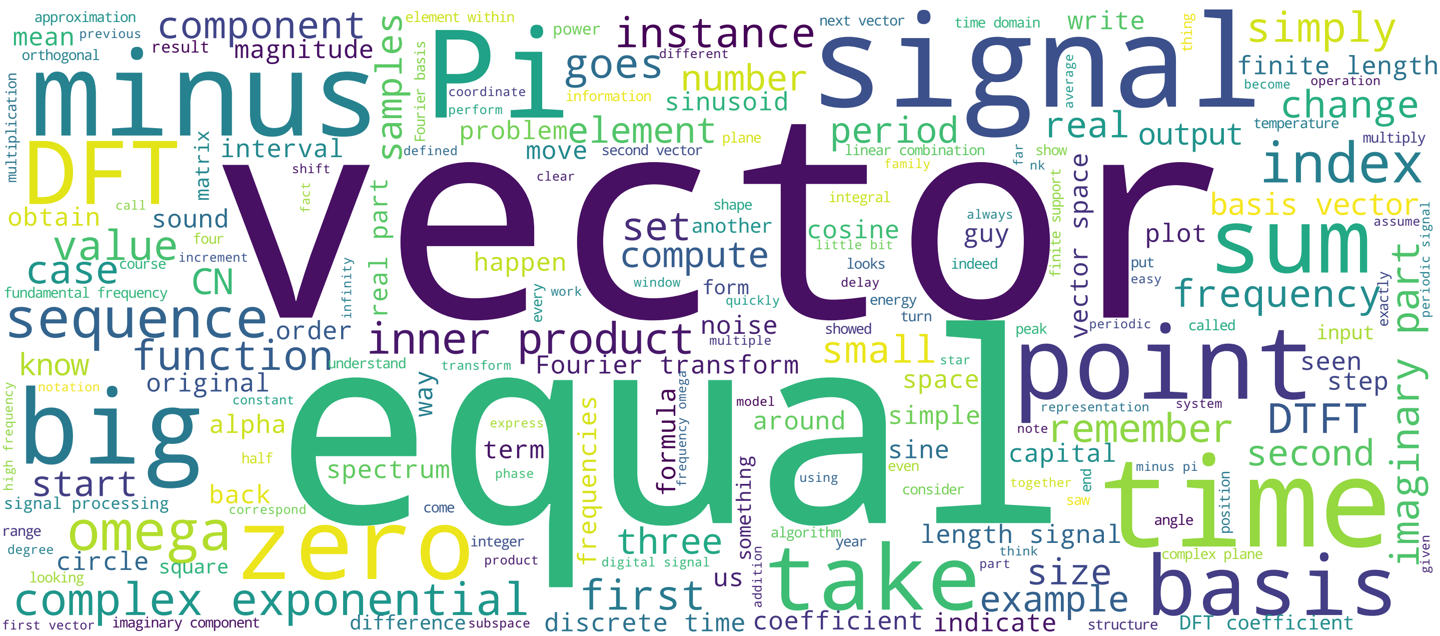
The mathematical setup

- let's start with finite-length signals (i.e. vectors in $\mathbb{C}^N$)
- Fourier analysis is a simple change of basis
- a change of basis is a change of perspective
- a change of perspective can reveal things (if the basis is good)



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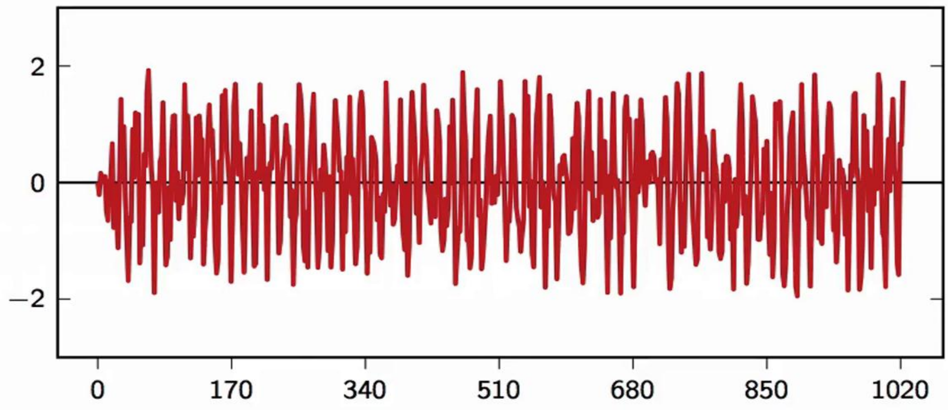
Video

