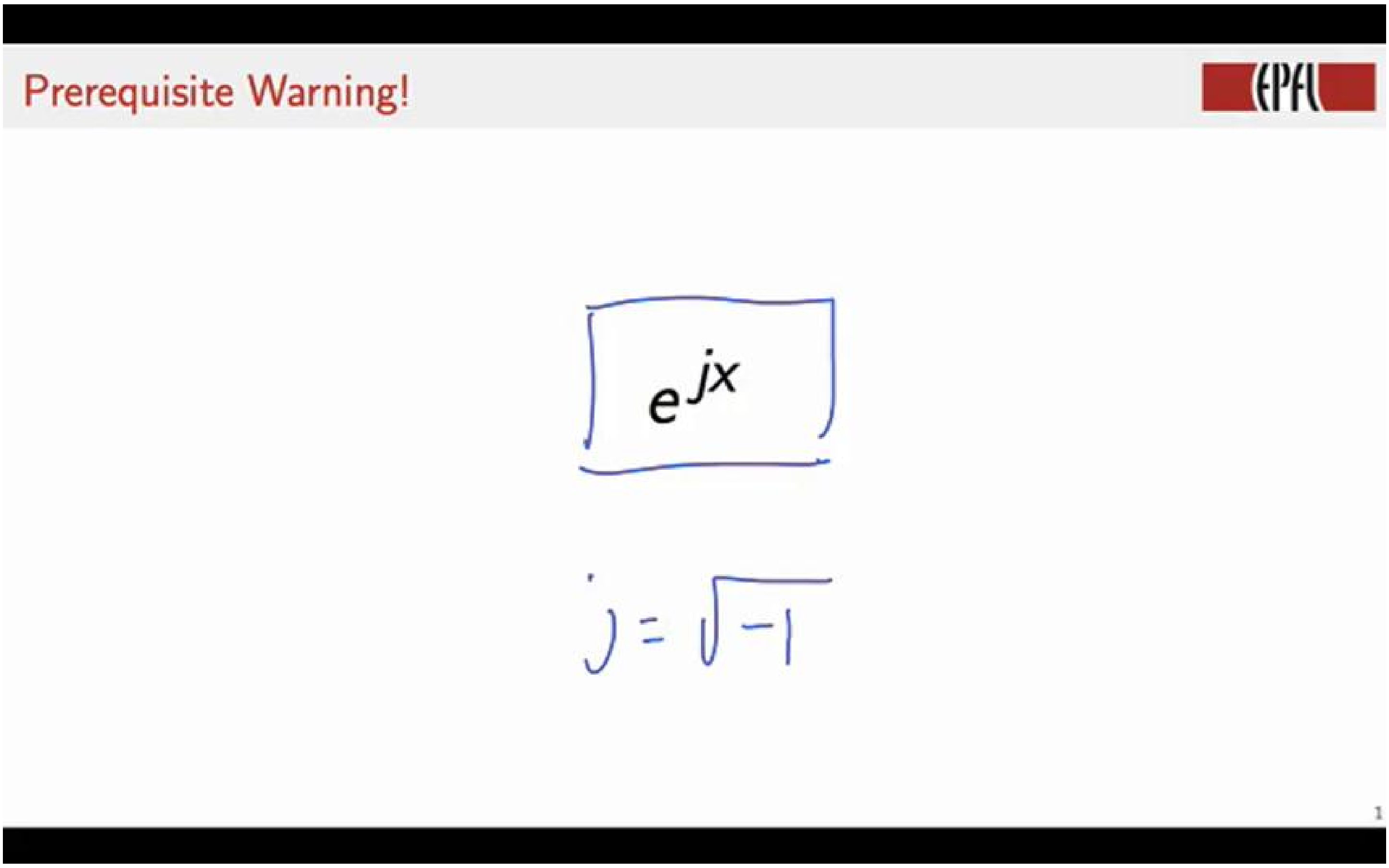
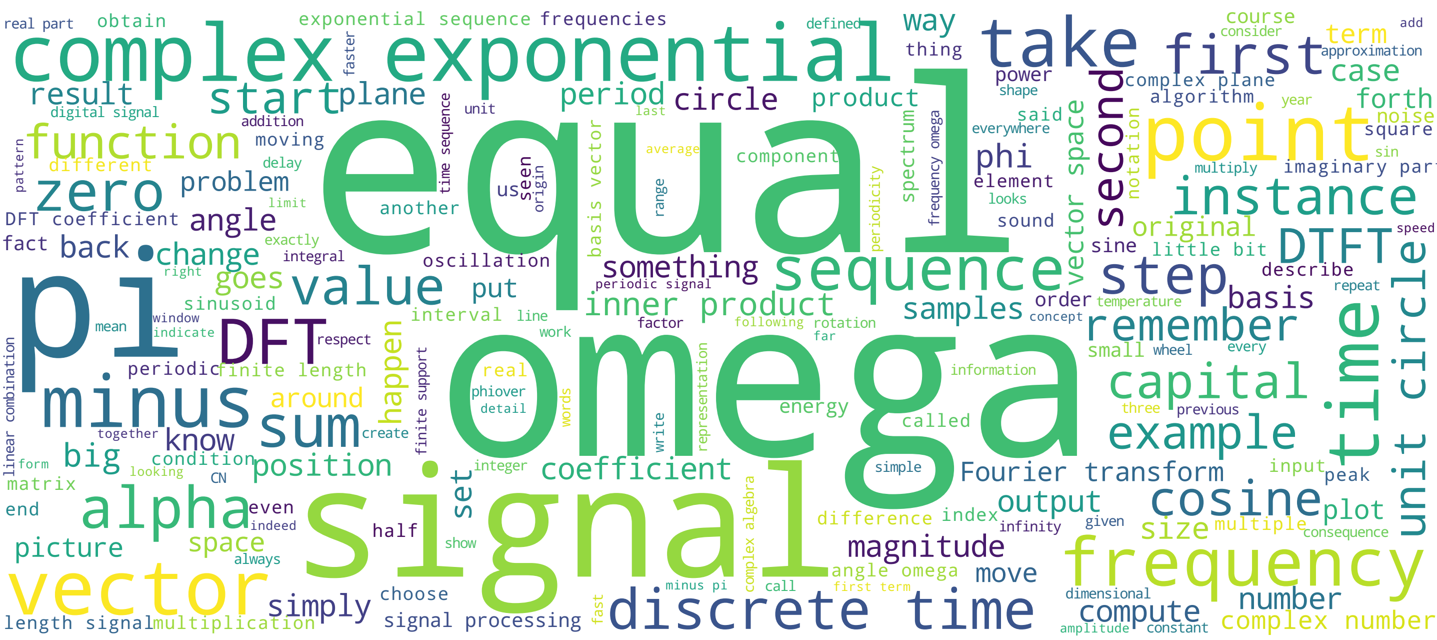




Search MOOC

Video

EPFL

Oscillations are everywhere!

Let's start with a little warning. To make the most of this lecture, you have to be familiar with the notation e to the power of JX, where j is the imaginary unit. If you're not, perhaps a little review of your complex algebra textbook is in order. We have said that in nature oscillation are everywhere from the heartbeat to engines to waves to musical instruments.

Notes

Summary

0m 00s

Oscillations are everywhere

- sustainable dynamic systems exhibit oscillatory behavior
- intuitively: things that don't move in circles can't last:
  - bombs
  - rockets
  - human beings...

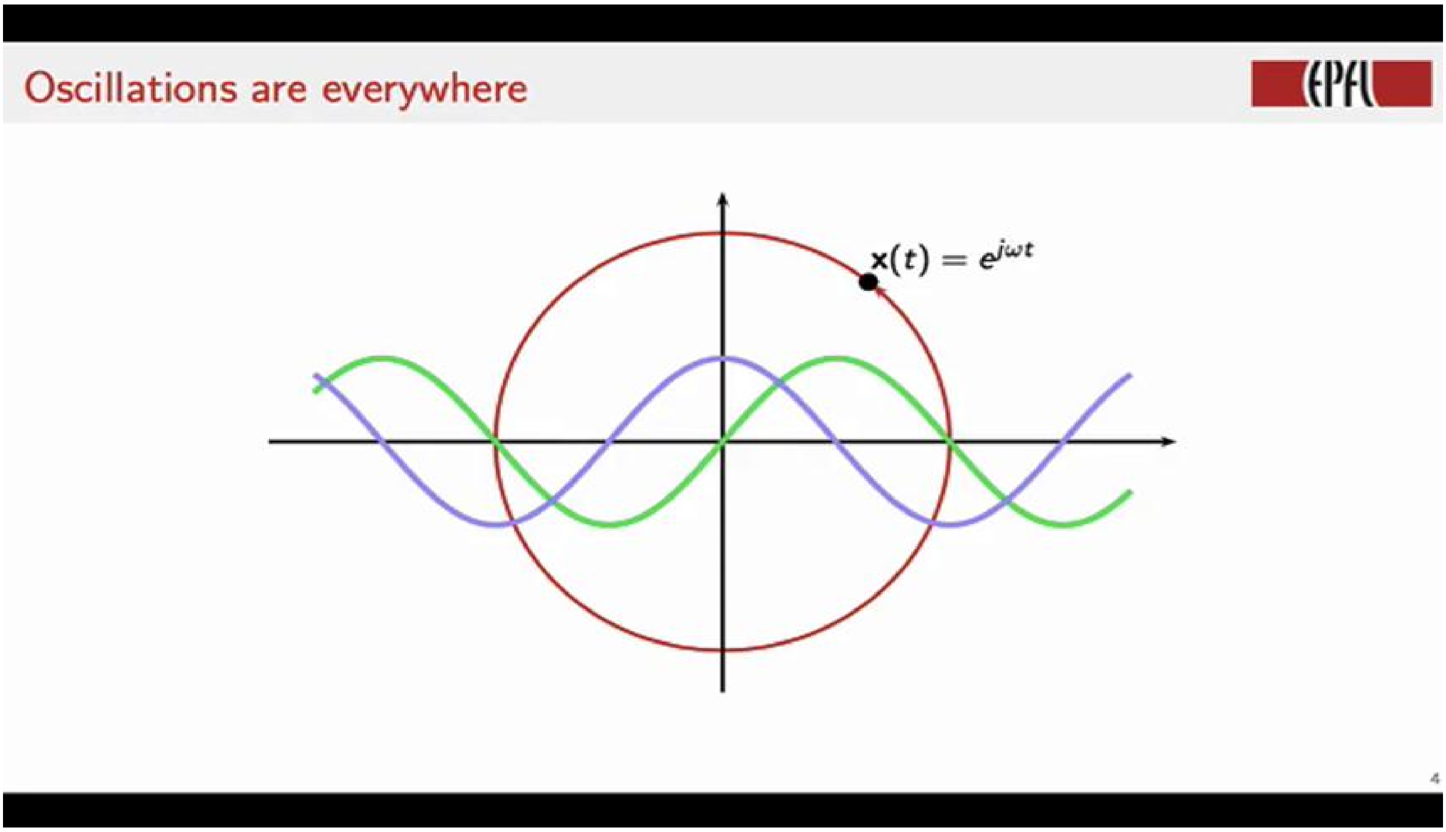
3

The fact is, the sustainable dynamic systems must exhibit an oscillatory behaviour. Somehow you have to have something that comes back to its initial state, because things that don't move in circle cannot really last for a long time. So for instance, a bomb has a lot of energy, but the release of energy is unidirectional and so not sustainable. Rockets same way, and unfortunately, human beings as well. Although many parts inside our bodies do work in circular patterns.

Notes

Summary

0m 24s



An oscillation is the product of a rotation. You take a point on the plane, you make it go around in a circle, you create an oscillatory behaviour. If you put a coordinate systems around the centre of the oscillation you can describe the vertical and horizontal displacement of this point as it goes around the circle with a cosine and a sine function. So you always have two trigonometric functions that work together to describe the position of something that turns around in circles. Perhaps it's easier to use a complex reference system centred on the origin of the oscillation and use a complex exponential to describe the position of the point on the plane. So we will say that the position of the point is described by a complex exponential e to the j omega t, where omega is the rotation frequency. So here we are in standard algebraic terms where t is a real variable that indicates time.

Notes

Summary

0m 53s

The discrete-time oscillatory heartbeat

Ingredients:

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A

$$
\begin{aligned}
x[n] &= A\underbrace{e^{j(\omega n+\phi)}} \\
&= A[\cos(\omega n+\phi)+j\sin(\omega n+\phi)]
\end{aligned}
$$

5

In discrete time, things are a little bit different. The discrete time oscillatory heartbeat has three fundamental ingredients a frequency omega, where the units are radiance and not radiance over seconds because our "time variable is a dimensional, it has an initial phase phi and an amplitude A," and the discrete time sequence is a times e to the j omega n plus phi. We can use Euler's formula to decompose this into a real and imaginary part, and so we have a times cosine of omega n plus phi plus j sin of omega plus phi.

Notes

Summary

1m 48s

Why complex exponentials?

