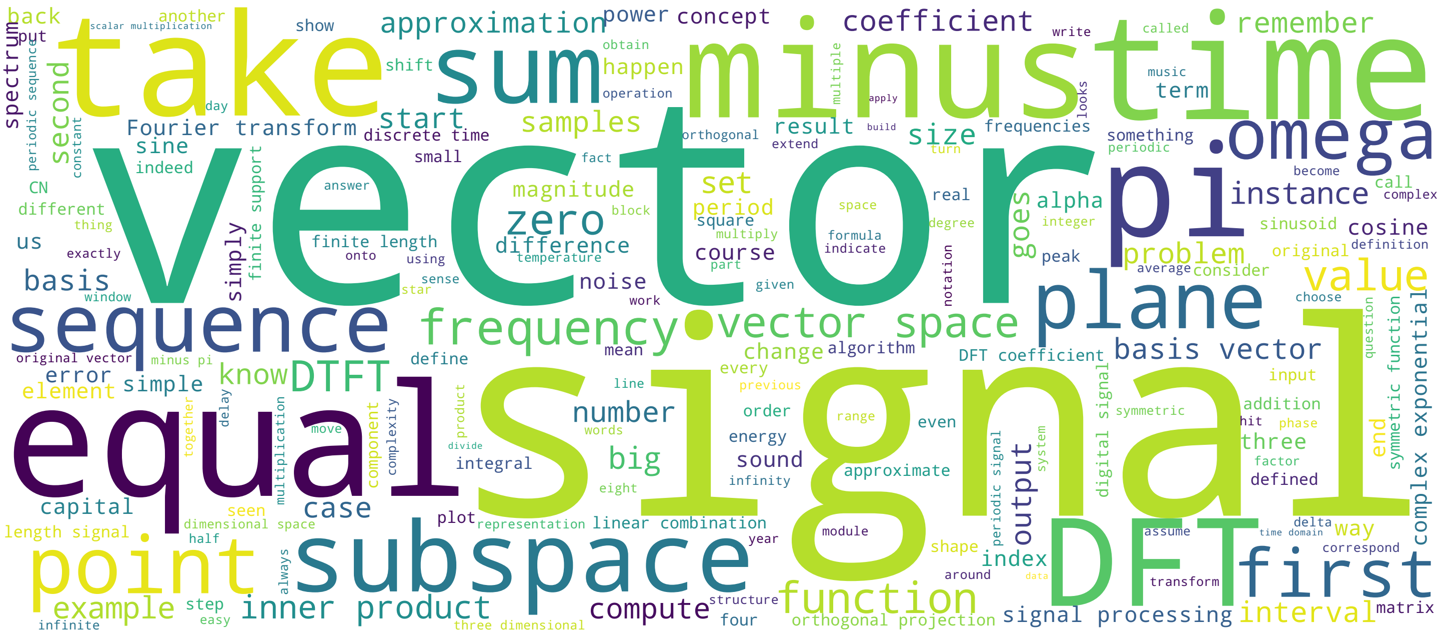
Digital Signal Processing

Subspace-based approximations



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Video

Outline

- vector subspace
- least squares approximations
- approximation with Legendre polynomials

1

One of the greatest achievements of digital signal processing is the ability to efficiently compress information prior to transmission, and this is why we can enjoy so much digital content on a variety of fixed and mobile devices. Here we will see how to cast the compression problem in terms of vector space operation. We will start by introducing the concept of vector subspace, talk about least squares approximations, and then show you how to approximate a function using Legendre polynomials rather than a Taylor series expansion.

Notes

Summary

0m 00s

Vector subspace

a subset of vectors *closed* under addition and scalar multiplication

2

The concept of vector subspace is very simple. It is a subset of vectors that is closed under addition and scalar multiplication.