Mystery signal



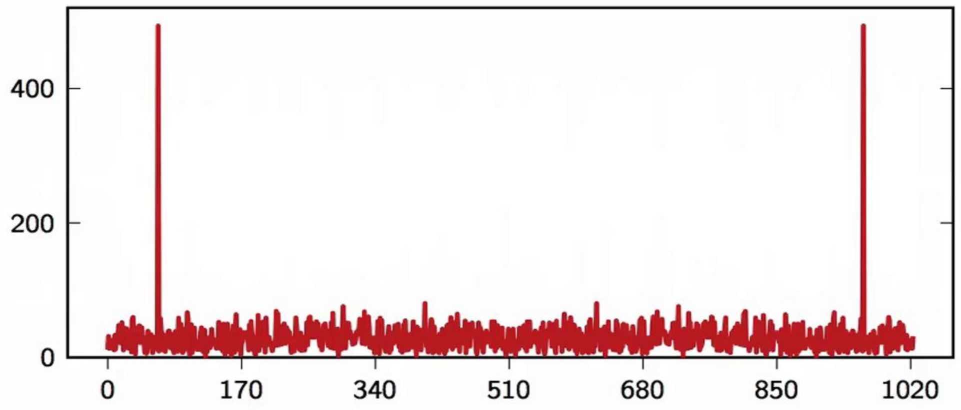
Let's start with finite-length signals, which are the easiest to handle. We have seen that a finite length signal is simply a vector in CN. And it turns out the free analysis in this case, is simply a change of basis in CN. Now remember, a change of basis is a change of perspective, and a change of perspective can reveal things. You're looking at things from a different angle. So for example, imagine you have a signal like this. This is a 1024-point signal, and as you look at it in the time domain, you can see some oscillatory behaviour, but it's not really clear what's going on. Well, if we perform an appropriate change of basis, and we will show shortly how to do that, the signal will look like this.