- we can use complex numbers in digital systems, so why not?
- it makes sense: every sinusoid can always be written as a sum of sine and cosine
- math is simpler: trigonometry becomes algebra

6

Why use complex exponentials instead of explicit sin and cosine functions? Well, first of all, we're all building our own digital signal processing world. We said we can use complex numbers in digital systems, so why not? But it makes sense, because every circular motion is always a sine and cosine intertwined and Euler's formula compactly brings them together in one single function. It makes math simpler because trigonometry becomes simple algebra.

Notes

Summary

2m 29s

The advantages of complex exponentials

Example: change the phase of a cosine the "old-school" way

$$\cos(\omega n + \phi) = a\cos(\omega n) - b\sin(\omega n), \qquad a = \cos\phi,\ b = \sin\phi$$

- we have to remember complex trigonometric formulas
- we have to carry more terms in our equations

7

For instance, let's try and change the phase of a cosine in the old school trigonometric way. So if we have cosine of omega n plus phi, then we have to remember the trigonometric identities for the sum of angles and we know that the result will be something like A times the cosine of the first term plus b times the sin of the first term. But do we remember if it's plus or minus? Or do we remember the values for a and b and so on and so forth? So it's kind of a mess and very much error prone.

Notes

Summary

3m 01s

The advantages of complex exponentials

Example: change the phase of a pure cosine with complex exponentials

$$\cos(\omega n + \phi) = \mathrm{Re}\{e^{j(\omega n+\phi)}\} = \mathrm{Re}\{e^{j\omega n}\, e^{j\phi}\}$$

- sine and cosine "live" together
- phase shift is simple multiplication
- notation is simpler

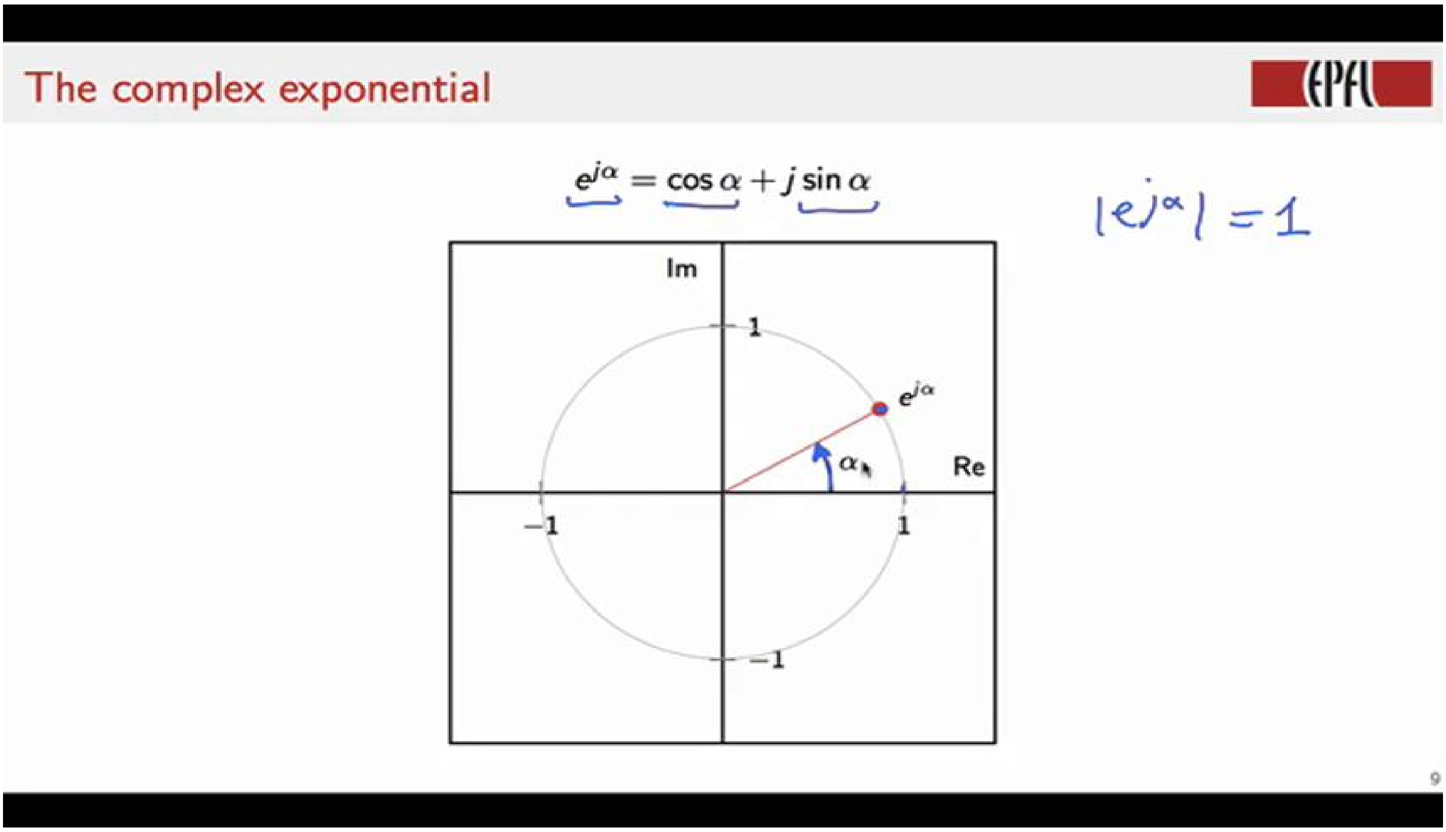
8

Conversely, if we try and change the phase using complex algebra, we have the cosine of omega n plus phi is simply the real part of e to the j omega n plus phi. Now we can decompose this complex exponentia l into the product of two complex exponentials. We have no problem performing the multiplication of two complex numbers when we separate the real imaginary parts and then we get the result without any mnemonic effort. So sin and cosines live together, phase shift becomes simple multiplication and the notation is way simpler. Okay, so the complex exponential is our friend, let's get to know it a little bit better.

Notes

Summary

3m 31s



The complex number e to the j alpha is a complex number with real part cosine of alpha and imaginary part sin of alpha. In magnitude e to the j alpha is always equal to one. So the point e to the j alpha always lives on the unit circle. How we find this point? Well, we travel along the unit circle counterclockwise until we reach an angle equal to alpha where alpha is expressing radiance and that is the position of our complex exponential.

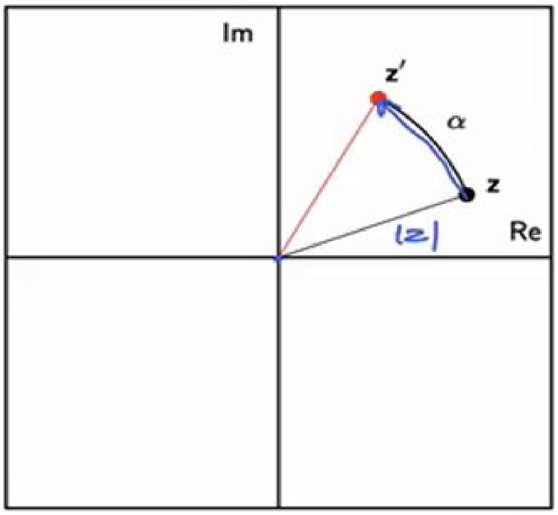
Notes

Summary

4m 12s

The complex exponential

rotation: $z' = z\,e^{j\alpha}$



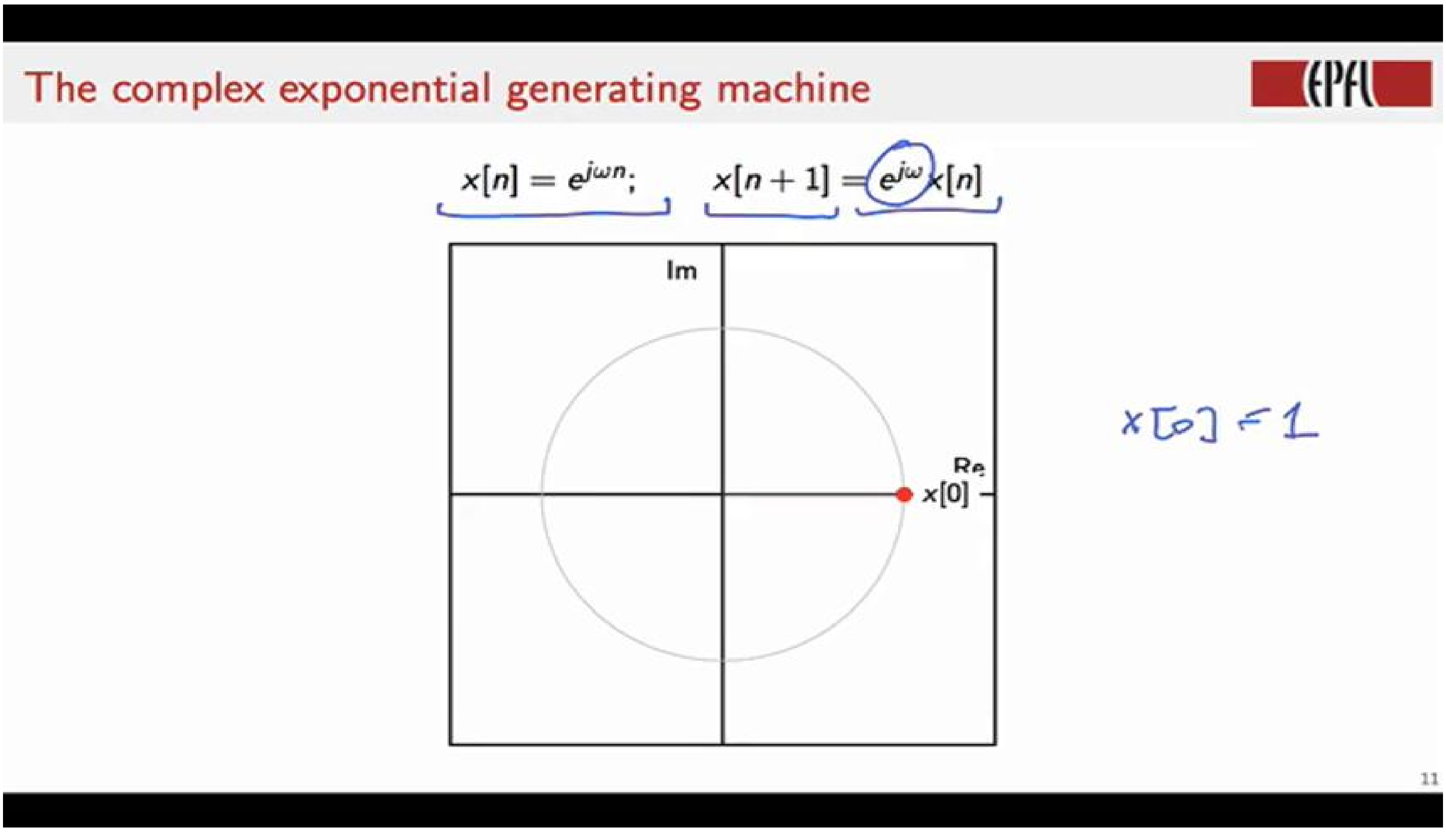
10

If we have any point on the complex plane and we multiply this point by e to the j alpha, we are rotating this point counterclockwise by an angle alpha on a circle centred in the origin and with radius equal to the magnitude of z. So we can use multiplication by a complex exponential to rotate any point on the complex plane.

