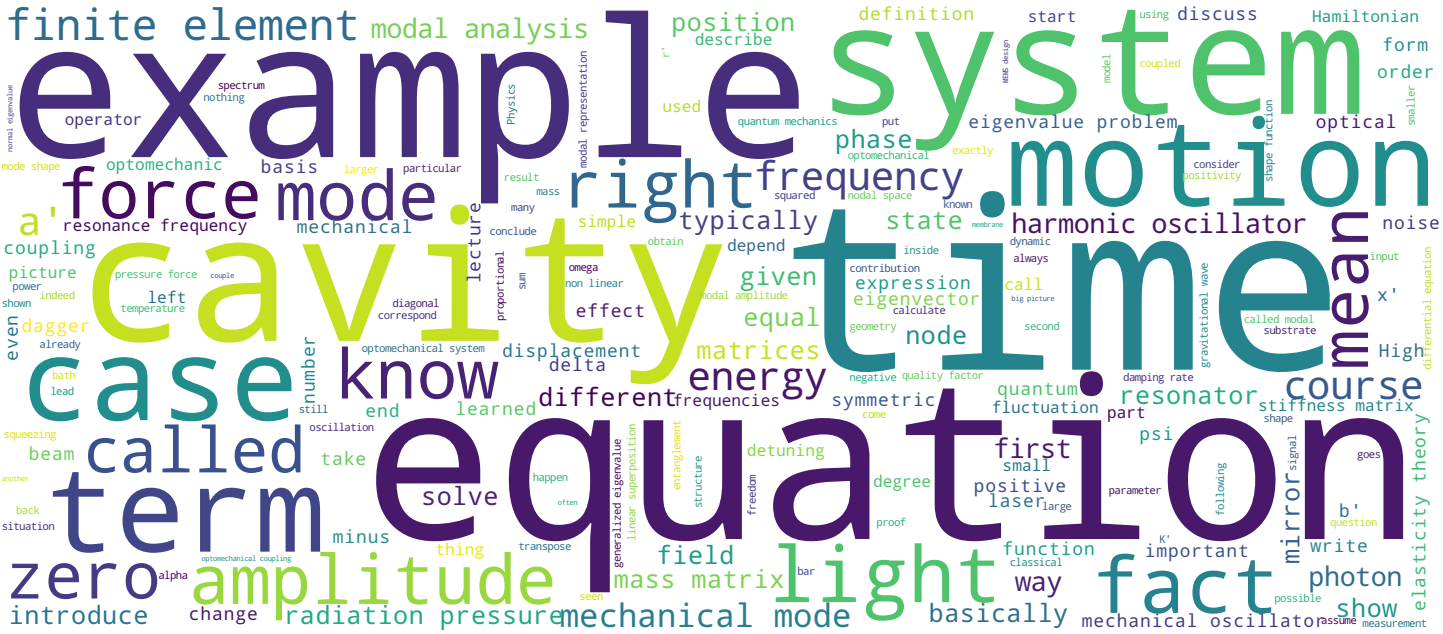

Search MOOC

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WELCOME & OUTLINE

- **Outline**
  - Intuition of mechanical modes and the **big picture**
  - Modal analysis in FEM
    - The generalized Eigenvalue problem
    - Properties of the global Mass and Stiffness matrices
    - The modal basis
    - Decoupling of the equations of motion
  - Summary & Conclusion

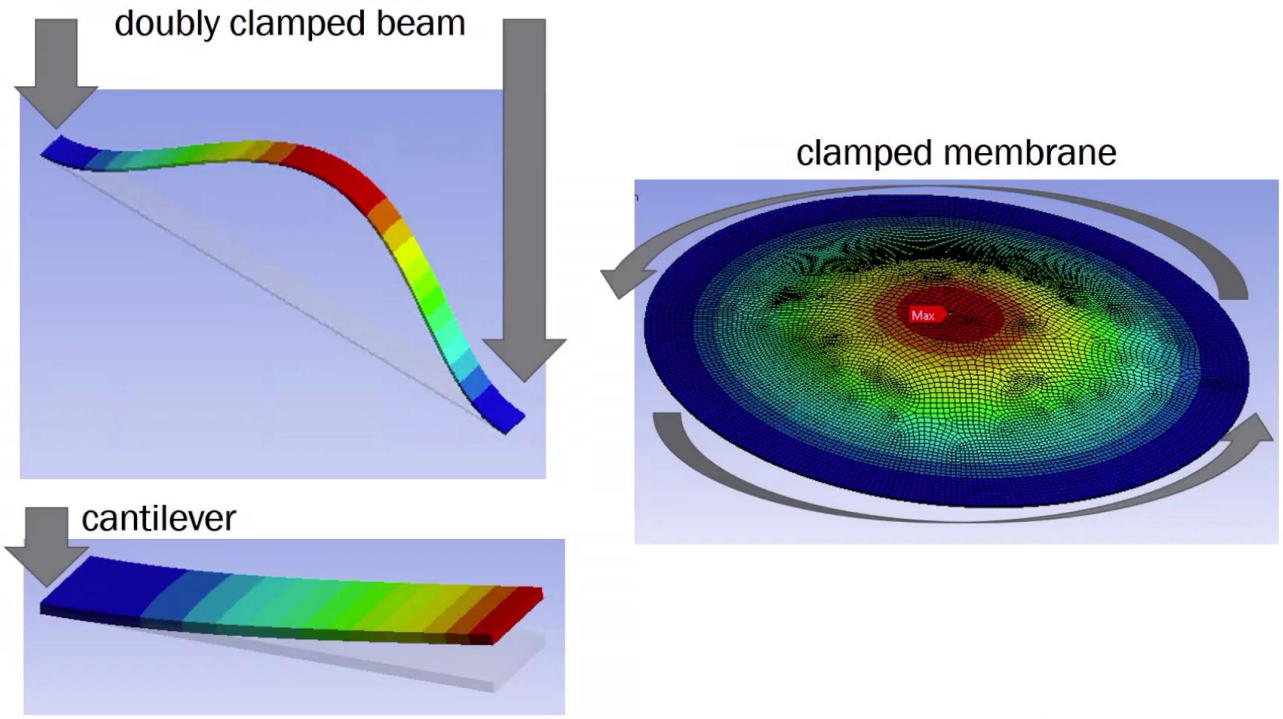
Hello everybody. My name is Dr. Peter Degenfeld-Shonburg. I am from a Corporate Research of Robert Bosch GmbH and today I will give a lecture on Modal Analysis in a Finite Element Simulation. It is the part of the optomechanics mooc. We will first start with an intuition of a mechanical mode. So in all what follows we will discuss mechanical modes. And then we will discuss the big picture and then talk about why we go through all the trouble with elasticity theory, finite element modeling and modal analysis. We'll dig into the details. So what is the generalized Eigenvalue problem in a modal analysis and then we will discuss some properties of the mass and most importantly we will show that the modes form a basis of your vector space, the so-called modal basis and if you rewrite the equations of motions we are able to decouple them into harmonic oscillators. At the end we will summarize and conclude.