Notes

Summary

0m 00s

Mystery signal in the Fourier basis

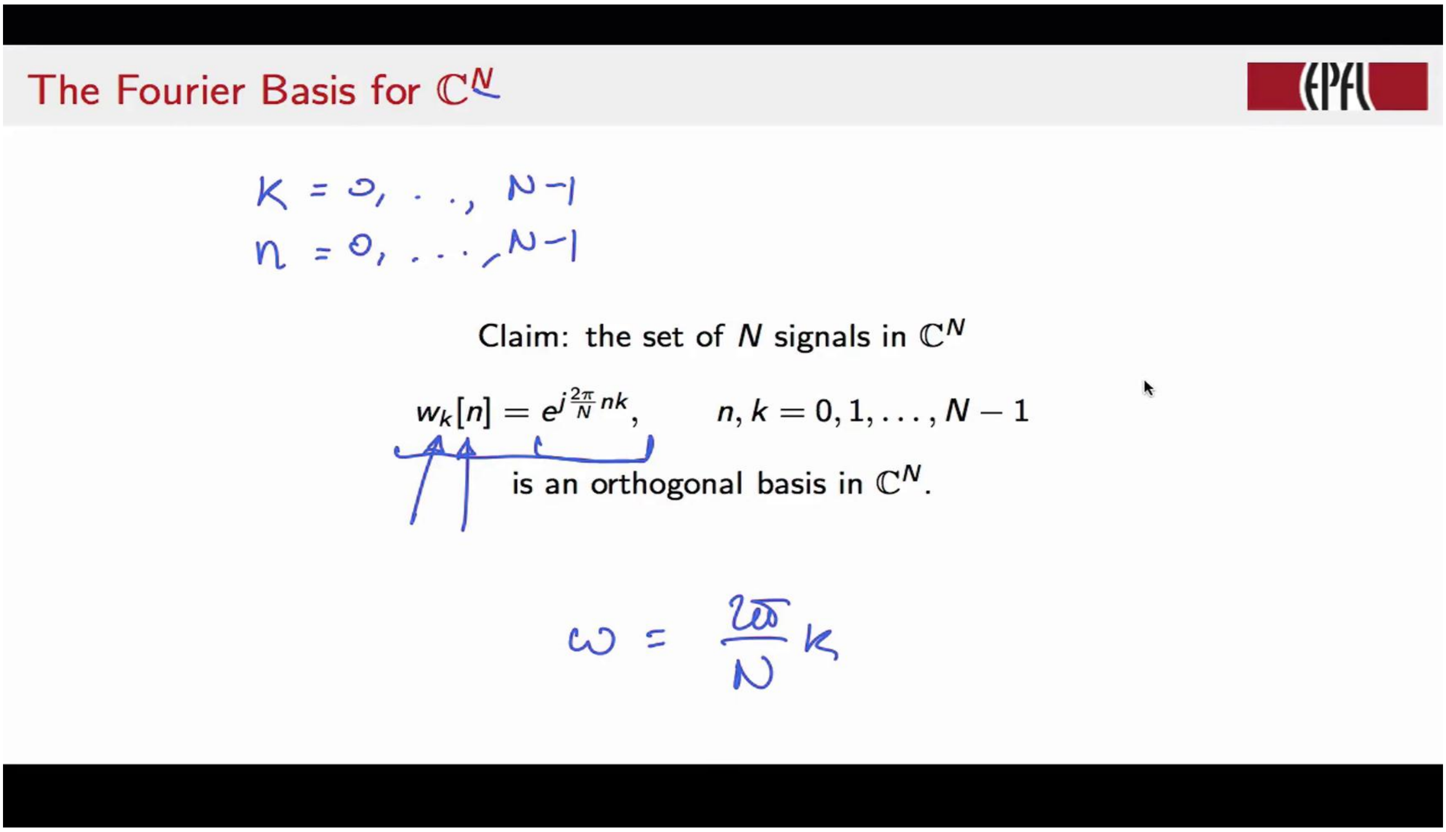


In the Fourier basis, the same signal will have this structure. It is clear here that we have two very different things going on at the same time. We have two spikes and we will see very shortly what the spikes mean, and then we have something that looks like noise in the bottom. Well, Fourier analysis allows us to separate these two elements, that in the time domain were indistinguishable.

Notes

Summary

0m 40s



So here is the Fourier basis for CN. Remember CN as I mentioned relative N. So we will need n different vectors, each one of which will have length n. So k will be the index that indicates different vector, and k will go from zero to big N minus one. And then we will use small n to indicate that the index of the element within each vector, and small n will also range from zero to big N minus one. The n signals will have this form. So remember, k is the index of the signal and small n is the index of the element within each signal, and each element is simply e to the j two Pi over big N nk. What does that mean? This is a simple complex exponential at a frequency omega, which is two Pi over big N times k. So the index of the vector or of the signal, gives you the fundamental frequency of the complex exponential. We claim that this constitutes a set of orthogonal vectors in CN.