Notes

Summary

4m 41s

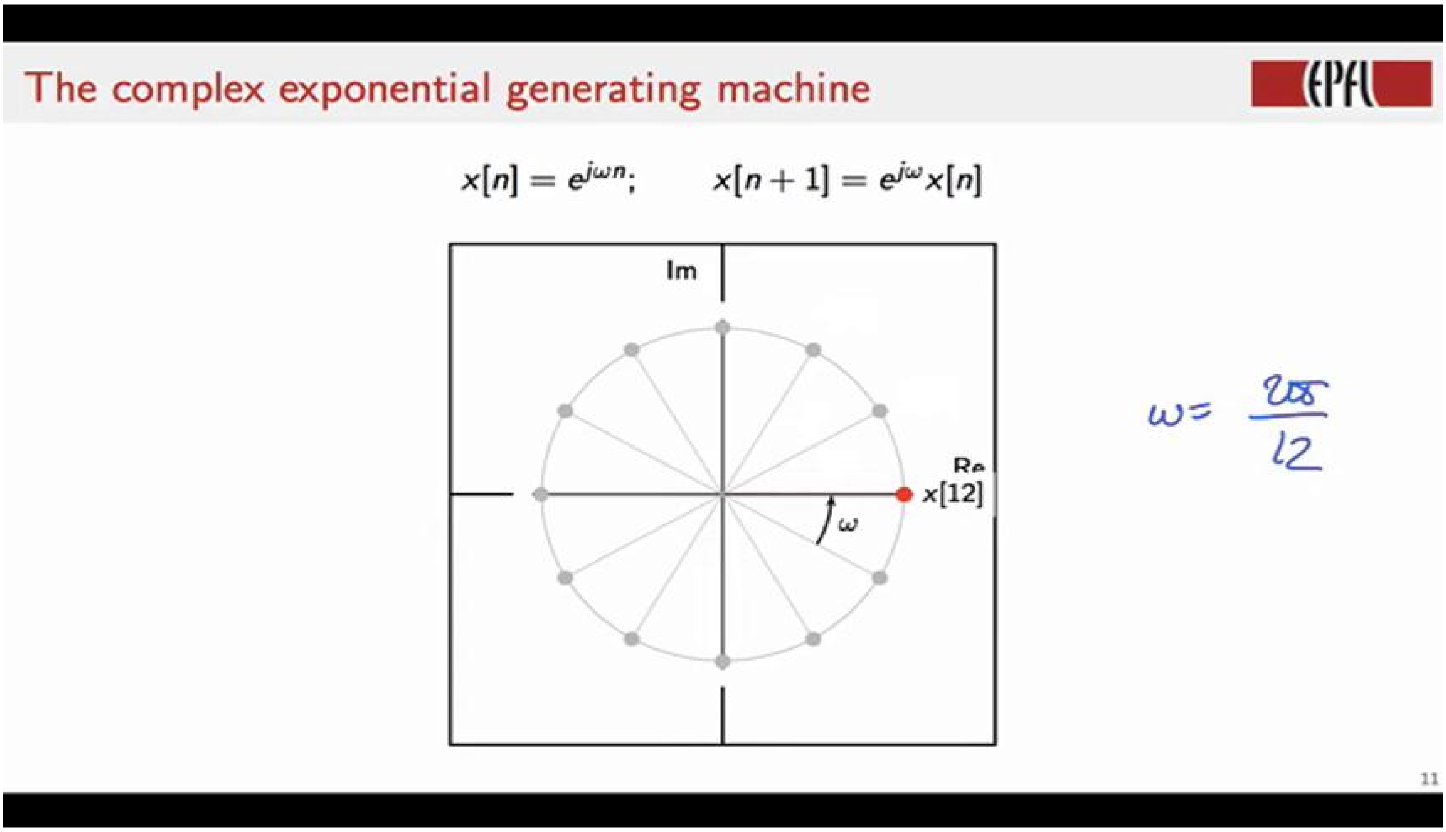


This is actually at the heart of the complex exponential generating machine that will create all these complex exponential signals that we will use in the future. So a sequence like x of n equal to e to the j omega n, where omega is the frequency of the complex exponential sequence, can be obtained at each step by multiplying the previous point in the sequence by e to the j omega. So this is how we recursively generate a complex exponential sequence. If we plot the sequence of points on the plane, it would all start at one location. So let's assume that x is zero is equal to one and so the next step will be one times e to the j omega and we will have moved here by an angle omega counterclockwise around the unit circle.

Notes

Summary

5m 10s



Then the next step, we will move by another angle omega and so on and so forth. That's how we generate a complex exponential signal. Here in this example you see that after 12 steps we go back to the original point. So clearly we can say that omega is equal to two phidivided by 12, because in 12 steps we have gone back to the starting point. After going around the circle, the sequence repeats.

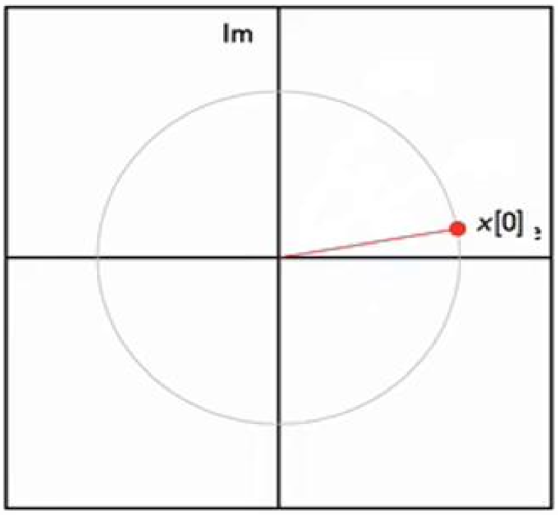
Notes

Summary

6m 01s

Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega} x[n], \quad x[0] = e^{j\phi}$$



12

Of course, the initial point need not be one. We can start at any arbitrary point on the complex plane and the complex exponential sequence will proceed exactly in the same way where we advance counterclockwise by an angle omega at each step.

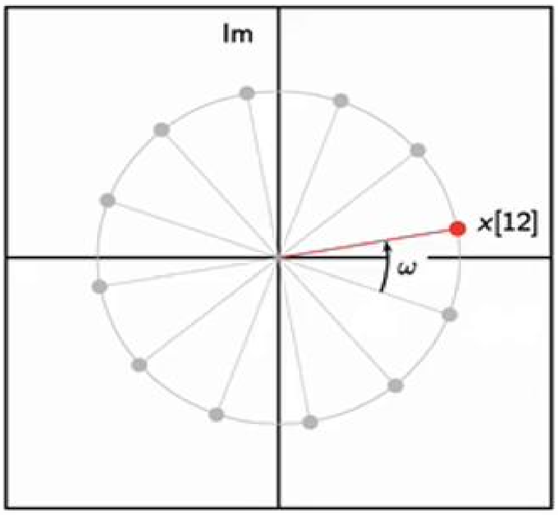
Notes

Summary

6m 29s

Initial phase

$$x[n] = e^{j(\omega n + \phi)}; \qquad x[n+1] = e^{j\omega}x[n], \quad x[0] = e^{j\phi}$$



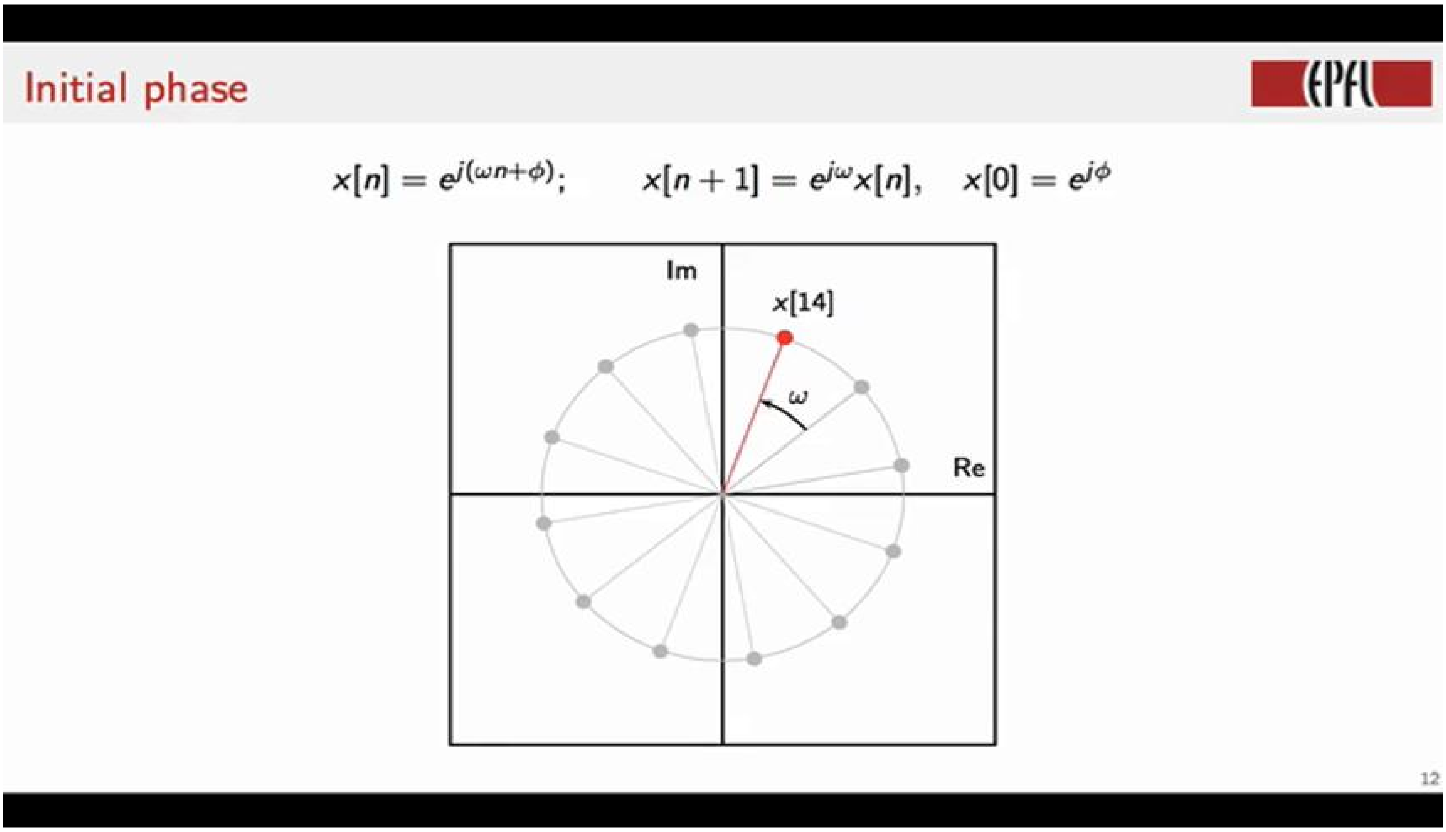
12

Here, for instance, we keep the same frequency of two phiover 12 and so we will repeat the pattern once again every 12 steps.

Notes

Summary

6m 45s

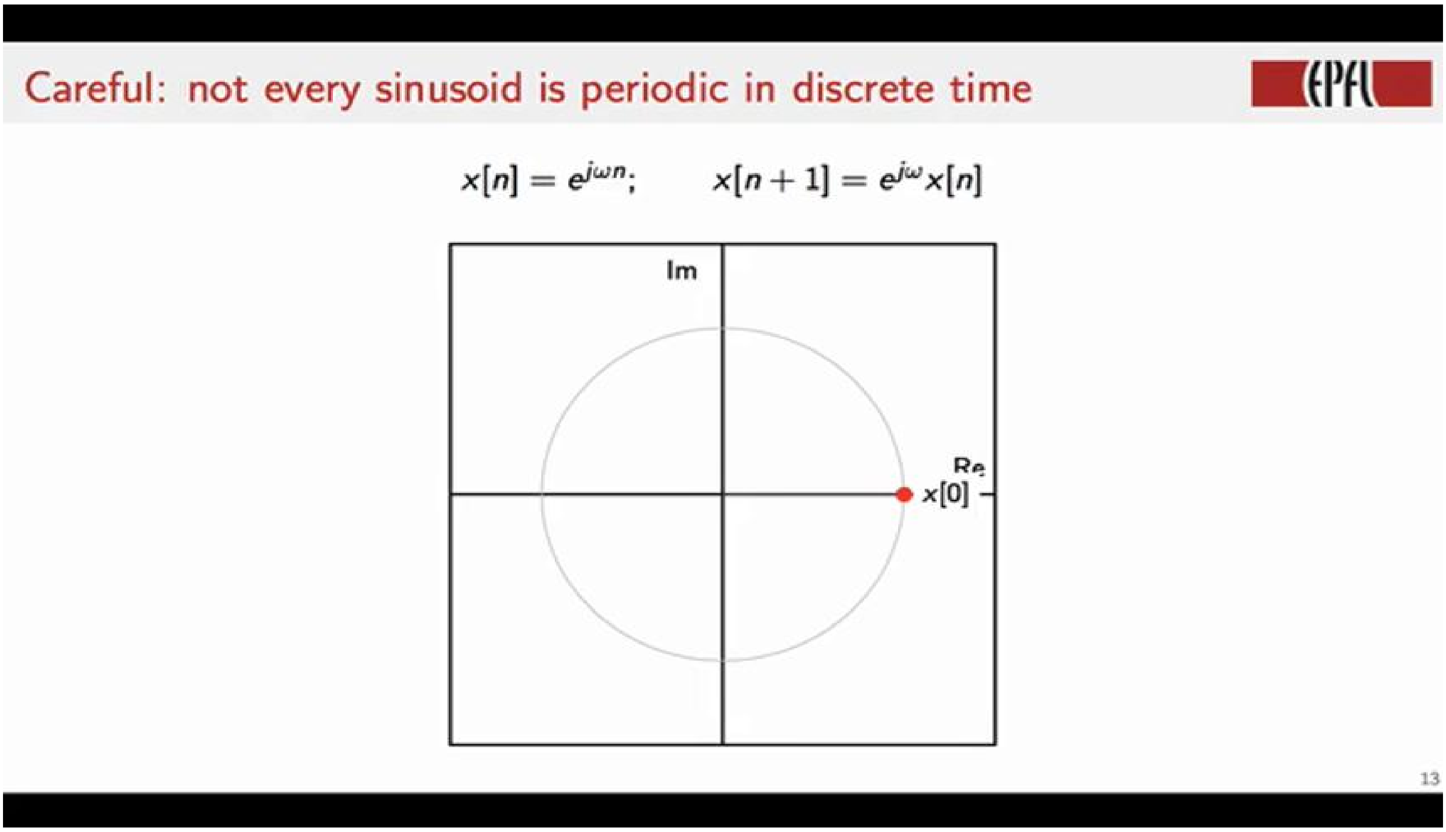


Now, so far there haven't been any surprises with respect to standard complex algebra. But here is one key fact about discrete time complex exponentials.