Notes

Summary

0m 00s

MOST FAMOUS MECHANICAL MODES



All right. Mechanical modes. What do we have in mind? So to me the most famous ones are the doubly clamped beam, the cantilever, and the clamped membrane. All right so now everybody has a picture in his mind of a certain modes and for this simple geometries there are analytic solutions existing.

Notes

Summary

1m 14s

WHY BOTHER WITH FEM?

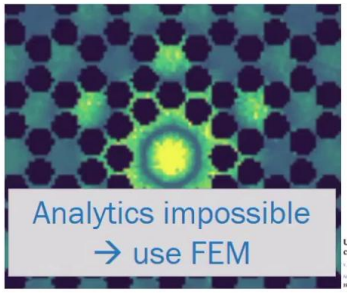
Analytic results from elasticity theory

- The EOM in nodal space for a membrane with $h \ll a$ reads

$$\mu_{2D}\,\ddot{z}(x,y,t) + d\,\dot{z}(x,y,t) + \frac{Eh^3}{12\,(1-\sigma^2)}\,\Delta^2 z(x,y,t) - T\Delta z(x,y,t) = F_{ext}$$

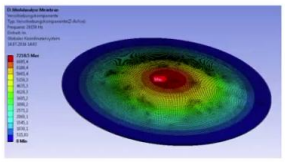
- The vertical displacement of the membrane is $z(x,y,t)$.
- The geometric nonlinearity comes from the tension T

$$T = Eh\frac{\delta A_0}{A_0} = \frac{Eh}{2A_0}\int dA_0\,\left(\vec{\nabla}\,z(x,y,t)\right)^2 \quad \rightarrow\rightarrow T \sim z^2$$ Nonlinear Stiffness

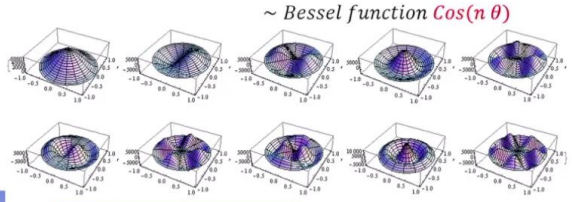


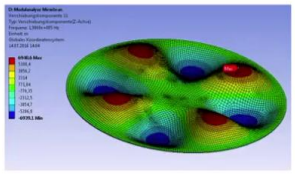
Ultracoherent nanomechanical resonators via soft clamping and dissipation dilution

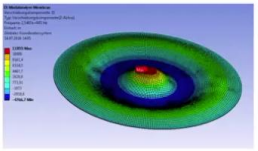
FEM vs analytics for a modal analysis



~ Bessel function $Cos(n\,\theta)$







Frequencies and maxima of mode shapes fit almost perfectly

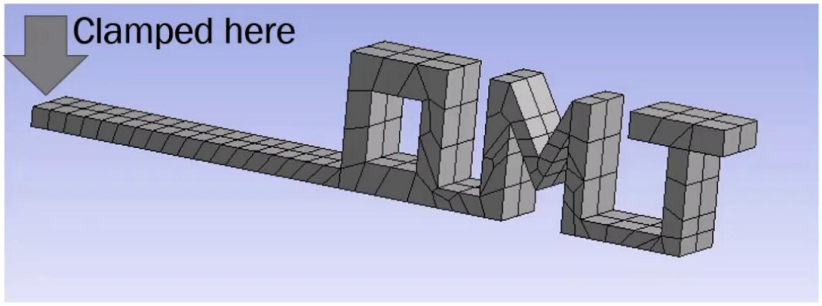
So if you compare your Finite Element Model with this analytic solution, you might wonder why bother at all with FEM, I mean, in contrast to this analytics the FEM is much more, well, you need to go through much more trouble to get the results and at the end you find that frequencies and mode shapes fit perfectly together. But, of course, the answer to this question is as soon as you leave this very simple geometries, for example, if you dig holes into your membrane then the analytics will be unsolvable. So what you need to do is you need to use FEM.