Notes

Summary

1m 03s

The Fourier Basis for $\mathbb{C}^N$

In vector notation:

$$\{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,N-1}$$

with

$$w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$$

is an orthogonal basis in $\mathbb{C}^N$

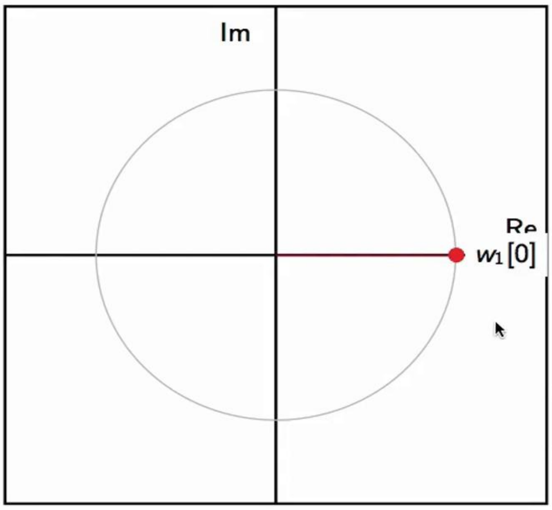
We can express the signals in vector notation. So remember our convention. We use bold phase letter to indicate vectors. A SuperScript in parenthesis will give you the index of a vector when the vector comes from a set, in this case it comes from a set of big N vectors, and a subscript will indicate the element within each vector. So again, w of k of n is simply equal to ∈ to the j two Pi over big N nk.

Notes

Summary

2m 06s

Recall the complex exponential generating machine...



$\tilde{\omega}^{(1)}$

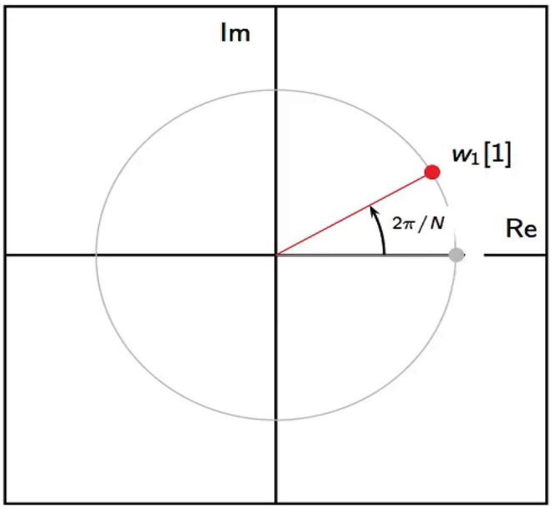
We can quickly get a feeling for what these vectors look like if we remember the complex exponential generating machine that we saw a few modules back. So here for instance, we show the components of w one, the second vector in our basis, and as we go through the components. So this would be the first component of the vector, it would be equal to one.

Notes

Summary

2m 35s

Recall the complex exponential generating machine...



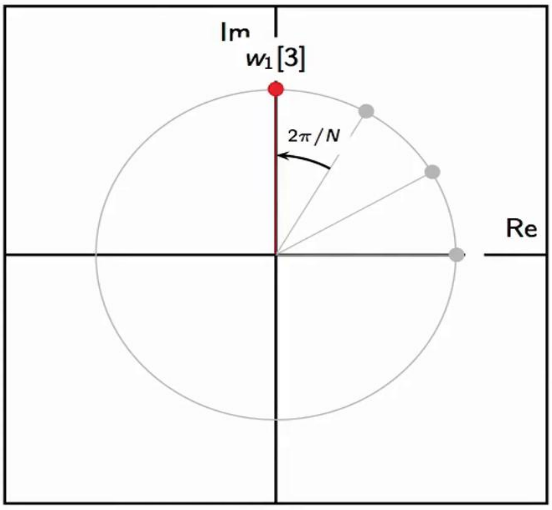
The second component would translate the point around the circle by an angle two Pi over n and so on and so forth.

Notes

Summary

2m 57s

Recall the complex exponential generating machine...