Notes

Summary

6m 53s

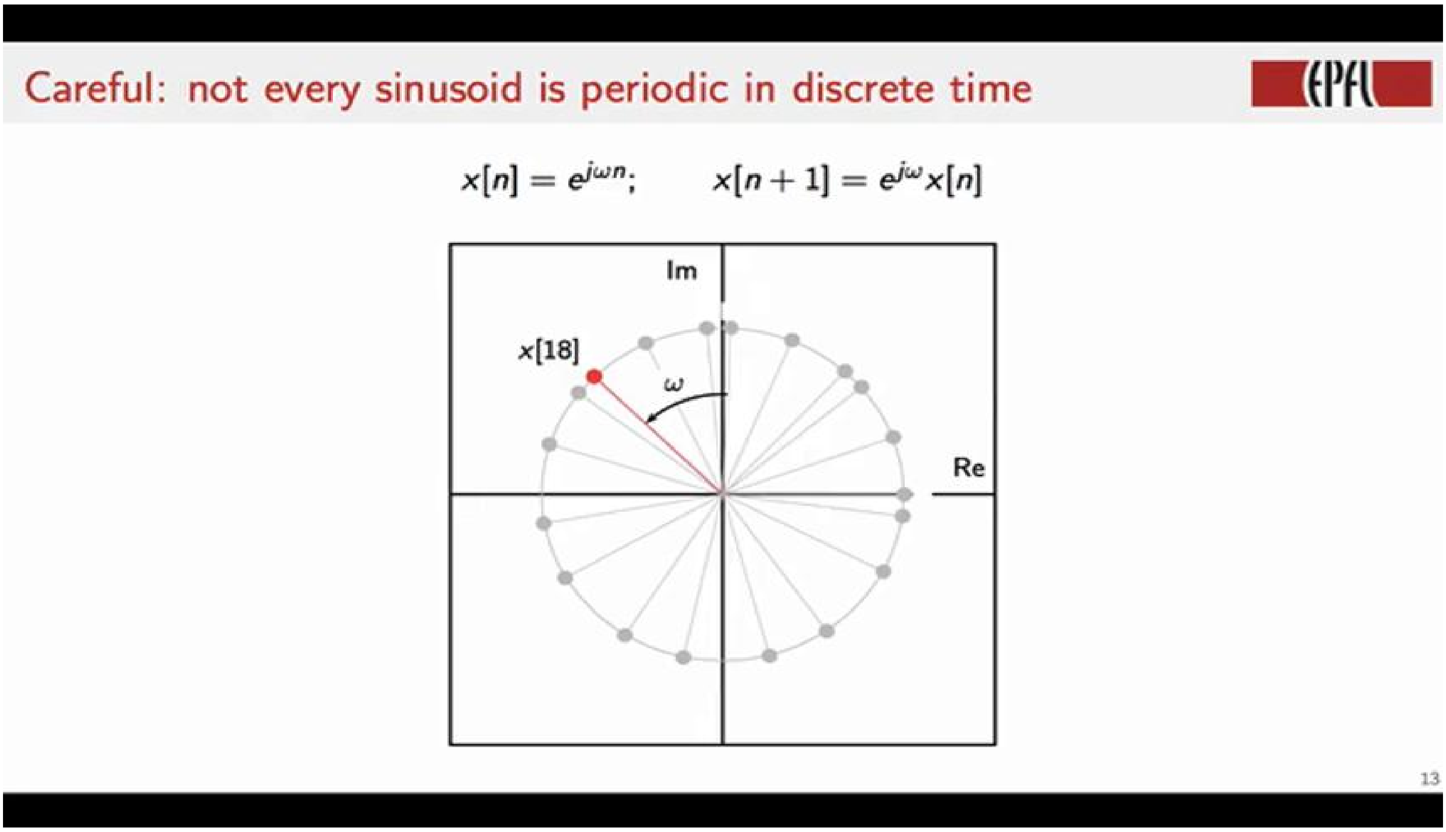


In discrete time, not every sinusoid is periodic.

Notes

Summary

7m 03s

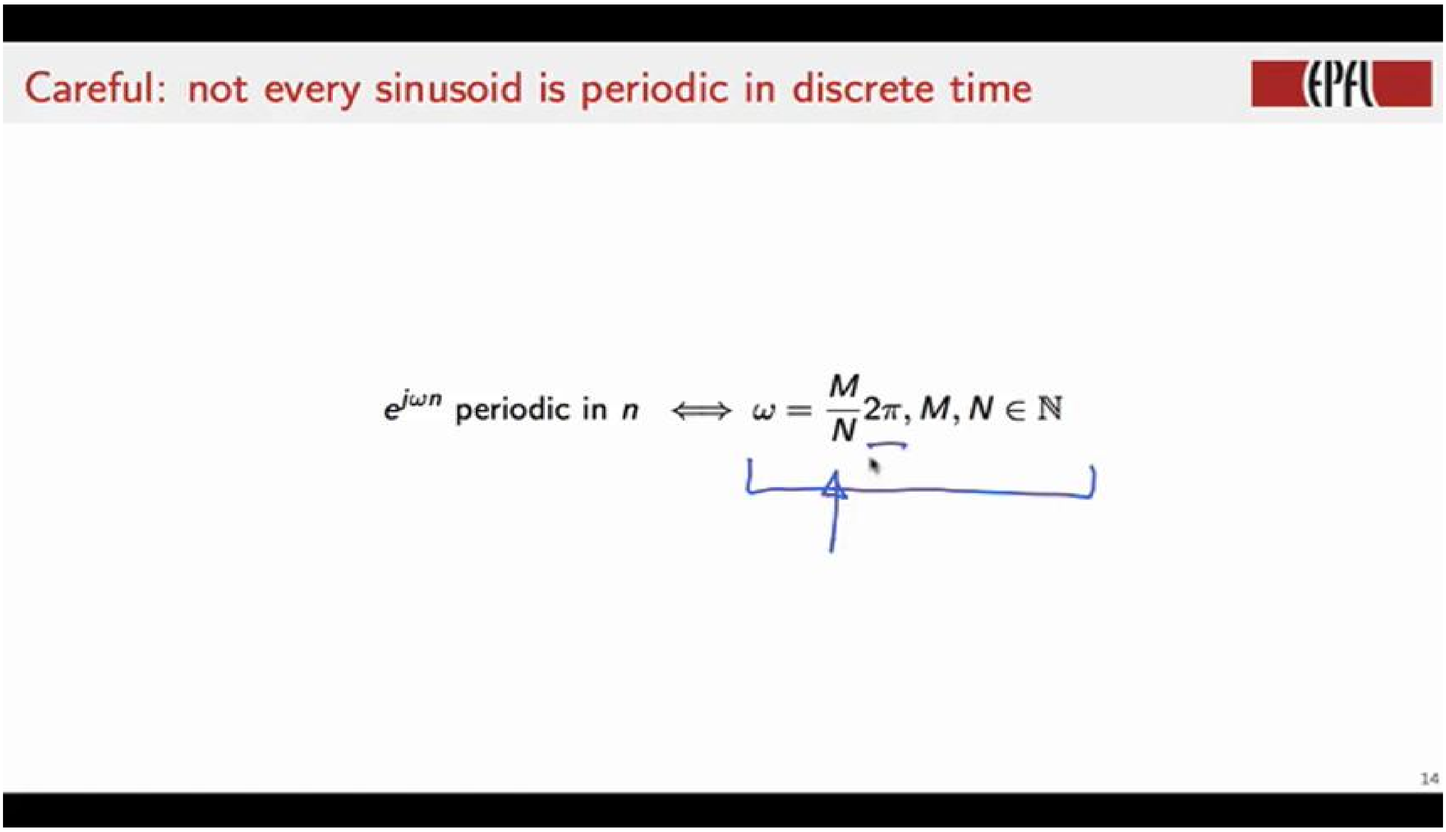


As a matter of fact, if you choose an arbitrary omega, there's a high probability that you will never end up on any of the previous points in your complex exponential sequence, as you can see in this picture here.

Notes

Summary

7m 07s

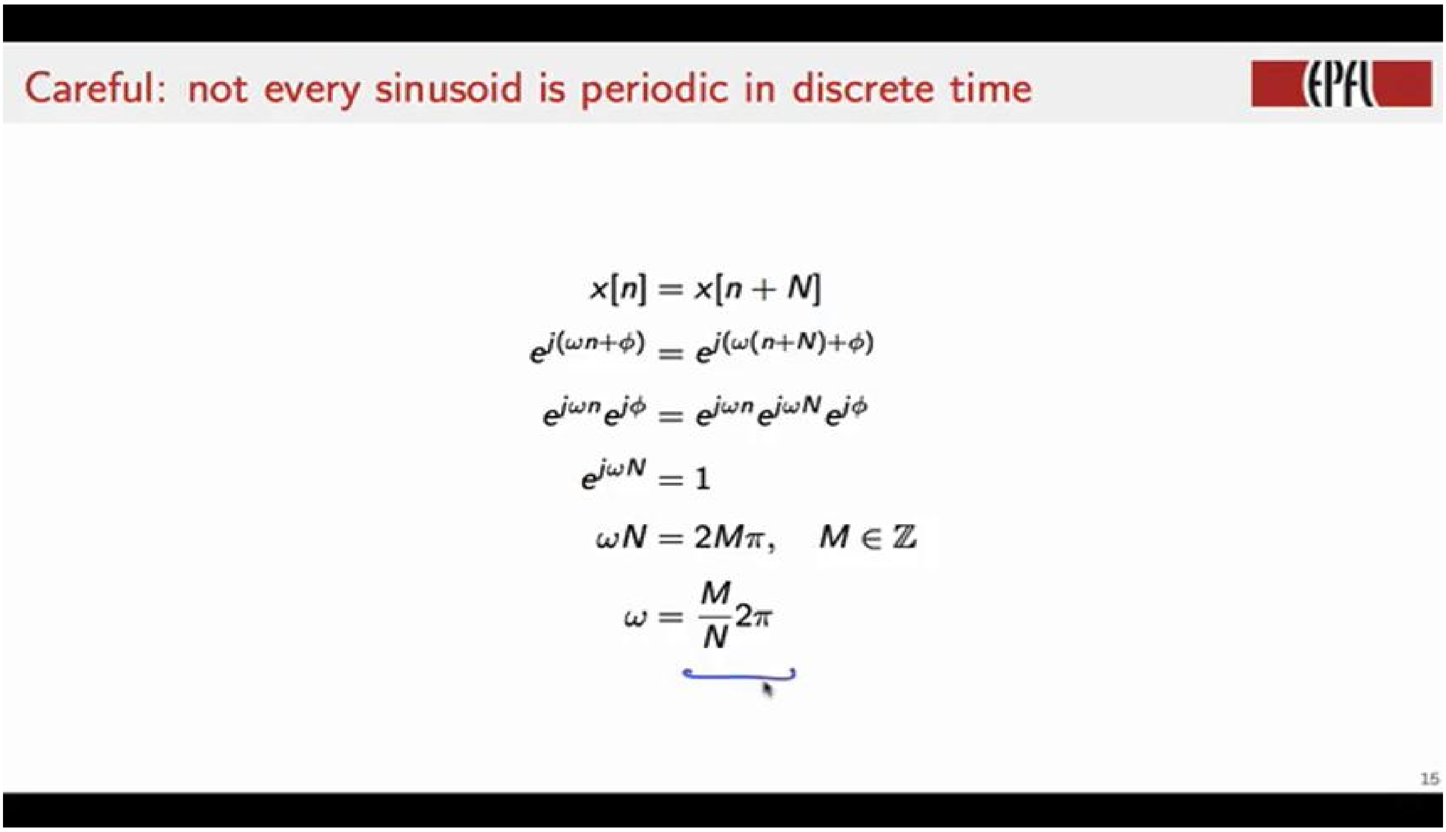


What happens in fact is that the complex exponential is periodic in n only if the frequency is a rational multiple of two pi. So it is of the form of a fraction a ratio of two integers times two pi.