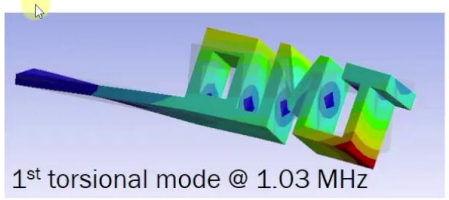
Notes

Summary

1m 44s

MODES OF THE OMT-BEAM





Take home message:

Mechanical modes are a certain "**shape of motion**" with a resonance **frequency** label

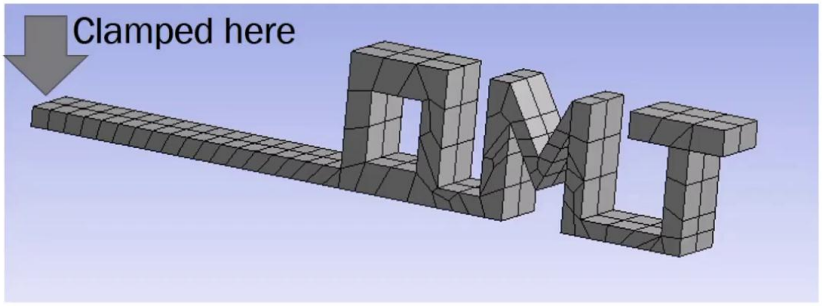
Another example of a certain geometry where there is no chance to solve it analytically is what I have actually designed for this lecture only to guide you through the examples is what we will call the OMT beam in the following. All right so what is the take-home message for a mechanical mode. So a mechanical mode has actually two things so one is the shape of motion so it actually shows you the deflection of every material point in your geometry from its resting position. But it's only a shape of motion. It does not give you an amplitude. So, in fact, you have to assign the amplitude of this shape yourself and typically you just assign this amplitude by solving the equation of motion. Apart from this shape of motion, you also have the resonance frequency and these two things together will make up what we call a mechanical mode. A few examples. These modes I think they are rather boring because, in fact, they look very much like candy level modes here. Also this torsional mode it does not appear in our early theory but it appears in a finite element solution of a cantilever.

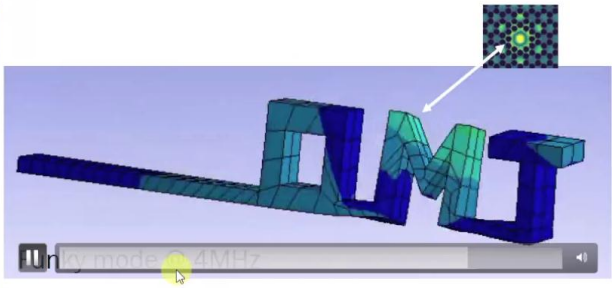
Notes

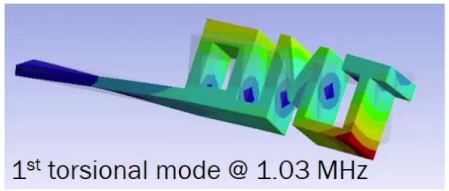
Summary

2m 29s

MODES OF THE OMT-BEAM







Take home message:

Mechanical modes are a certain “**shape of motion**” with a resonance **frequency** label

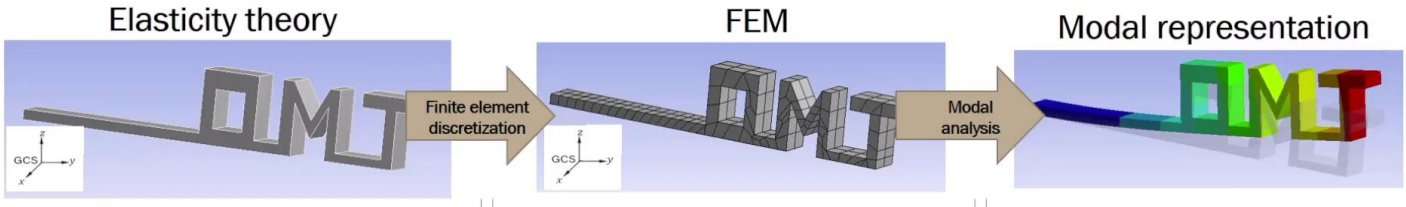
But already at higher frequencies you can have, let's say, funky modes or more interesting modes that is this ‘M’, you know, is doing all the shaking here and somehow reminds me of this localized movement here in this membrane but that's just exaggerating. The main point is that in order to solve for such a modes you really need finite element modeling. There is no analytic hope.

Notes

Summary

3m 53s

MECHANICAL MODES: THE BIG PICTURE



Task: Solve partial differential equations for the field variable $\boldsymbol{u}(\boldsymbol{x}, t)$ which denotes the time dependent displacement of every material point at position $\boldsymbol{x}$ from its position at rest.

Navier-Cauchy equation:

$$\mu\nabla^2\mathbf{u} + (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \mathbf{F} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

Source: https://en.wikipedia.org/wiki/Linear_elasticity

Impossible to solve for arbitrary geometries and boundary conditions

Task: Solve ordinary differential equations for the vector-valued variable $\vec{u}(t)$ which denotes the time dependent displacement of every node of the FE mesh from its position at rest $\vec{x}$.