$w_n^{(1)} = e^{j\frac{2\pi}{N}n}$

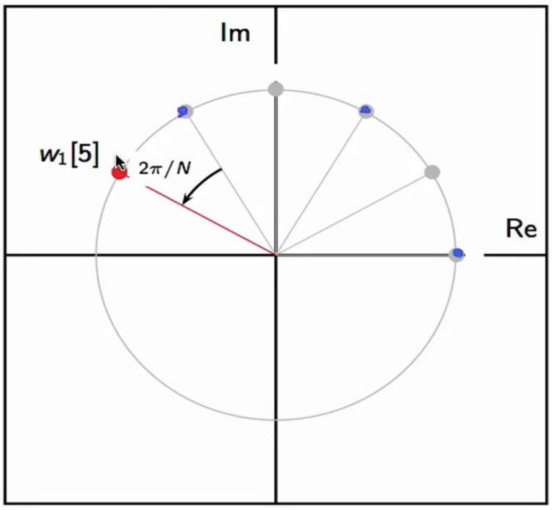
Remember w one of N would be equal to ∈ to the j two Pi over big N. So we're going around the circles in increments of two Pi over big N.

Notes

Summary

3m 05s

Recall the complex exponential generating machine...



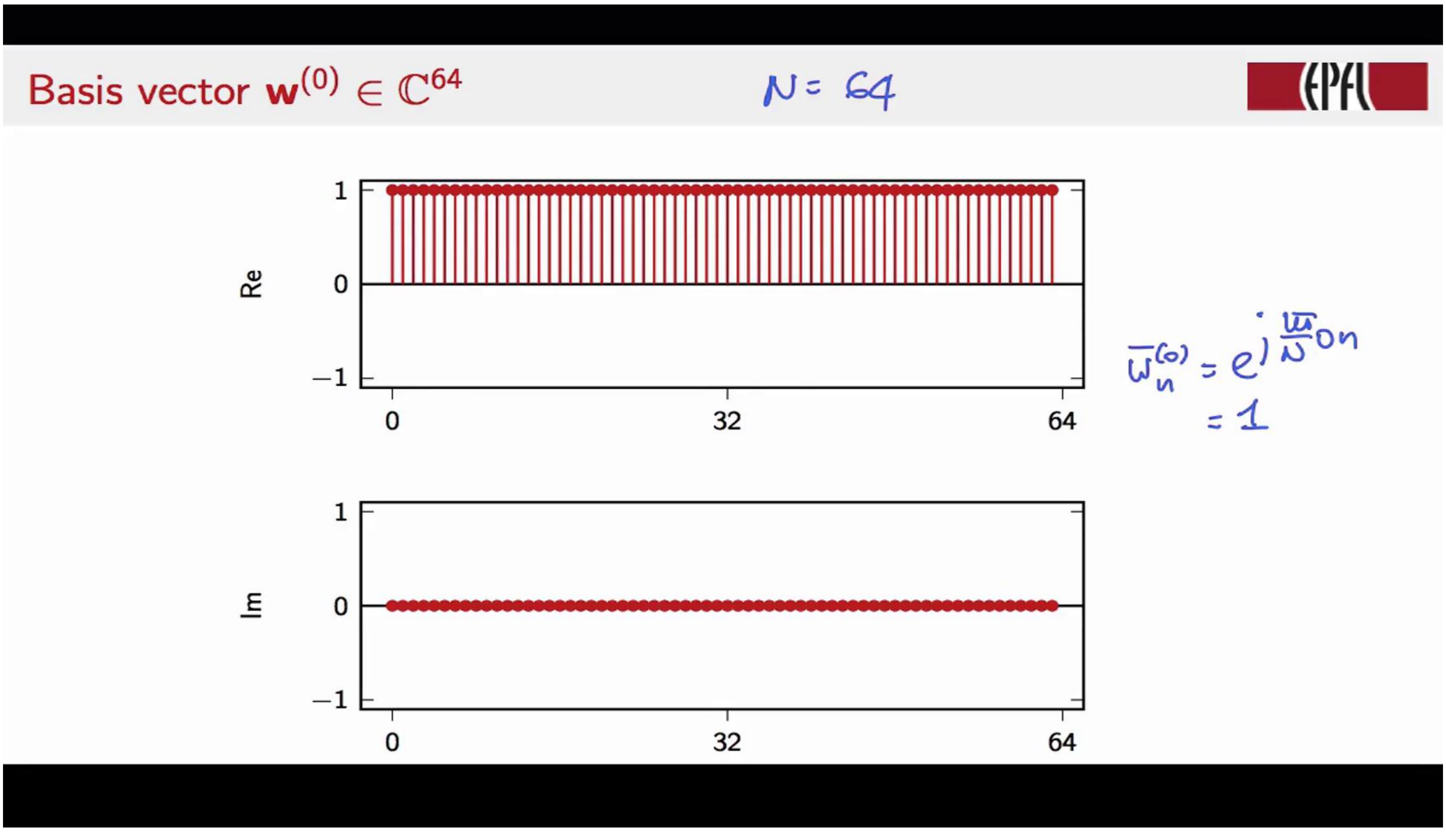
$w_n^{(2)} = e^{j\frac{2\pi}{N}2n}$

Now, the second vector in the basis w two of N will be of the form ∈ to the j two Pi over big N, two n. And so this vector will move in increments of two Pi over N times two. So the points for that would actually be these guys and so on.

Notes

Summary

3m 21s



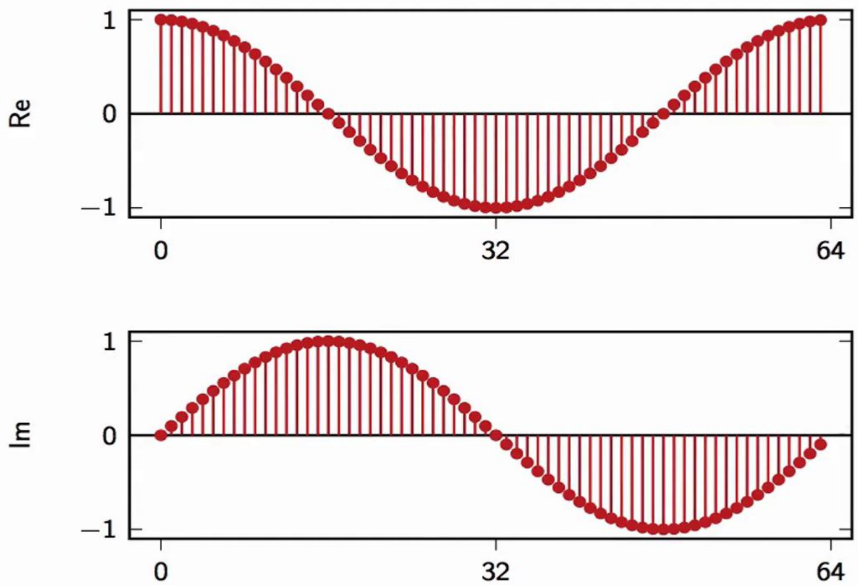
Okay, so let's look at a concrete example by setting big N equal to 64 and look at the Fourier basis of the space C64. Here we're looking at the real and imaginary components, of each basis vector. This, if you think of the point moving around in circles in the complex plane, will be the cosine and sine of its substandard angle as the point moves along the circle. So here we have the first vector in the basis w zero. And each component of this vector will be simply ∈ to the j two Pi over big N zero time n. And because of the multiplication by zero the exponent will be always zero. And so this vector is identically equal to one. Its real component therefore is one for all indices, and its imaginary component is zero everywhere.