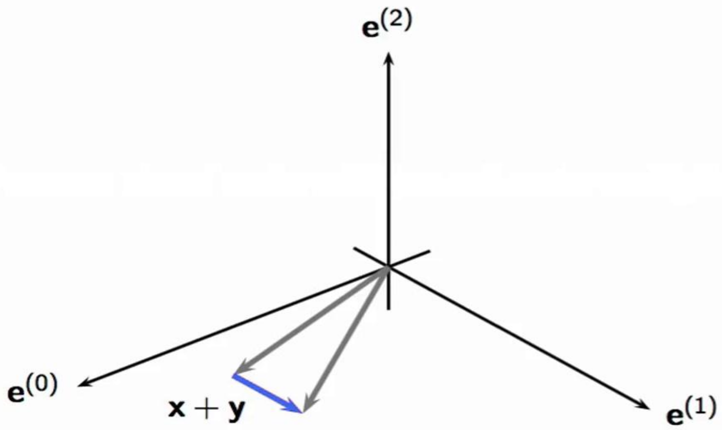
Notes

Summary

0m 33s

Example in Euclidean Space

intuition: $\mathbb{R}^2 \subset \mathbb{R}^3$



3

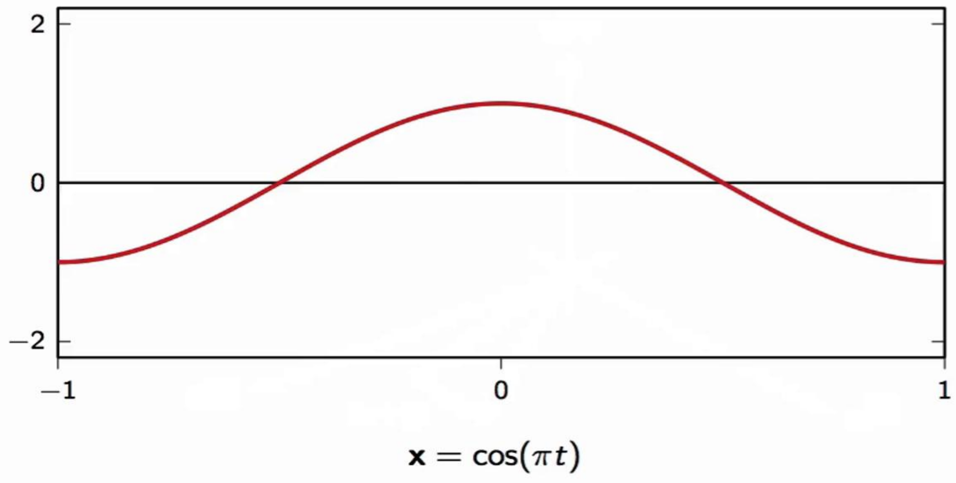
Classic example is a plane in three-dimensional space. A plane is a subspace because if you combine together vectors lying on the plane, you will remain in the plane. You will never be able to leave the plane to the third dimension. This is actually the theme of a very famous book called Flatland, published at the end of the 19th century, which I encourage you to check out if you don't know it already.

Notes

Summary

0m 41s

Subspace of symmetric functions over $L_2[-1, 1]$



4

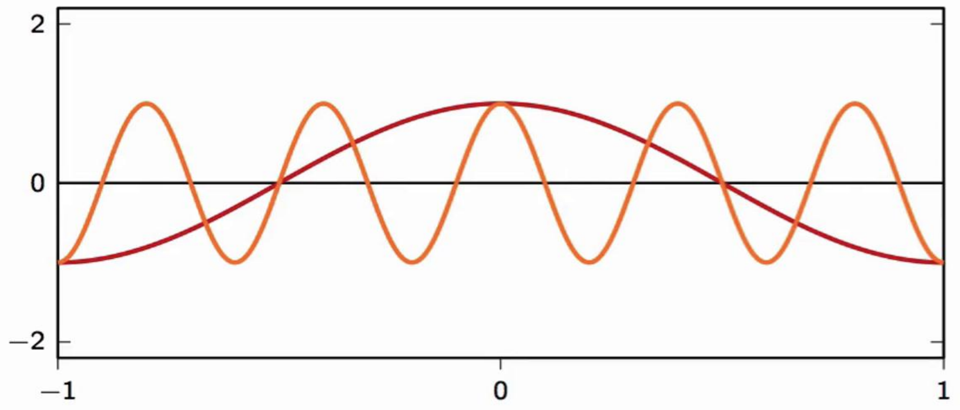
We can extend the concept of subspace to more complicated vector spaces such as functional vector spaces. Take, for instance, the set of symmetric function over the interval. This is a subspace in the sense that whenever we take a linear combination of symmetric functions over this interval, we will always end up with a symmetric function. Here, for example, you have cosine of pi of t.