Notes

Summary

7m 19s

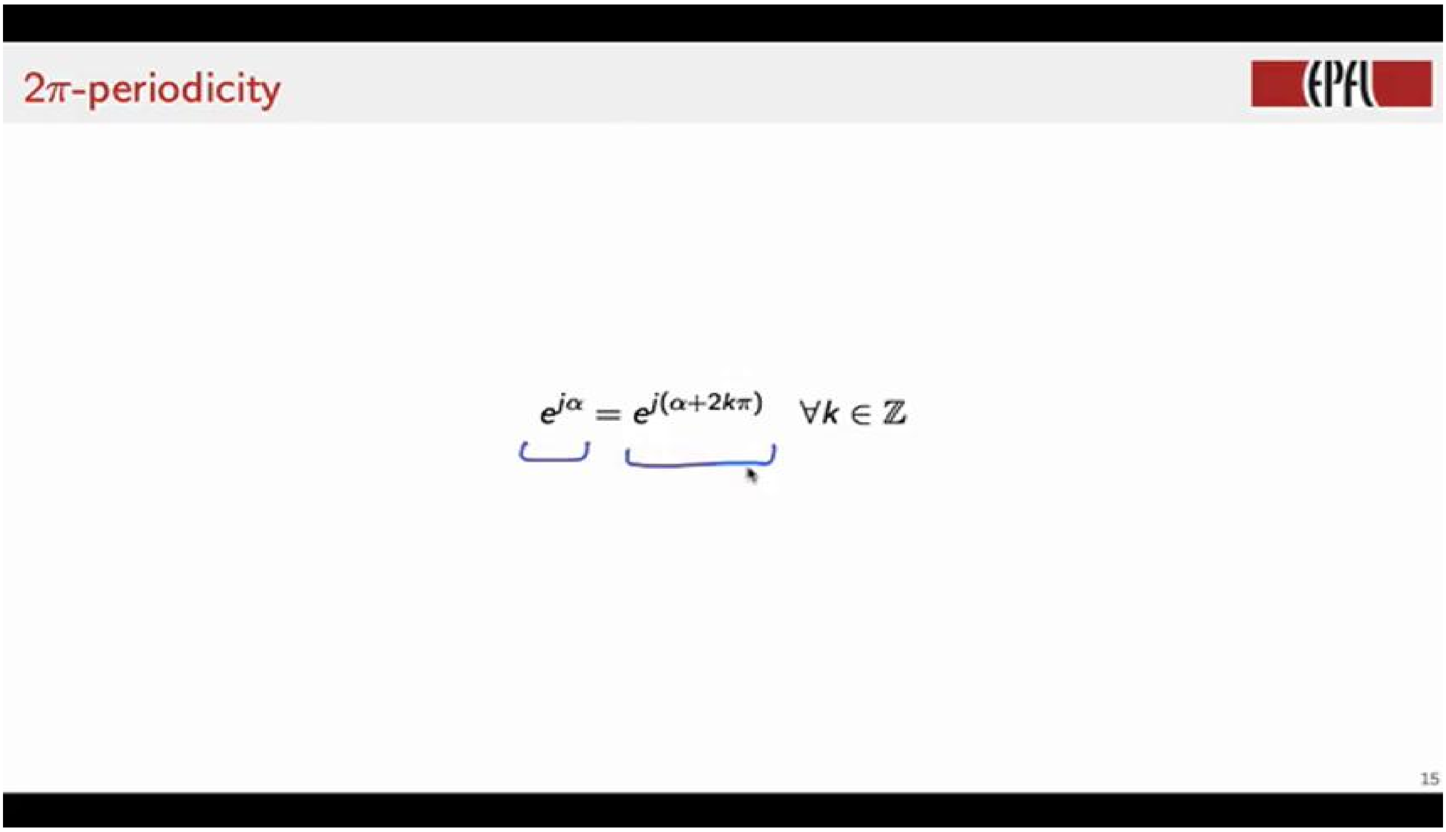


It is rather easy to see why it must be so. Remember, the condition for periodicity for a discrete time sequence is the following x of n equal to x of n plus capital n, where capital n is an integer indicating the period of the sequence. If we now use complex exponential sequences explicitly, we have e to the j omega n plus phi equal to e to the j omega n plus capital n plus phi. So omega and phi arbitrary frequencies and phase. We can now dismantle this equation by splitting the complex exponentials into products of individual terms like in this line here. If we now simplify the terms that are left and right, we obtain that the condition of periodicity is simply e to the j omega capital n equal to one. This poses a condition on omega, because this expression is true only when omega n is a multiple of two pi. That's when the complex exponential hits the unit on the complex plane. So we can rewrite this as omega n equal to an arbitrary multiple of two pi. Let's call that two capital m phiwith m an integer again. We obtain finally the condition on the frequency for the complex exponential which is what we said, a rational multiple of two pi.

Notes

Summary

7m 34s

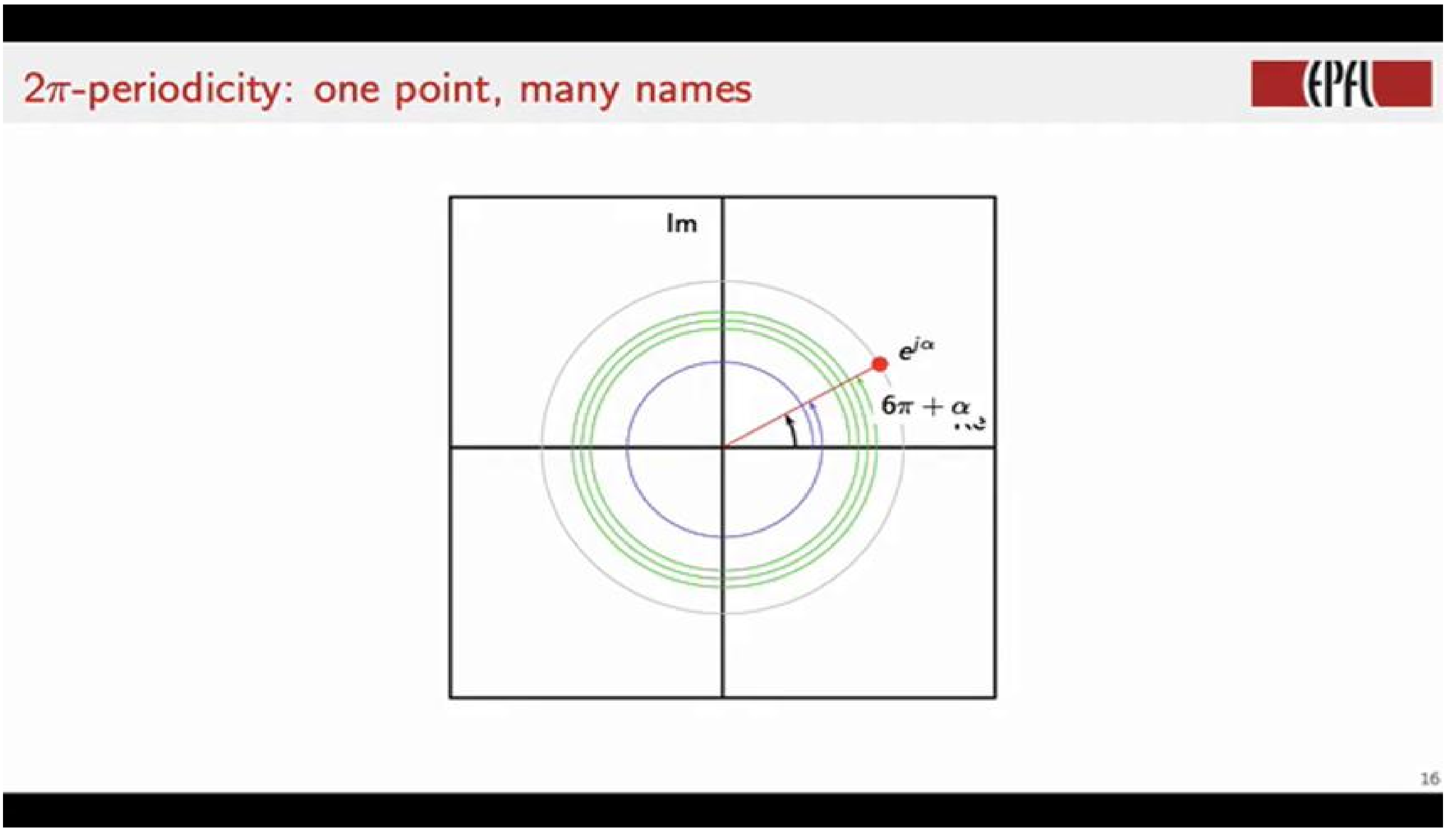


Now let's go back to an arbitrary point on the unit circle e to the j alpha. It is clear that we can add any multiple of two pi, positive or negative two alpha, and land exactly in the same position.

Notes

Summary

8m 57s



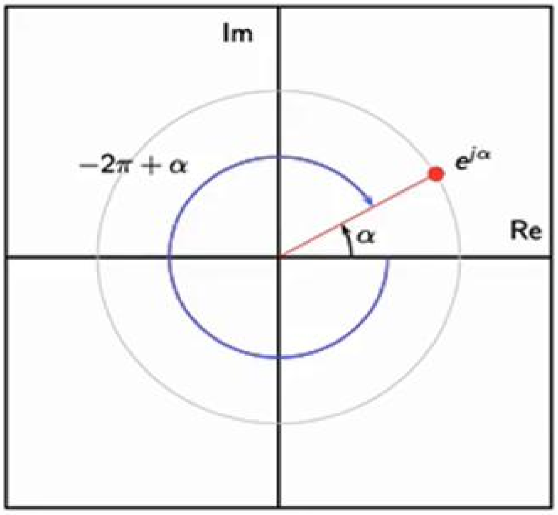
In other words, the same point on the unit circle can have many names. It could be e to the j alpha or e to the j two phiplus alpha, or say e to the j 6phi plus alpha.

Notes

Summary

9m 12s

One point, many names : ALIASING



17

Alternatively, it could be e to the j minus two phiplus alpha, and so on and so forth. So one point many names, this we call aliasing. The consequence of aliasing the consequence of this natural property of the complex exponential is that in discrete time it puts a limit on how fast we can go around the unit circle with a discrete time signal. You may have experienced the consequences of this upper limit on the speed that we can achieve in discrete time when looking, say, at an old western movie, and you have a stage coach that rolls into the frame of the movie.

Notes

Summary

9m 26s

How “fast” can we go?

2.2

33

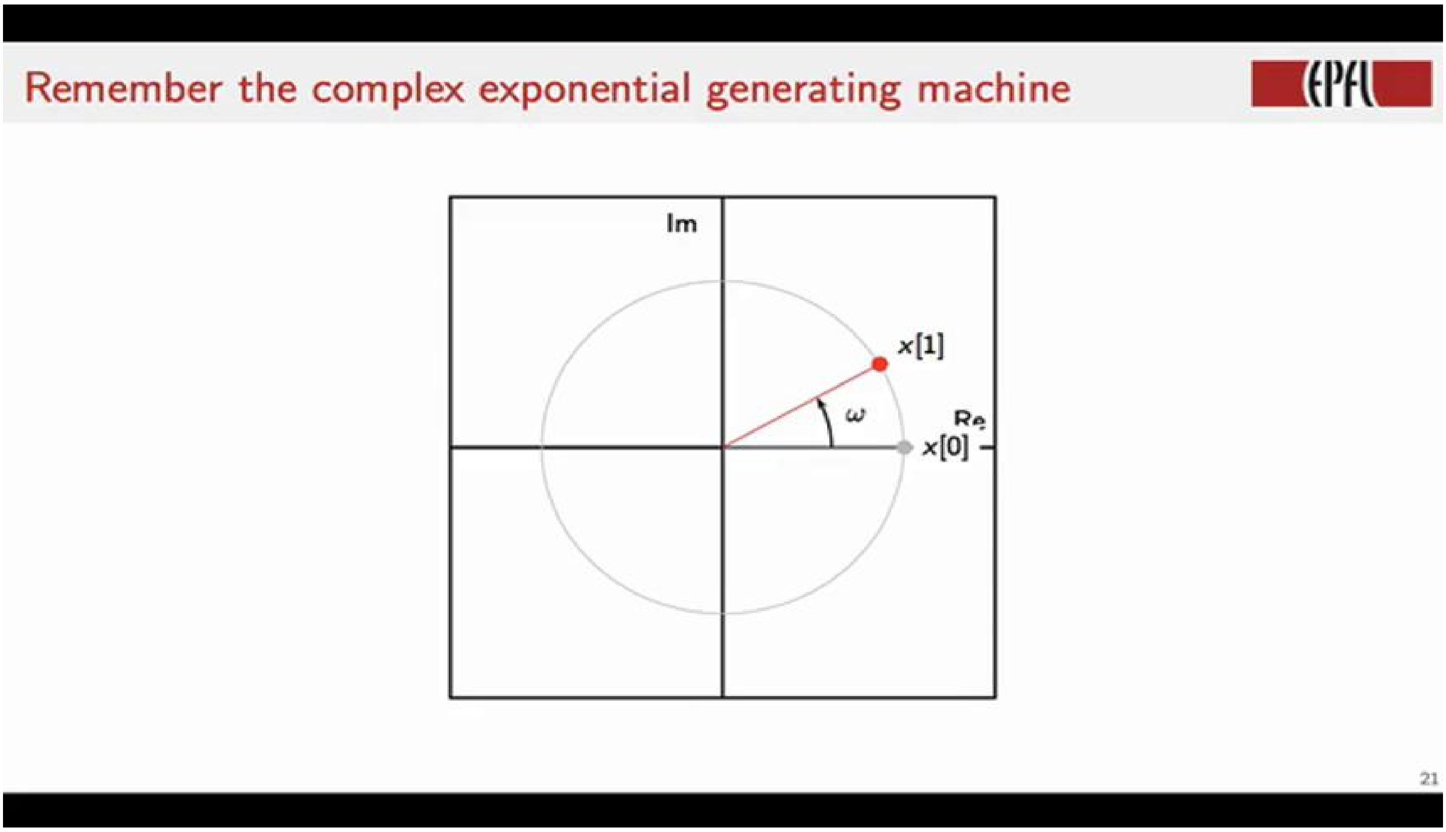
Although the stagecoach is moving in one direction, the wheels appear to be moving alternately backwards and forwards in a sort of unpredictable pattern, as in the example that we're playing right now.