$$M\,\ddot{\vec{u}} + D\,\dot{\vec{u}} + K\,\vec{u} = \vec{F}$$

Solvable with numerical effort

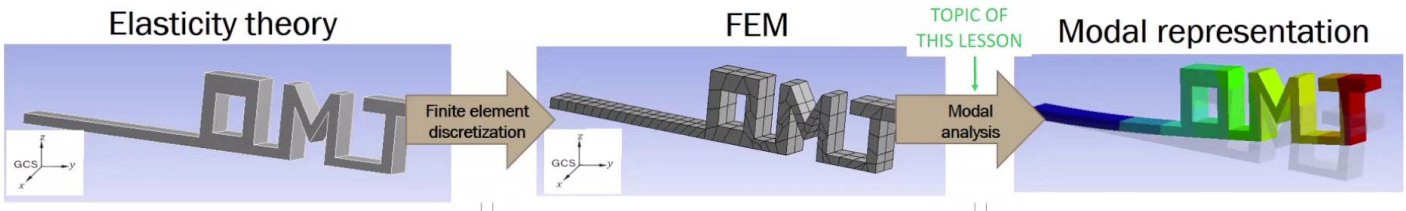
Okay. Let's come to the big picture. So in every Finite Element Model there is typically a fundamental physical theory beneath it. So for mechanical modes there is elasticity theory beneath it and so why is elasticity theory so problematic? Why is it not useful? Well. In fact, it's just impossible to solve elasticity theory for arbitrary geometries and boundary conditions. Why? Because you need to solve the partial differential equation for a field variable. The field variable is the deflection of every material point at coordinate point 'x' and in time 't'. So basically it's just very hard to do if not impossible. And then we entered this great idea of finite element discretization. In finite element discretization, you don't solve for the field variable 'u' so you do not care of the displacement at every material point but instead you introduce a mesh of your model and this mesh contains nodes and at these nodes you actually account or calculate the deflection 'u'. And now only at these nodes it suffices to to solve an ordinary differential equation for vector-valued variable. This vector might be large but still you just need some numerical efforts and then you can solve well, the finite element version of your elasticity theory equations.

Notes

Summary

4m 24s

MECHANICAL MODES: THE BIG PICTURE



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Solvable with numerical effort

Task: Solve ordinary differential equations for the scalar-valued variable $q_1(t)$ which denotes the time dependent modal amplitude.

TOPIC OF THIS LESSON

$$\vec{u}(t) = \sum_{n=1}^{DOF} q_n(t)\vec{S}_n \approx q_1(t)\vec{S}_1$$

Extremely good appr. for high-Q oscillation

Few- or even one-mode problem solvable for both the classical and quantum domain

These typically look like this so you have a map usually you have element matrices, the mass, the stiffness. You also put some external forces into your system and then that basically makes up your dynamics so your equations of motion. Another remark. In-between these nodes, of course, you can also get the deflections but basically it's just clever fitting functions between two nodes what you implement in a Finite Element Model. All right and the next step is again a very nice magic. It's the modal representation. So imagine that you are studying a High-Q oscillatory problem. That's what we typically study in optomechanics and also very often study in MEMS design. And in this case and that's the topic of this lesson, we can show that this displacement ‘u’ which still might be a very large vector, you can represent by the so-called modal representation so by a linear combination of mode shifts. And it turns out that for High-Q systems it actually suffices to only account for the amplitude of a single mode. And then your differential equation is very simple.

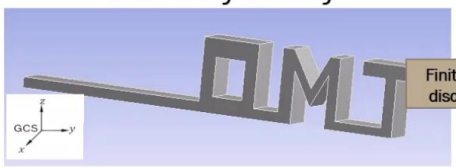
Notes

Summary

6m 12s

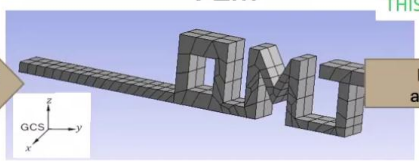
MECHANICAL MODES: THE BIG PICTURE

Elasticity theory



Finite element discretization

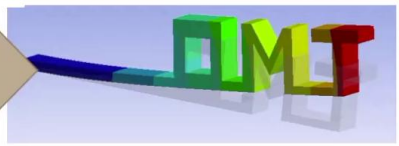
FEM



TOPIC OF THIS LESSON

Modal analysis

Modal representation



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$$\vec{u}(t) = \sum_{n=1}^{DOF} q_n(t)\vec{S}_n \approx q_1(t)\vec{S}_1$$

Extremely good appr. for high-Q oscillation

Few- or even one-mode problem solvable for both the classical and quantum domain

It's just an ordinary differential equation for the scalar variable, the so-called modal amplitude and this is enabling for not only classical problems but especially for problems in the quantum domain because, you know, in the quantum domain it's hard enough to solve a single mode problem or maybe a two or three mode problem but it's certainly is impossible to solve for so many degrees of freedom as encountered in the modal representation here. Yeah, that's the big picture. That's a huge deal. So the complex elasticity theory we are able to represent by a single mode and its modal amplitude.

Notes

Summary

7m 39s

MODAL ANALYSIS IN MEMS DESIGN

MEMS design relies on the accuracy and efficiency of the modal analysis

Source: https://www.youtube.com/watch?v=XsjvaYAFN1M

Also in industrial applications so in what we call MEMS design what we do is a modal analysis so we try to design a geometry which fulfills certain functionalities and that means, in fact, that we need to design a geometry where certain modes have certain functions and then you really can say that MEMS design heavily relies on the efficiency so meaning that the modal analysis needs to be fast enough so that you can do try and *[inaudible]* here and we also rely on the accuracy of the modal analysis because we have certain specification limits for frequencies and so on. Yeah, so modal analysis is a very important tool for MEMS design.

Notes

Summary

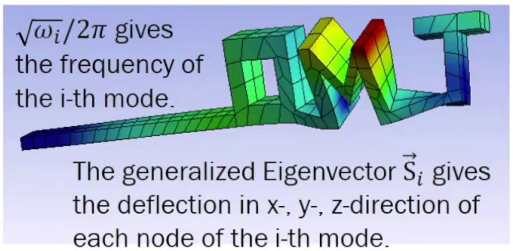
8m 19s

MODAL ANALYSIS IN FEM

In a **FE modal analysis** we solve for the **generalized** Eigenvalue problem (GEP) defined by:

$$\left(K - \omega_i^2 M\right)\vec{S_i} = \vec{0},$$

with $i \in \{1,2,3,\dots,n\}$, where n is some number set by FE user.