Notes

Summary

1m 05s

Subspace of symmetric functions over $L_2[-1,1]$



$\mathbf{y} = \cos(5\pi t)$

4

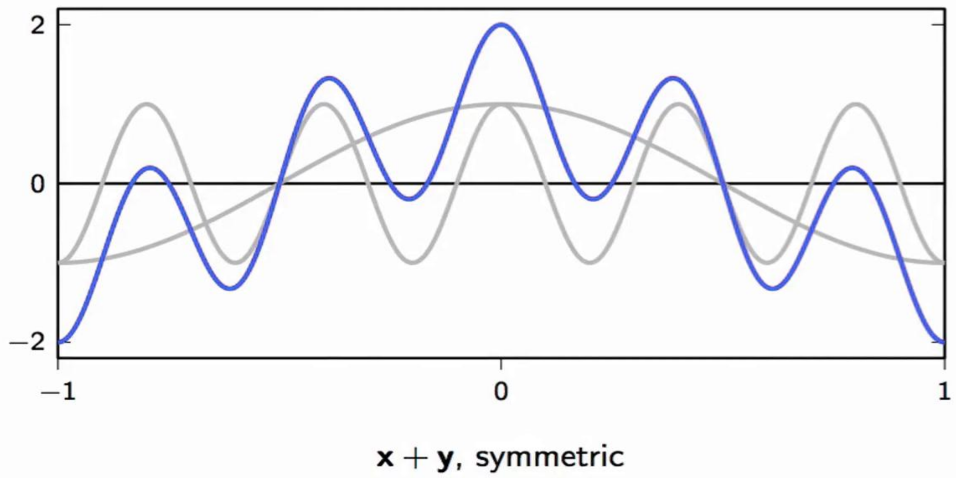
We take a second function, cosine of 5 pi of t.