Notes

Summary

10m 03s

Remember the complex exponential generating machine

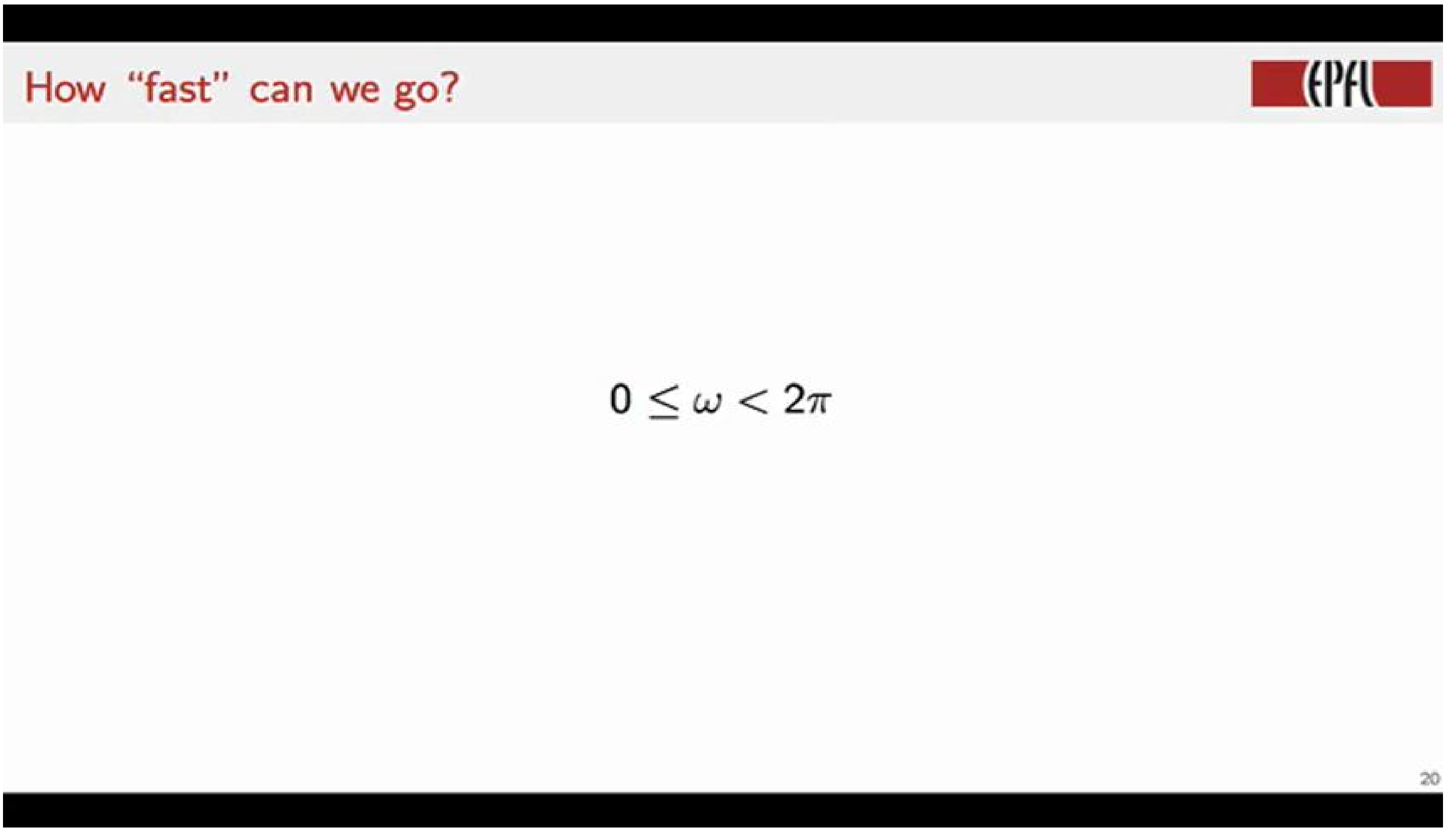


To see why this happens, let's go back to the concept of this complex exponential generating machine. It is rather intuitive to see that the frequency of this machine is restricted to the interval zero to two pi.

Notes

Summary

10m 17s



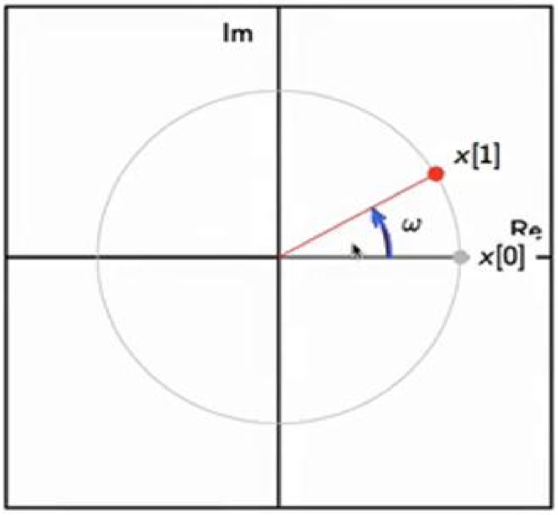
If we try and use a frequency that is larger than two pi, what happens is we incur automatically into allusion.

Notes

Summary

10m 31s

Remember the complex exponential generating machine



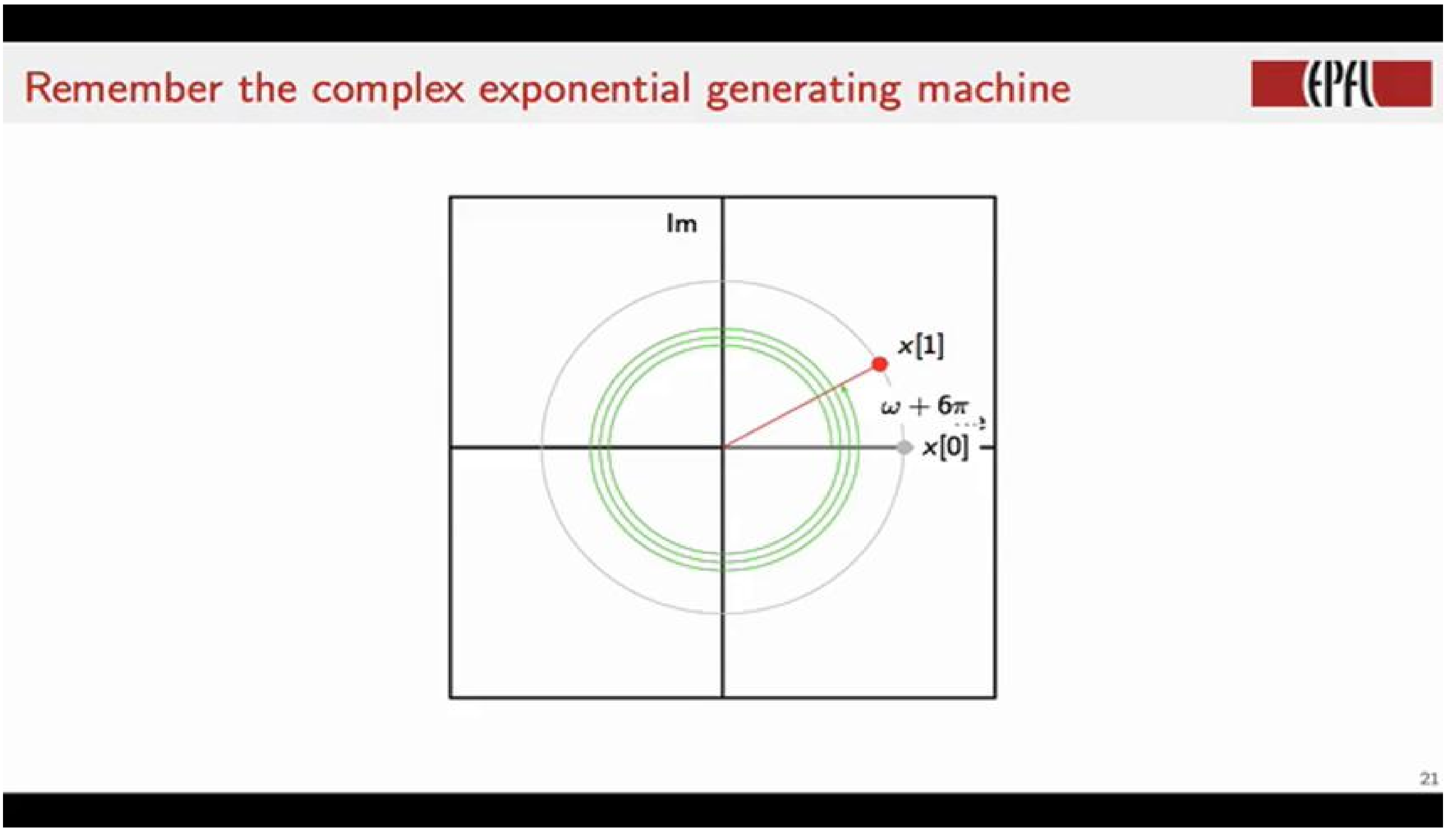
21

Remember, for instance, if at each step we're moving by an angle omega in creating the next sample, well,we would have exactly the same result if we use a step which is omega plus an arbitrary multiple of two pi.

Notes

Summary

10m 39s

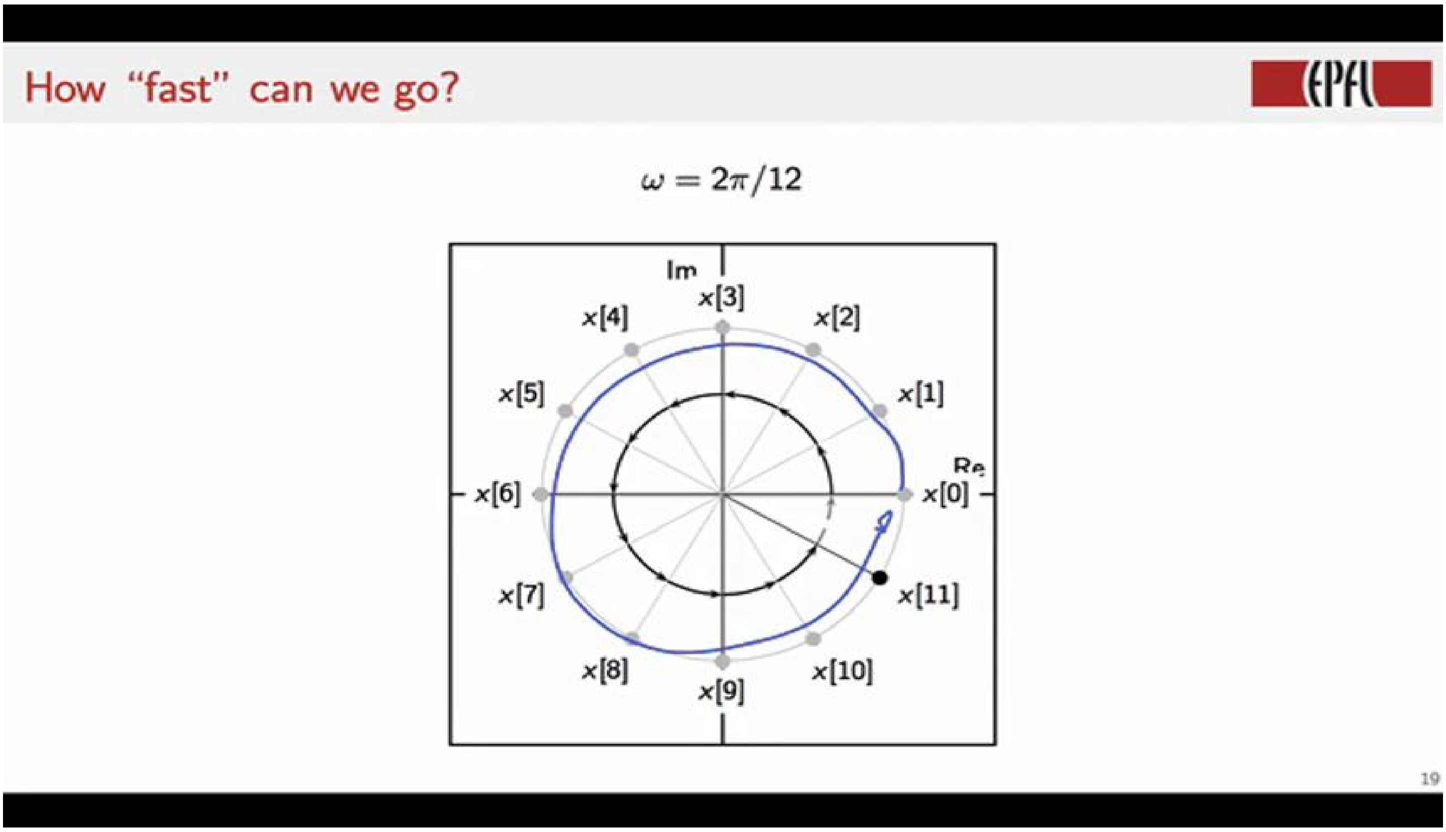


So whenever we go outside of the zero two phirange, because of the inherent two pi periodicity of the complex exponential, we fall back via a modular operation into the zero two phirange. But even within that range, we have to be careful about the backwards and forwards motions. So let's analyse this in more detail.