The generalized Eigenvector $\vec{S}_i$ gives the deflection in x-, y-, z-direction of each node of the i-th mode.

The matrices M and K are called the **global mass and stiffness matrices**, respectively.
They depend on:

- the density (mass matrix) and elastic constants such as Young's Modulus & Poisson's ration (stiffness matrix).

Okay so this was the big picture. Let's now go into the mathematics, into the details and basically what we're doing here is linear algebra. So a modal analysis is nothing but the solution to a generalized eigenvalue problem. So this generalized eigenvalue problem has two matrices 'K' and 'M' and it has eigenvalues and eigenvectors and the eigenvalues correspond directly to the eigenvector so to the resonance frequencies of the mode and the eigenvectors they correspond to the mode shape. And typically you just calculate a handful of modes so definitely a lot less number of modes than the number of degrees of freedom otherwise the modal analysis does not really help you in terms of efficiency. Okay. So back to these matrices 'M' and 'K'. They are called global mass and stiffness matrices and they are constructed by the FEM in a certain way which we cannot discuss in detail but let me tell you some basic things that you should know. So the mass matrix definitely depends on the density so that's a material property that enters the mass matrix and the stiffness matrix depends on all the elastic constants in your model such as Young's Modulus, for example.

Notes

Summary

9m 10s

MODAL ANALYSIS IN FEM

In a **FE modal analysis** we solve for the **generalized** Eigenvalue problem (GEP) defined by:

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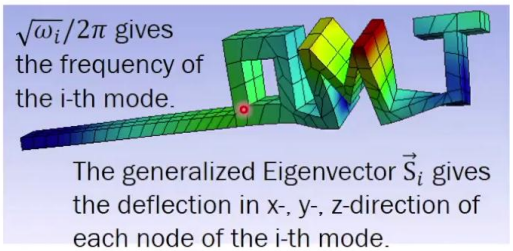

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The matrices M and K are called the **global mass and stiffness matrices**, respectively.
They depend on:

- the density (mass matrix) and elastic constants such as Young's Modulus & Poisson's ration (stiffness matrix).
- the element type (10-node tetrahedral, 20 node hexahedron, ...).
- boundary conditions (fixed support, ...).
- mesh coordinates (and connectivity) of the nodes.

Then the matrices also depend on the element types so I don't know if you have worked a lot with finite element, you find a large number of different element types. Here I just stated the two most really known state-of-the-art elements so the 10-node tetrahedral and 20-node hexahedron which is basically used here. Of course, it also depends on the boundary conditions so fixed support etc. And interestingly so the stiffness matrices depends on the coordinates addressed so the mesh coordinates addressed and, of course, also on the connectivity of the nodes so which nodes do you happen connected over two elements.

Notes

Summary

10m 37s

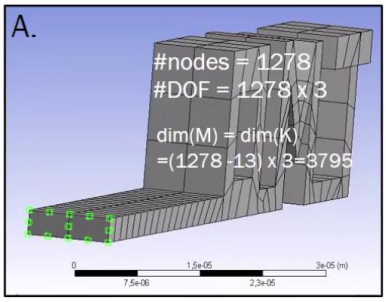
MATHEMATICAL PROPERTIES OF THE GLOBAL MASS AND STIFFNESS MATRICES

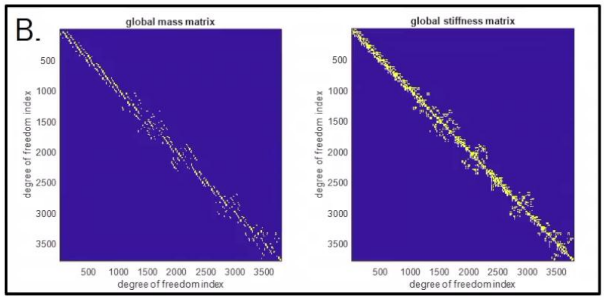
By construction of the FEM and physical reasons M and K are:

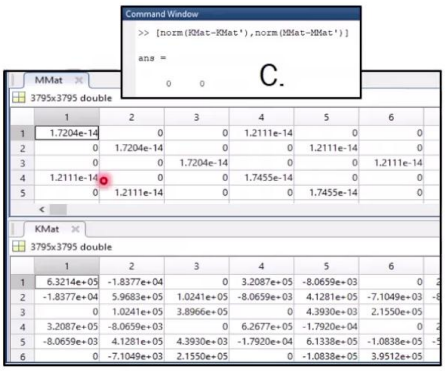
A. quadratic matrices with dimensions as DOF in the FEM (minus boundary DOF).

B. Highly sparse
(but M not necessarily diagonal)

C. Symmetric







Okay. So what are the mathematical properties that arise by construction and physical reasoning? Okay so one is the these matrices are quadratic and their dimensions correspond to the degrees of freedom so typically it's the number of nodes times three but if you wonder why your matrix dimensions are smaller in most cases than your degrees of freedom. Well, it's because you have to exclude the nodes at the boundary. All right. The next one is these matrices are highly sparse. Here I show you a MATLAB plot of where the matrices are different from zero and interestingly what you see is that also the mass matrix is sparse but it's not diagonal. So this is a typical misleading idea so sometimes people think that nodes have a certain mass and this certain mass you just write into the diagonals of a mass matrix. But that is indeed not the case so in the general finite element formulation, the mass matrix is off-diagonal. All right. Both matrices are symmetric by construction. Here just a screenshot. As you can see, these matrices are symmetric and if I take the norm of the matrix minus its transpose, it's actually identically zero.