Notes

Summary

1m 30s

Subspace of symmetric functions over $L_2[-1, 1]$



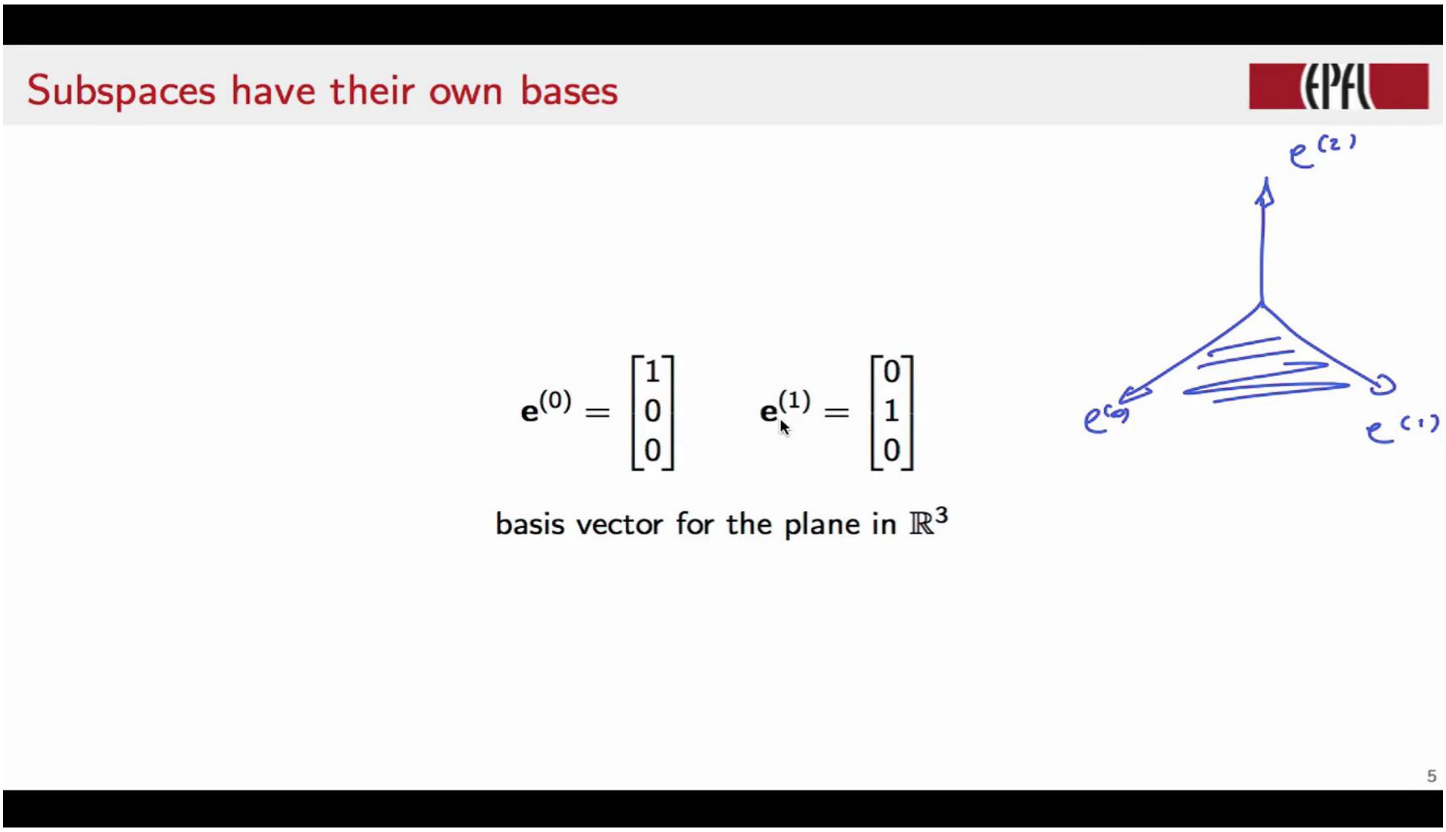
4

Both are symmetric around zero, and if we sum them together, the result, of course, will be symmetric as well.

Notes

Summary

1m 34s



Subspaces have their own basis. For instance, consider the plane in three-dimensional space. The three basis vectors in the canonical basis are e0, e1, and e2. And if we consider just this plane here, our basis for the plane will be e0 and e1.

Notes

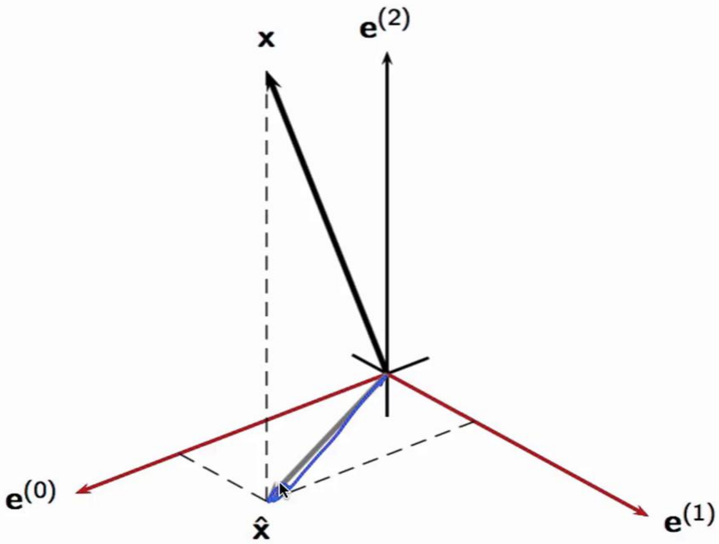
Summary

1m 40s

Approximation

Problem:

- vector $\mathbf{x} \in V$
- subspace $S \subseteq V$
- approximate $\mathbf{x}$ with $\hat{\mathbf{x}} \in S$



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Here we got to the problem of approximation. Suppose you have a full-fledged vector in your vector space, —in this case, a vector in three-dimensional space— you have a subspace S —let's take, for instance, the plane here— and the question is, how do I approximate x using only elements from this subspace? Intuitively, the answer is, I take the projection of this vector over the subspace and this will be my approximation. The power of the vector space paradigm for signal processing is that we can extend this geometric intuition to arbitrarily complex vector space. Now we will define the concept of orthogonal projection for approximation in abstract terms.