Notes

Summary

10m 54s



Here in the picture you have the first 12 samples of a discrete time complex exponential at a frequency two phiover 12. This sequence takes 12 samples to go one full revolution around the unit circle.

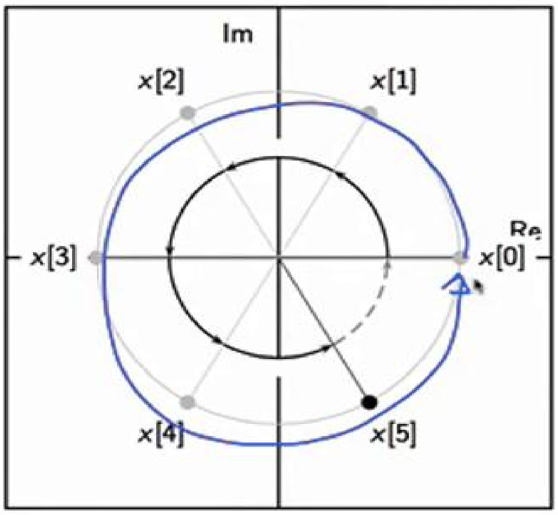
Notes

Summary

11m 16s

How "fast" can we go?

$$\omega = 2\pi/6$$



20

If we want to go faster, then we have to choose a faster frequency. So for instance, if we pick omega equal to two phiover six, it will only take six steps to complete a full revolution. Let's go even faster.

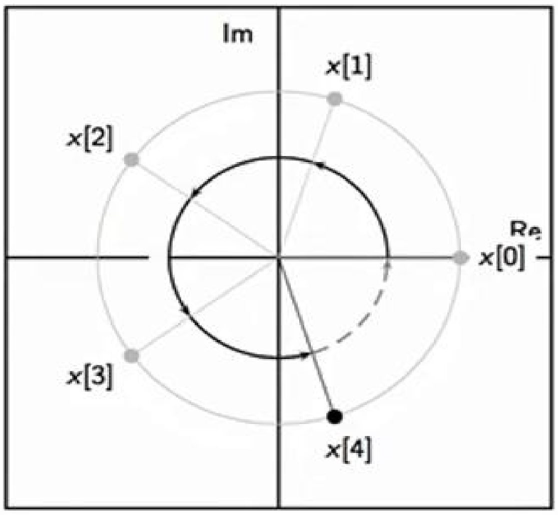
Notes

Summary

11m 30s

How “fast” can we go?

$$\omega = 2\pi/5$$



21

Two phiover five will require five steps.

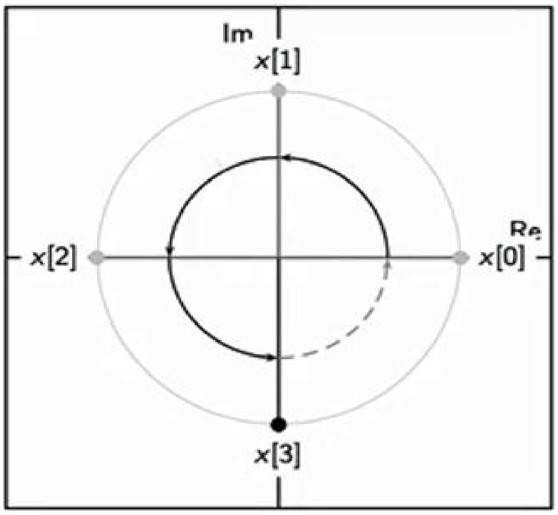
Notes

Summary

11m 44s

How “fast” can we go?

$\omega = 2\pi/4$



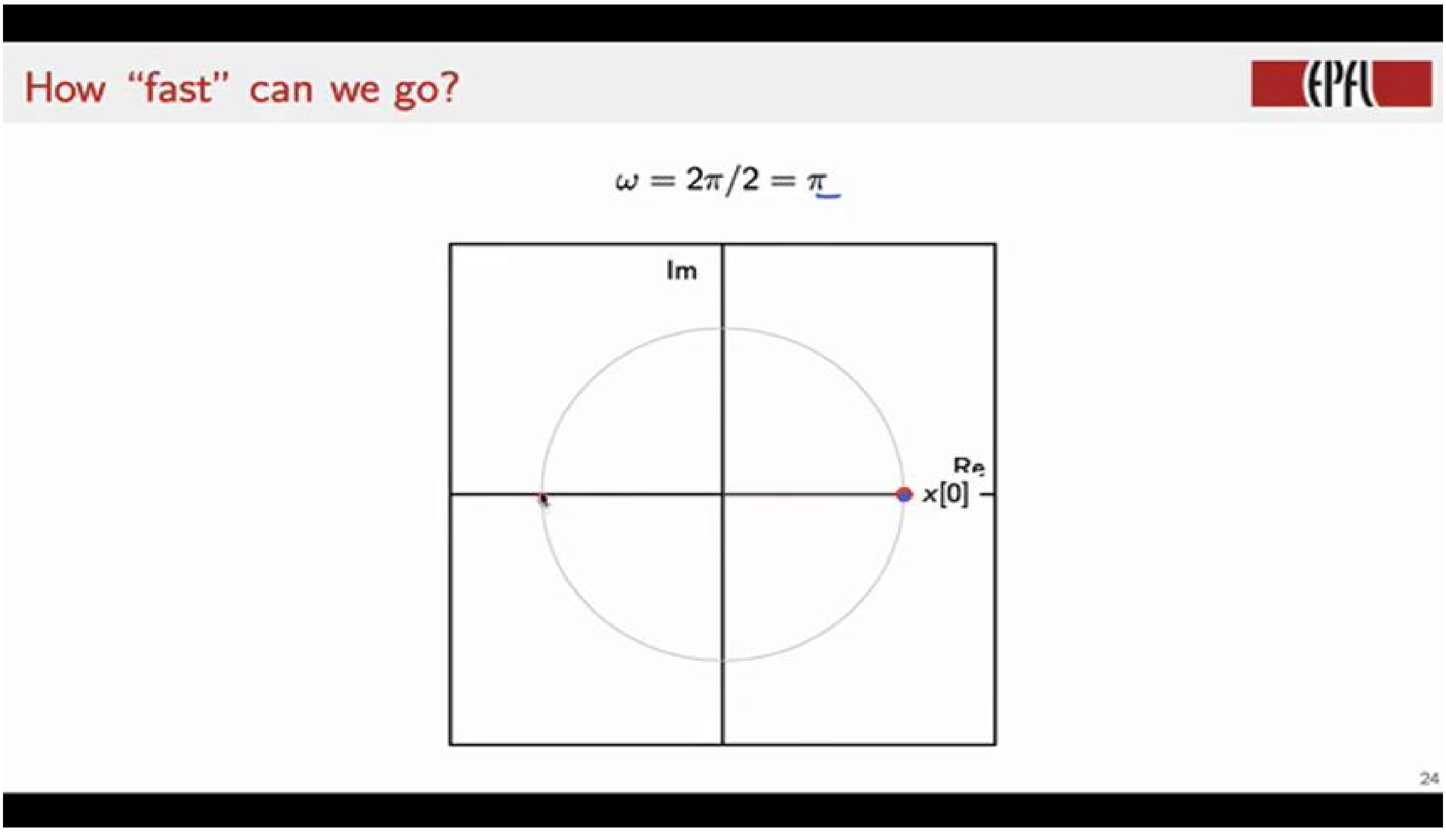
22

Two phiover four will require four steps.

Notes

Summary

11m 46s

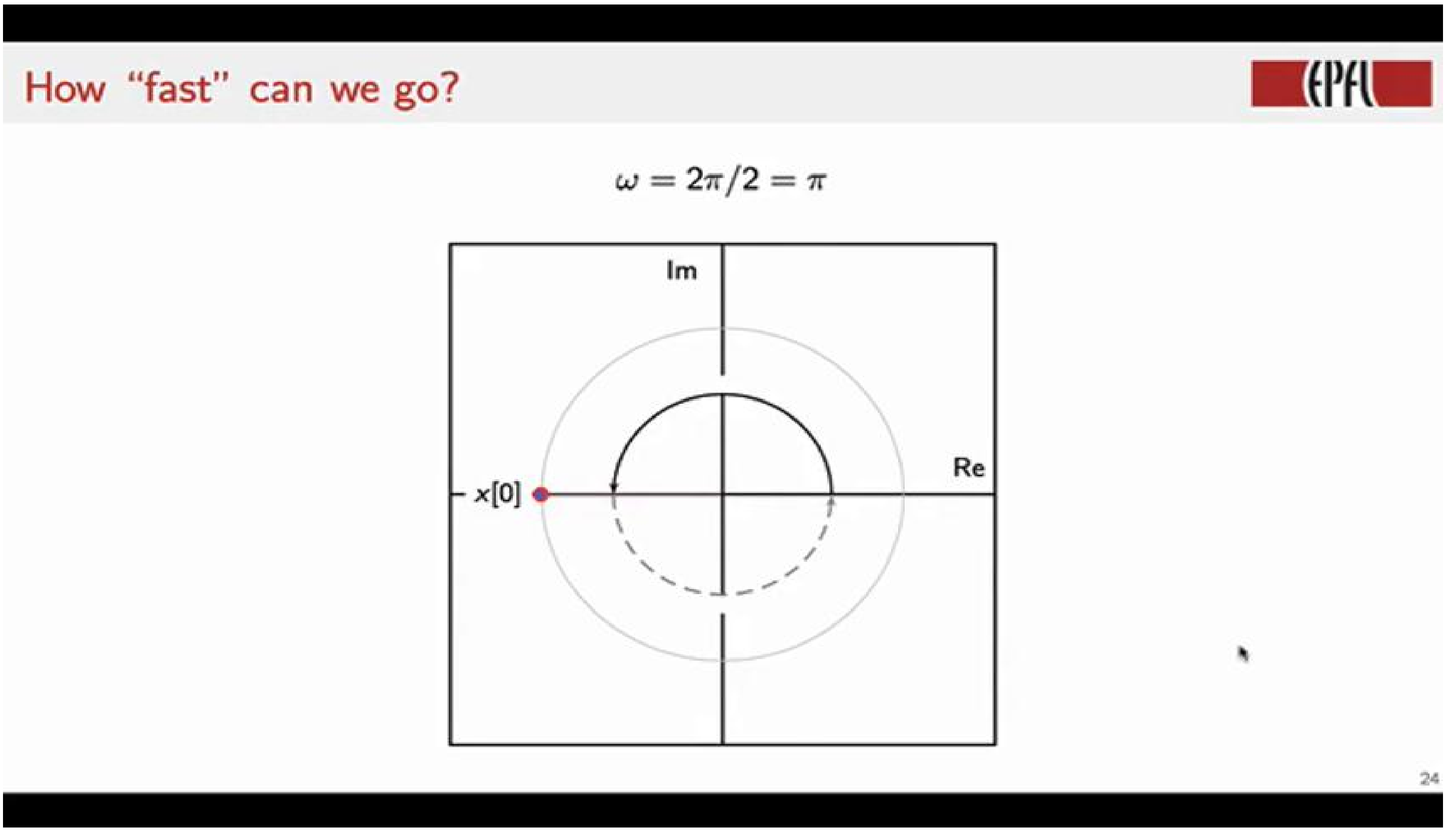


Two phiover three, three steps and finally, if we take a frequency of two phiover two equal to pi, we have a rotation that is not really a rotation. We're just oscillating between this point and this point.

Notes

Summary

11m 49s



So we're completing half of the circle at each step.