Notes

Summary

3m 40s

Basis vector $\mathbf{w}^{(1)} \in \mathbb{C}^{64}$



$\omega = \frac{2\pi}{N}$

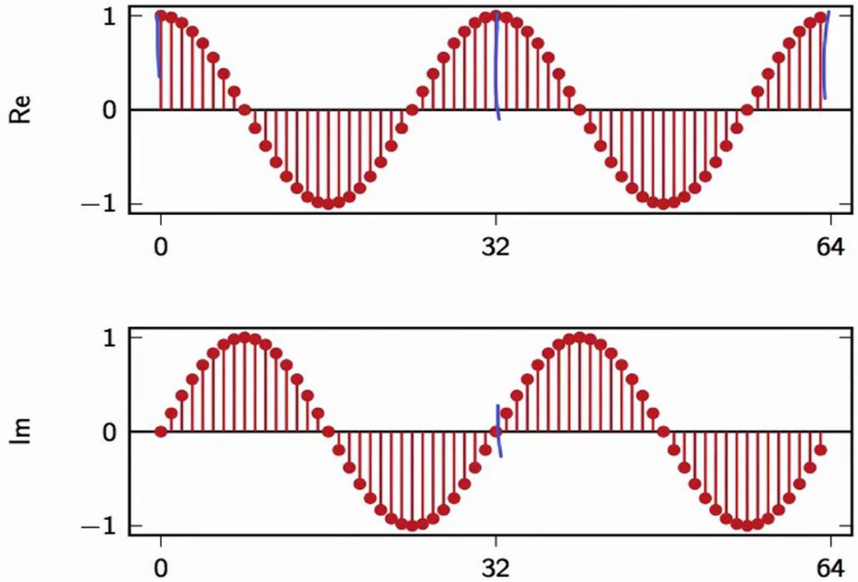
The next vector is w one, and here the fundamental angle of the complex exponential is omega two Pi over big N. What this means is that the point will complete one full revolution over 64 points, and we can see this in the plots of its real and imaginary components. Here we have a full cosine period for the real part, and a full sine period for the imaginary part.

Notes

Summary

4m 32s

Basis vector $\mathbf{w}^{(2)} \in \mathbb{C}^{64}$



$\omega = \frac{2\pi}{N} 2$

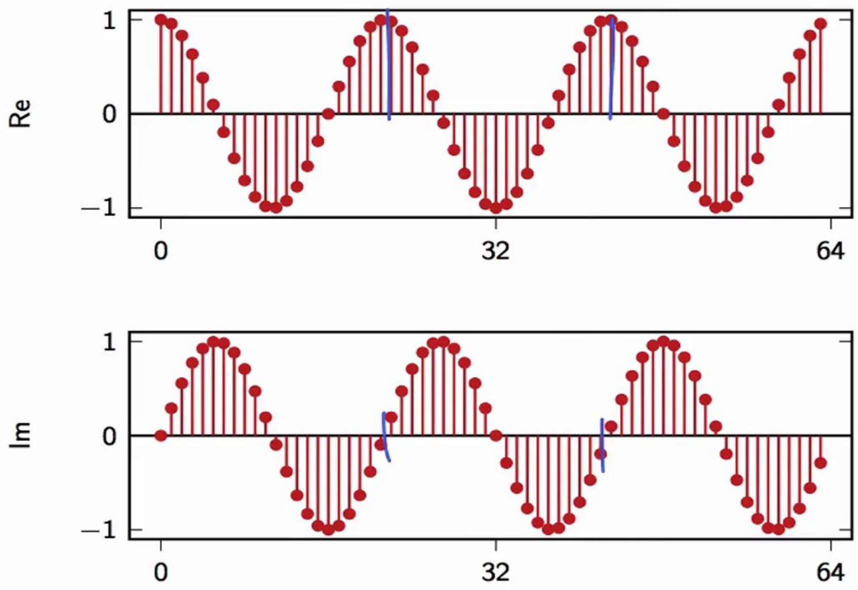
The next vector goes twice as fast. Now the fundamental frequency of the complex exponential will be two Pi over big N times two. And indeed we have two periods of the cosine and two periods of the sine in the imaginary part.