Notes

Summary

1m 59s

Least-Squares Approximation

- $\{\mathbf{s}^{(k)}\}_{k=0,1,\ldots,K-1}$ orthonormal basis for S
- orthogonal projection:

$$\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s}^{(k)}, \mathbf{x} \rangle \, \mathbf{s}^{(k)}$$

orthogonal projection is the "best" approximation over S

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Consider a north normal basis for the subspace S, call this s of k. The orthogonal projection of a vector x onto this subspace is defined like so. This is like a partial basis expansion where we take the coefficients as inner products between the original vector and the basis for the subspace. The orthogonal projection —defined like so— is the best approximation for a given vector x onto the given subspace S.

Notes

Summary

2m 41s

Least-Squares Approximation

- orthogonal projection has minimum-norm error:

$$\arg\min_{\mathbf{y}\in S} \|\mathbf{x} - \mathbf{y}\| = \hat{\mathbf{x}}$$

- error is orthogonal to approximation:

$$\langle \mathbf{x} - \hat{\mathbf{x}}, \hat{\mathbf{x}} \rangle = 0$$

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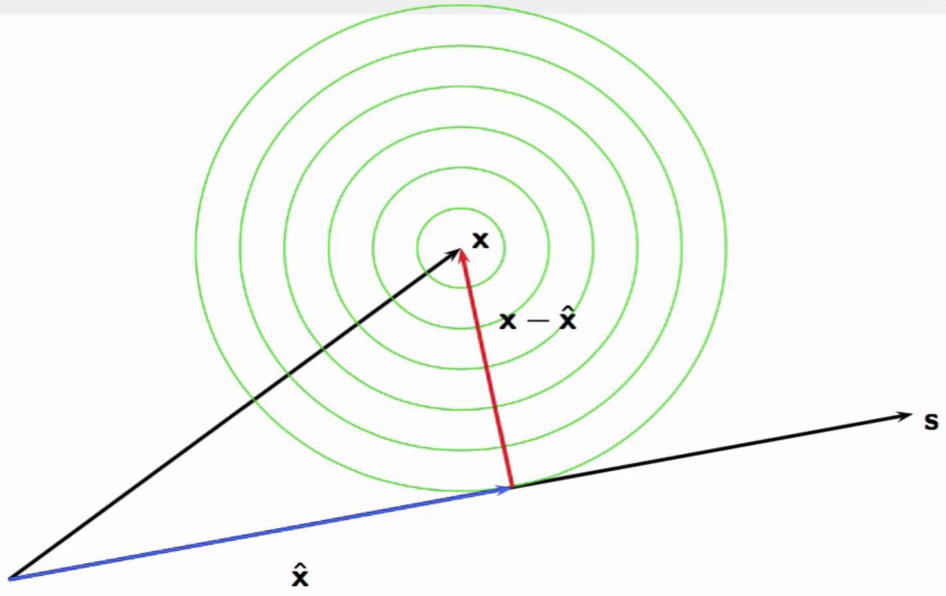
It is the best because it has two fundamental properties. First of all, it has minimum norm error. You remember that we define the distance between two vectors as the norm of the difference between the vectors. So what we're saying here is that the orthogonal projection of the subspace is the vector that minimizes the distance between the approximation and the original vector. The second fundamental property is that the error is orthogonal to the approximation. What that means is that the error and the set of basis vectors for the subspace are maximally different. In other words, they're uncorrelated. The basis vectors for the subspace cannot capture any more information contained in the error.

Notes

Summary

3m 10s

Least Squares Approximation



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We can try and illustrate this with a minimum dimensionality example like so, where in the plane, our subspace is one-dimensional subspace S. Assume this vector here is unit norm, so it is its own orthonormal basis, and we want to approximate the vector x line on the plane. The way we can find a minimum distance approximation is to consider curves of equal distance emanating from x until we hit the approximation subspace. If we do that geometrically, we build circles like so, and when we hit S, we can see that indeed, we are at a minimum distance from x, and that the error, —this red vector here— is indeed orthogonal to the subspace that we are approximating from.

Notes

Summary

3m 53s