Notes

Summary

11m 26s

MATHEMATICAL PROPERTIES OF THE GLOBAL MASS AND STIFFNESS MATRICES

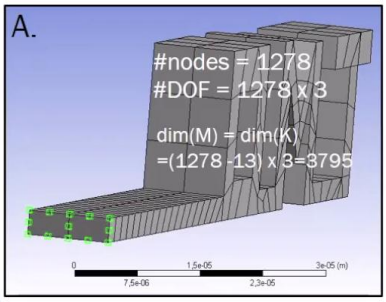
By construction of the FEM and physical reasons M and K are:

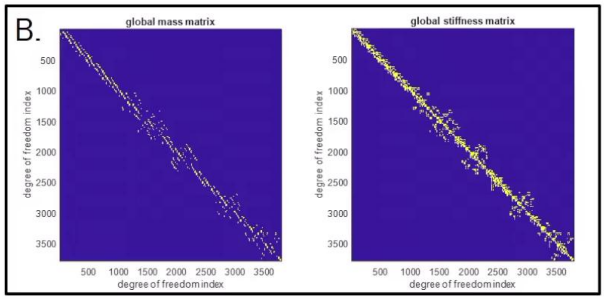
A. quadratic matrices with dimensions as DOF in the FEM (minus boundary DOF).

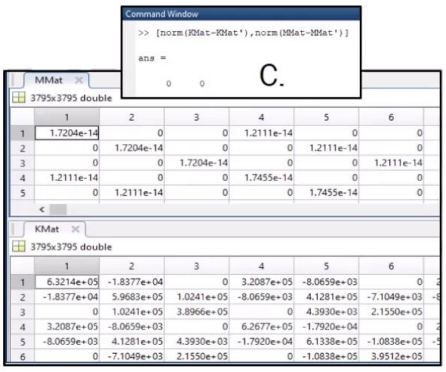
B. Highly sparse
(but M not necessarily diagonal)

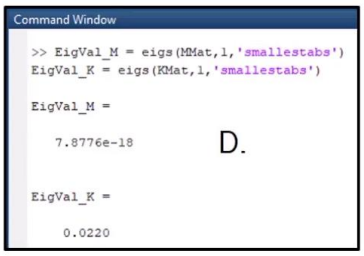
C. Symmetric

D. Positive definite









And then the last one which is very important is the positivity of the matrices so both matrices are positive definite which we can here now quickly show by calculating the smallest eigenvalues and as you can see these smallest eigenvalues are positive and together with this property that these matrices are symmetric the smallest eigenvalue being positive is a direct proof of the positivity. So these matrices are positive and symmetric and highly sparse.

Notes

Summary

12m 59s

MODAL BASIS

$$\left(K - \omega_i^2 M\right)\overrightarrow{S_i} = \vec{0} \quad \text{(GEP)}$$

How can we transform the GEP to a “normal” Eigenvalue problem?

Reminder: in many physical domains we finds Eigenvalue problems (electro dynamics, quantum mechanics, ...)

For example the stationary Schrödinger equation: $H|\phi_n> = E_n|\phi_n>$

Step 1 Use positivity of M for Cholesky decomposition, i.e. $M = LL^T$, with invertible matrices L, L^T.

```
>> L = chol(MMat,'lower'); norm(L*L'-MMat)

ans =

   7.1909e-29
```

```
>> norm(L/L - eye(3795))

ans =

   1.3608e-15

>> norm(L'/L' - eye(3795))

ans =

   3.0937e-15
```

Okay. Let's go to the idea of a modal basis. So let's recall the generalized eigenvalue problem and now you know that the mass matrix 'M' is not diagonal. So the first question I had when I studied this first coming from my university studies of Physics is, can I transform this equation to a normal eigenvalue problem? Why? Why should I ask myself this question? Well, the answer is because in Physics you always find normal eigenvalue problems so, for example, in the Helmholtz equations of electrodynamics or the stationary Schrodinger equation of quantum mechanics. So here, for example, what you have is a normal eigenvalue problem so your Hamilton operator or in finite-dimensional Hilbert spaces, it's just a matrix is actually Hermitian and we know that Hermitian matrices have eigenvectors which form a basis. So can we do the same here and the answer is yes. We can do that by doing three steps. So step one. We use the positivity of the mass matrix to do a so-called Cholesky decomposition. That means that we write 'M' like this and the most important fact is that this matrix 'L' and its transpose, they are regular matrices so they are, in particular, invertible.

Notes

Summary

13m 32s

MODAL BASIS

$$\left(K - \omega_i^2 M\right)\vec{S_i} = \vec{0} \quad \text{(GEP)}$$

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Step 2 Multiply L^{-1} from the left to the (GEP), introduce $A \overset{\text{def}}{=} L^{-1}KL^{-T}$ and $\vec{\psi}_i \overset{\text{def}}{=} L^T\vec{S}_i$, and arrive at $\left(A - \omega_i^2\, 1\right)\vec{\psi_i} = \vec{0}$ with $A = A'$

→ Since A is symmetric, $\{\vec{\psi}_i\}_{i\in\{1,2,\dots,DOF\}}$ form an orthonormal basis of the nodal space.