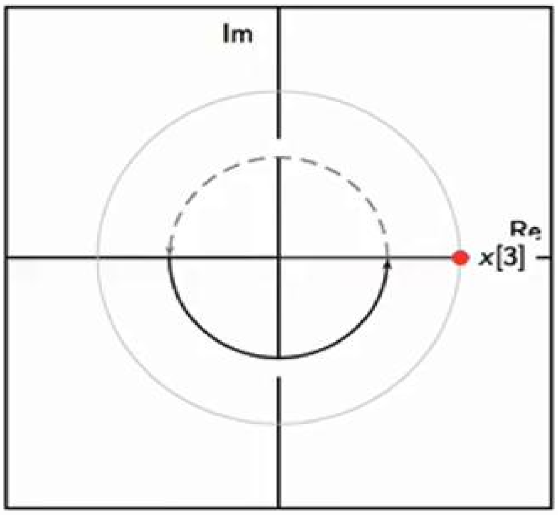
Notes

Summary

12m 02s

How "fast" can we go?

$$\omega = 2\pi/2 = \pi$$

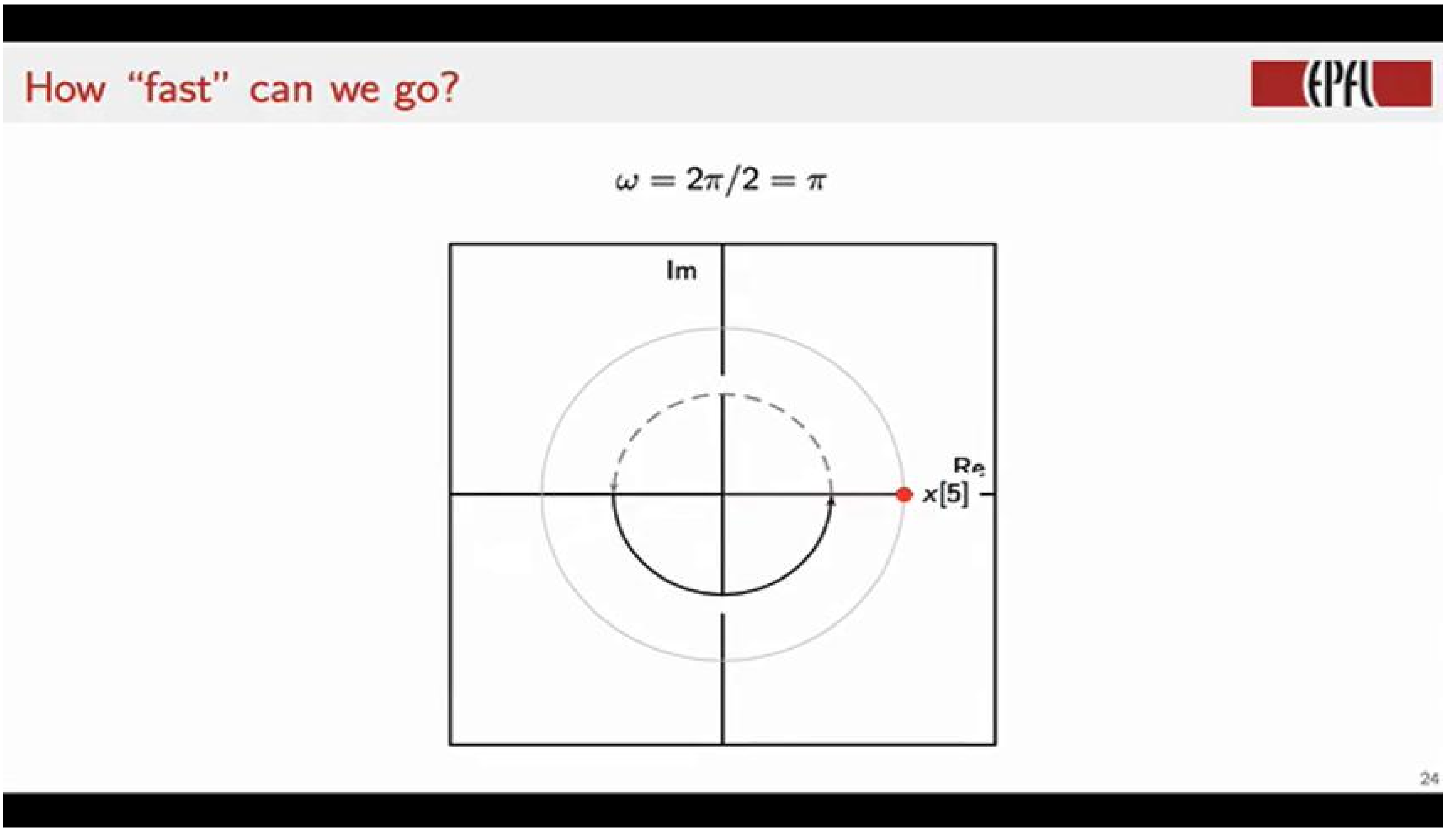


Can we go any faster than that?

Notes

Summary

12m 05s



Well, the answer is not really.

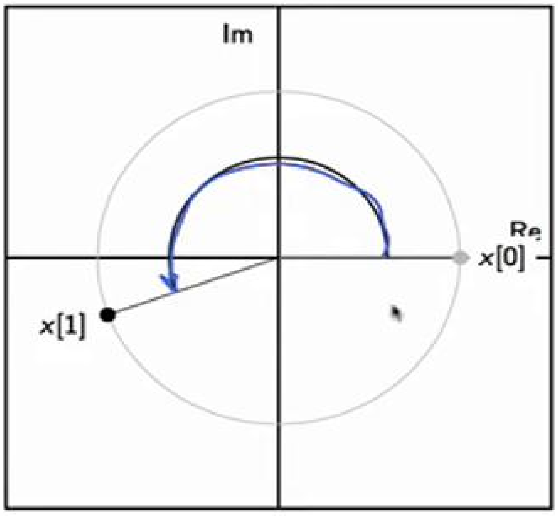
Notes

Summary

12m 08s

What if we go "faster"?

$\pi < \omega < 2\pi$



25

If we try to take a frequency that is bigger than pi, so say omega bigger than phiand less than two pi, we will be travelling more than half the circle in one step. But another way to look at this move, if you want, is that the point has travelled omega minus two pi in the clockwise direction instead.

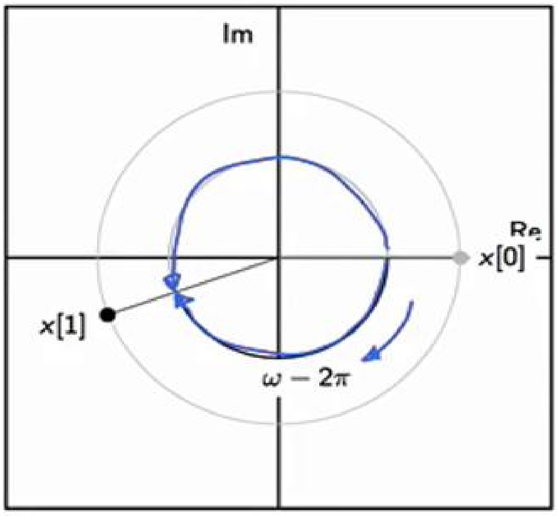
Notes

Summary

12m 09s

What if we go "faster"?

$$\pi < \omega < 2\pi$$



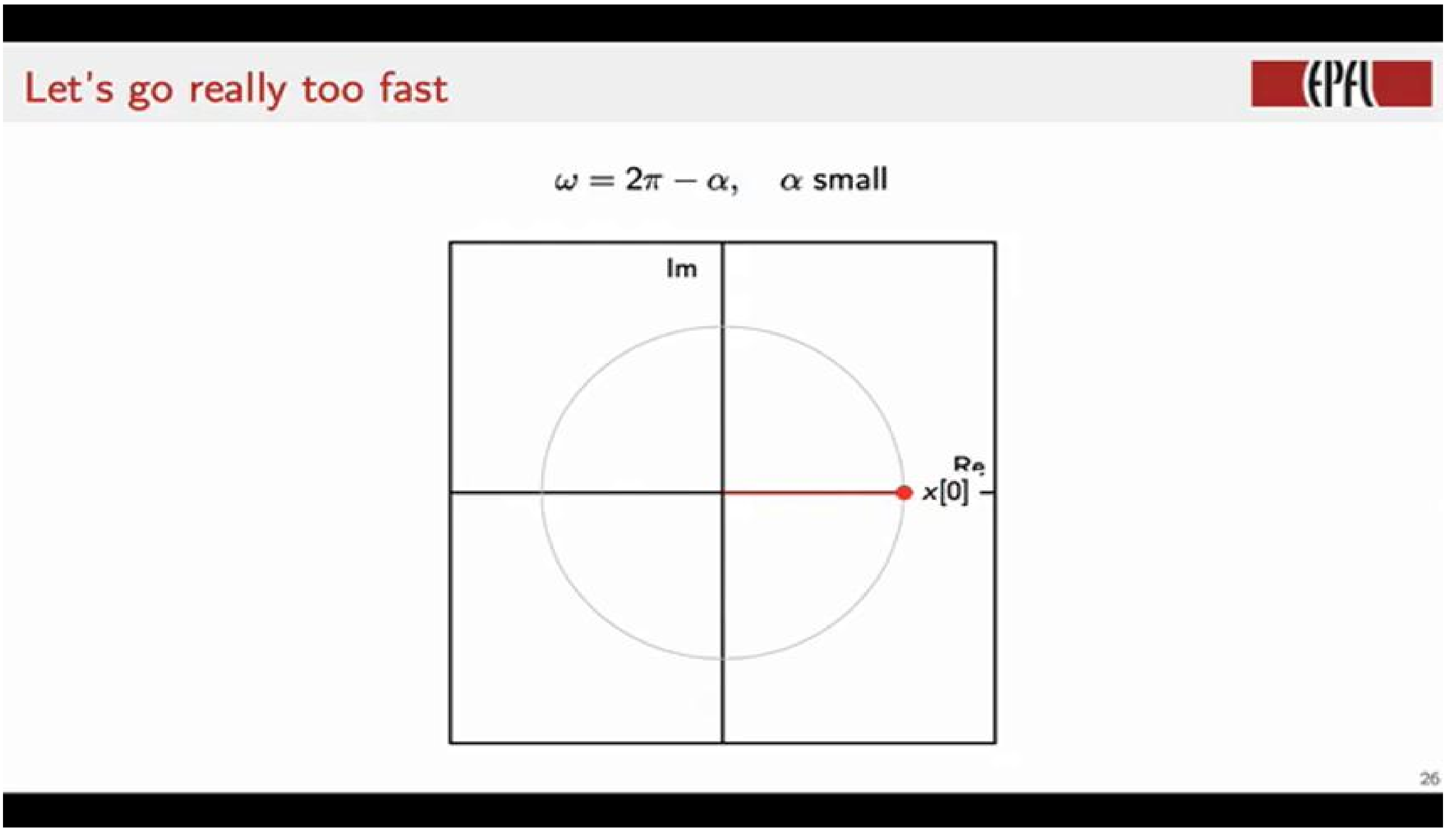
25

This angle is a magnitude smaller than this angle. So in a sense, trying to go faster than phiat each step corresponds to going slower in the other direction.

Notes

Summary

12m 31s



The thing is exacerbated if we take frequency omega, which is equal to two phiminus alpha with alpha small. Now, at each step we're going almost a full circle.

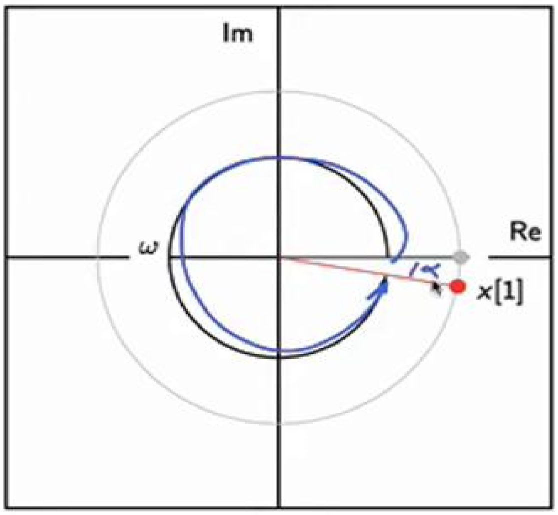
Notes

Summary

12m 45s

Let's go really too fast

$\omega = 2\pi - \alpha, \quad \alpha$ small



26

The first step will be like so we go all the way around minus a little tiny alpha here, and then we continue.

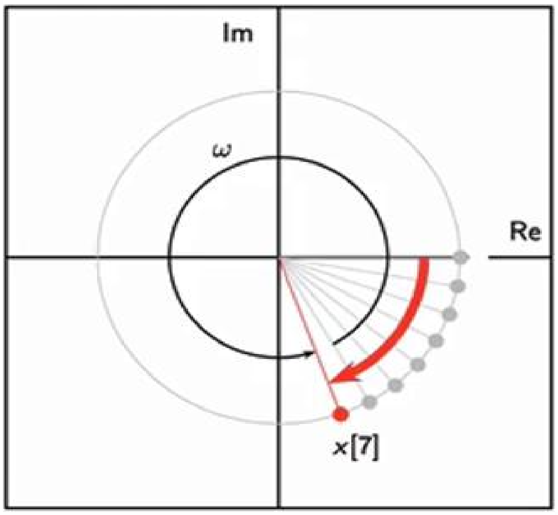
Notes

Summary

12m 56s

Let's go really too fast

$$\omega = 2\pi - \alpha, \quad \alpha \text{ small}$$



26

At each step we repeat the same large movement. As we accumulate these angles, you can see that the apparent motion, even at the slow speed at which I'm changing the slides, correspond to a clockwise motion at the speed of alpha. Remember, alpha is very small. That's really what's happening to the wheels of the stage coach. They're moving too fast with respect to the frame rate of the film camera.

Notes

Summary

13m 03s

To illustrate Aliasing in more dramatic detail, here's a video obtained by artificially rotating the picture of a bicycle wheel with four spokes. This wheel has a fourfold symmetry, and the alloys and angles will alternate faster than if we had a single point rotating around the unit circle. As the angular velocity increases and the angular velocity is displayed in degrees per frame in the top left corner of the video, we can see that the motion alternates between a forward motion to a backwards motion and then a sense of stationarity as the angle is around 45 degrees, and then forward motion again, and so on and so forth. But if we look at the hub of the wheel, which is not perfectly symmetric in the picture, then we can see that the actual motion of the frames is always forward at an increasing angular speed.

Notes

Summary

13m 28s