Again I have used this OMT beam and done the numerics with this element or matrices of it and indeed I can do this decomposition hence there is also an inverse and perfectly fine. All right so then the second step is to introduce two new quantities so one is this matrix ‘A’. It's defined like this. Stiffness matrix here in-between sandwiched from left and right hence this vector psi which is basically the the shape vector 'S' multiplied by this transpose. And you can do it on the piece of paper. It's done quickly. You just multiply 'L' to the minus one from the left to the GEP. Introduce this definitions and then you arrive at this equation and this equation here is exactly what we wanted. It's a normal eigenvalue problem. And on top of it we have a matrix 'A' which indeed is symmetric because 'K' is symmetric. And then we know from all we have learned from linear algebra and quantum mechanics and so on that if or since 'A' is symmetric, we know that its eigenvectors form an orthonormal basis of the nodal space so the vector space of all the degrees of freedom. Yes and we actually normalize this so these vectors are orthogonal and we normalize them to one for what follows.

Notes

Summary

15m 06s

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→ Since A is symmetric, $\{\vec{\psi}_i\}_{i\in\{1,2,\dots,DOF\}}$ form an orthonormal basis of the nodal space.

Step 3 From linear indepence of $\{\vec{\psi}_i\}_{i\in\{1,2,\dots,DOF\}}$

→ Linear indepence of $\{\vec{S}_i\}_{i\in\{1,2,\dots,DOF\}}$

→ $\{\vec{S}_i\}_{i\in\{1,2,\dots,DOF\}}$ form a basis of the nodal space (not necessarily orthogonal).

Proof: $0 = \sum_{i=1}^{DOF} a_i \vec{\psi_i} = \mathrm{L}^{\mathrm{T}} \sum_{i=1}^{DOF} a_i \vec{S_i}$ if and only if $a_i = 0\ \forall\, i$.

All right and in the third step we can argue that since the size form a basis they are linear independent and here is the proof of the next conclusion, is that since these are linear independent, we can write down this if and only if all the a's in this are linear so superposition as zero and then we can introduce for psi we can introduce this definition, ‘L’ transpose times shape function. And now since we know that 'L' is invertible, we can move it to the other side so it will goes away. It goes to zero. And then we are left with this superposition of the shape functions which are zero and 'ai' are all zero. So it means that the shape functions form linearly independent and, therefore, they form a basis of the nodal space. However, now the shape functions are not necessarily orthogonal.

Notes

Summary

16m 43s

DECOUPLED HARMONIC OSCILLATORS

Reminder: From elasticity theory we end up with the EOM in the FEM:

$$M\,\ddot{\vec{u}} + D\,\dot{\vec{u}} + K\,\vec{u} = \vec{F}$$

Since $\{\vec{S}_i\}_{i\in\{1,2,\dots,DOF\}}$ form a basis of the nodal space, we can always write

$$\vec{u}(t) = \sum_{i=1}^{DOF} q_i(t)\vec{S}_i \equiv U\vec{q}(t)$$

Modal amplitude

All right. How do we now do the next magical step? How do we decouple the equations of motion into harmonic oscillators? Well, let us remind ourselves this was the equation of motion in the finite element notation. And now since the shape functions form a basis of the nodal space, we can always write this thing here and what this thing tells you is that every arbitrary deflection 'u' which is a large vector as a function of time, you can write as a linear superposition of modes and the mode shapes they don't contain the time dependence and only the, you know, scalar valued quantities 'qi' which are the so-called modal amplitudes contain the time dependence. And I want to stress the fact that this modal representation, this linear superposition of modes always works because 'Si' is the basis or forms the basis so you can always do this. Sometimes people confuse the fact that this is a linear superposition with issues in non-linear equations. So even if the matrix ‘K’ is non-linear as a function of 'u', or the forces are non-linear as a function of 'u' even then, even if you have non-linear problem then you can write this linear superposition.

Notes

Summary

17m 43s

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(Modal amplitude)

$$\vec{q}(t) = \{q_1(t), q_2(t), \dots, q_{DOF}(t)\}$$

$$U = \{\vec{S}_1, \vec{S}_2, \dots, \vec{S}_{DOF}\}$$

Rewrite the nodal EOM into the **modal EOM**:

$$\underbrace{U^T M U}_{=M_m}\ddot{\vec{q}} + \underbrace{U^T D U}_{=D_m}\dot{\vec{q}} + \underbrace{U^T K U}_{=K_m}\vec{q} = \underbrace{U^T\vec{F}}_{=\vec{F}_m}$$

Diagonalization of M and K:

$$\delta_{i,j} = \vec{\psi}_i^T \cdot \vec{\psi}_j = (L^T\,\vec{r}_i)^T \cdot (L^T\,\vec{r}_j) = \vec{r}_i^T \cdot (L\,L^T\,\vec{r}_j) = \vec{r}_i^T \cdot (M\,\vec{r}_j) = (U^T M\,U)_{i,j} = (M_m)_{i,j}$$

It's convenient to rewrite this linear superposition into this form with this modal amplitude vector 'q' which just contains the modal amplitudes like this and this matrix ‘U’ which contains the shape vectors in its columns. In the next step, we write our equation of motion and then so the nodal one into so-called modal equation of motion. Like this we just introduce the definition of 'U' in here and take advantage of the fact that the time derivative only acts on 'q' and then we multiply the transpose of 'U' from the left. And then we have this modal matrices. Here the modal mass matrix, the modal stiffness matrix and the modal forces. And why do we do this transformation in such a way? Well, we do it because in this way we can show that our modal matrices become diagonal so, in fact, the modal mass matrix here in red depicted. It is proportional to the ident... so it is, in fact, equal to the identity matrix that's what these two red things are saying. The proof in black is in-between. The same thing holds true for the stiffness matrix so if the modal stiffness matrix becomes diagonal and its diagonal elements just have the omega squared in it and again the proof is in black here.

Notes

Summary

19m 14s

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Modal EOM as a bunch of **decoupled harmonic oscillators**:

(with small disclaimer for the damping):

$$\ddot{q}_m + \frac{\omega_m}{Q_m}\dot{q}_m + \omega_m^2\,q_m = F_m$$
$$\forall m \in \{1, 2, \dots, DOF\}$$

You can do it. You can pause the video and do it more slowly. Let's go quickly through the proof. So we use the definition of the psi so these are orthonormal so they are orthogonal and normalized to one. So if this equality is true and down here we use the fact that the omegas are the eigenvalues of 'A' with these eigenvectors. Okay and then we just start plugging back in definition so for psi we put the 'L' transpose times 'o r', Sorry that's it. I put this. This 'r' should be an 'S'. I'm sorry so this small 'r' should be the 'S' here. Okay. Apart from this typo that's one 'm'. All right then and in this next thing equality here we use all we know from the things we have learned from the bracket notation in quantum mechanics so we bring this over here then it loses its transpose. And here as well. Then we realize that this and this is nothing but the mass and the stiffness matrix. And from there we conclude the proof. Again just pause the video and then do it slowly. All right. So and then the magic happens so we have managed to rewrite the nodal equations of motion into modal equations of motions which is nothing but a bunch of harmonic oscillators. Of course, there's a small disclaimer for the damping.

Notes

Summary

20m 48s

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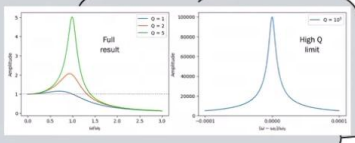
Reminder: Lecture by Gary Steele on the damped harmonic oscillator



Typically the damping can be off-diagonal depending on how you model it but for now we just assume it's a very good approximation to make it diagonal which, in fact, it is very often the case. Almost always, let's say, and we can introduce this very important quantity, the so-called quality factor for every mode and basically this equation we can solve analytically. So let us remind ourselves of what was the analytic solution of the harmonic oscillator and its most important feature. So we have learned that in oscillatory systems where you drive your system with a single frequency with this oscillatory force with a single frequency then your system reacts according to this transfer function so it means that the amplitude of oscillation will be the force divided by stiffness multiplied by this transfer function. So whenever the external excitation frequency matches your resonance frequency, you will have what is known as the q-factor enhancement here.

Notes

Summary

22m 25s

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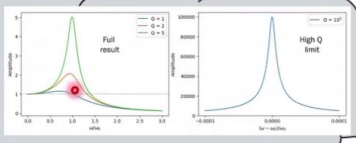


So in typical studies of High-Q systems we have the High-Q because we want to introduce only small forces and that's what happened and what's happening so we introduce small forces and almost every mode is just deflected by, well, it's static deflection without any q-factor enhancement, but if you hit one of the modes, you know, by hit I mean if the resonance frequency of the drive equals the resonance frequency of the mode, then this mode will oscillate which with an amplitude which is enlarged by the quality factor so and then in High-Q systems we are talking about ten to the four or even ten to the nine.

Notes

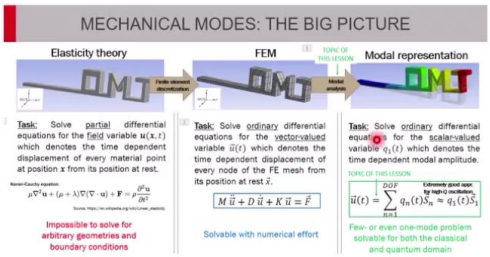
Summary

23m 40s

CONCLUSION / SUMMARY

- Summary of key points
- The modal analysis is a generalized Eigenvalue problem (GEP).
- The Eigenvectors (Eigenvalues) of the GEP give the mode shapes (resonance frequencies).
- The global element matrices are symmetric and positive definite.
- The mechanical modes form a basis of the nodal space.
- The linear equations decouple completely into harmonic oscillators.

- Conclusion
- For high-Q oscillatory systems it is possible to transform the complex equations of motion of elasticity theory to simple harmonic oscillators. FEM enables the transformation for any geometry via a modal analysis.



So before I present the important conclusion, summarize quickly so we have we have learned about the generalized eigenvalue value problem which is the description of... which is the foundation of the modal analysis and we have seen that the eigenvectors and eigenvalues of this GEP just correspond to the mode shapes and the resonance frequencies. Now we have learned some important properties about the global element matrices like symmetricity and positivity. Most importantly we have understood that mechanical modes form a basis for the nodal space. And by rewriting the equations of motions in the nodal space into this modal basis we can find simple equations where we just have harmonic oscillators which are all decoupled in the linear regime. So now let me conclude with what is fascinating here. So this erratic construct is fascinating because we have managed to go from a very complex elasticity theory to a very simple representation of the simple harmonic oscillator just by doing a modal analysis to find the modal representation and by doing all these mathematical tricks. So yeah, this is a very fascinating concept and it's enabling to do quantum mechanics with mechanical modes.

Notes

Summary

24m 31s

CONCLUSION / SUMMARY

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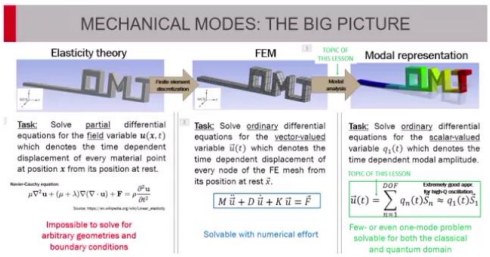

But I also want to stress again the fact that the final element is what allows us to do this for, you know, to do this fascinating modal transformation for any arbitrary geometry and we can do it by what is known as the modal analysis.

Notes

Summary

26m 05s