Notes

Summary

4m 59s

Basis vector $\mathbf{w}^{(3)} \in \mathbb{C}^{64}$



$\omega = \frac{2\pi}{N} 3$

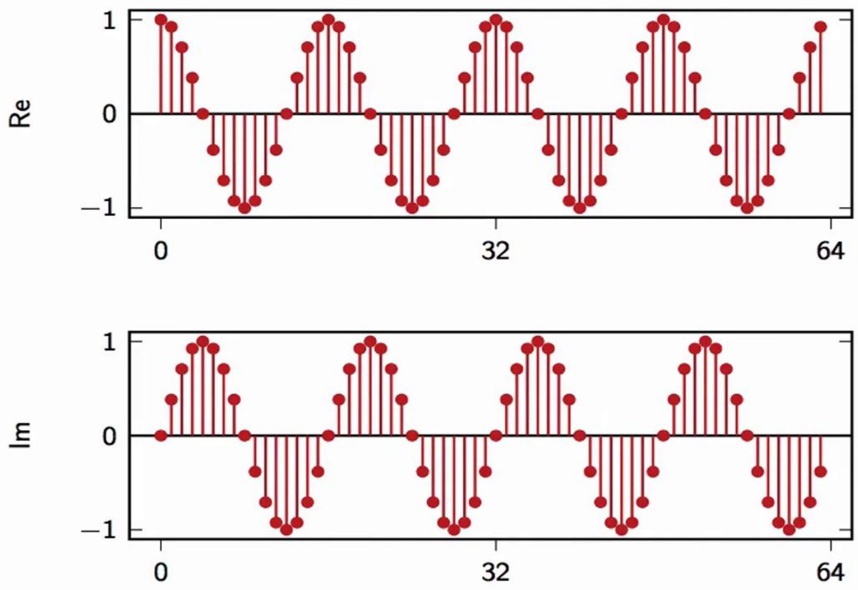
The next vector is w three and the fundamental frequency will be two Pi over big N times three. So we have three periods, both in the real and in the imaginary part.

Notes

Summary

5m 19s

Basis vector $\mathbf{w}^{(4)} \in \mathbb{C}^{64}$



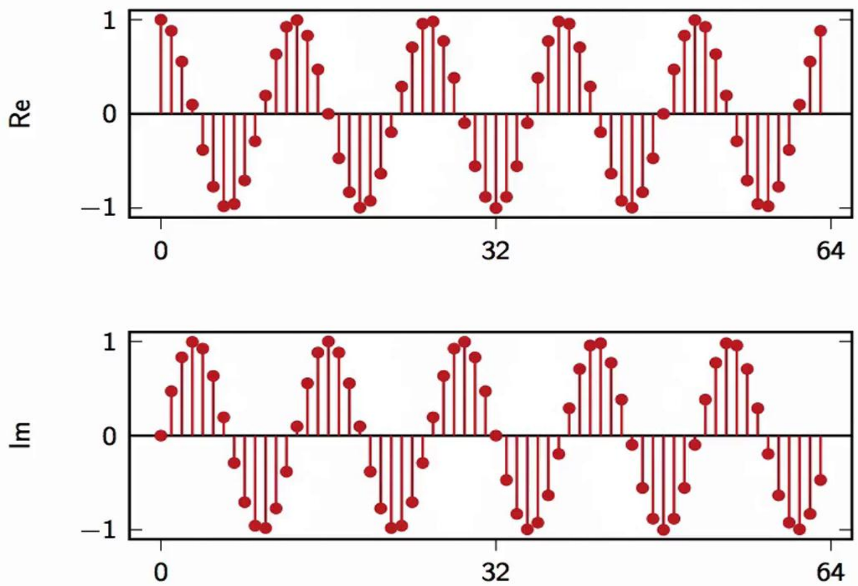
But now you understand what's going on.

Notes

Summary

5m 31s

Basis vector $\mathbf{w}^{(5)} \in \mathbb{C}^{64}$

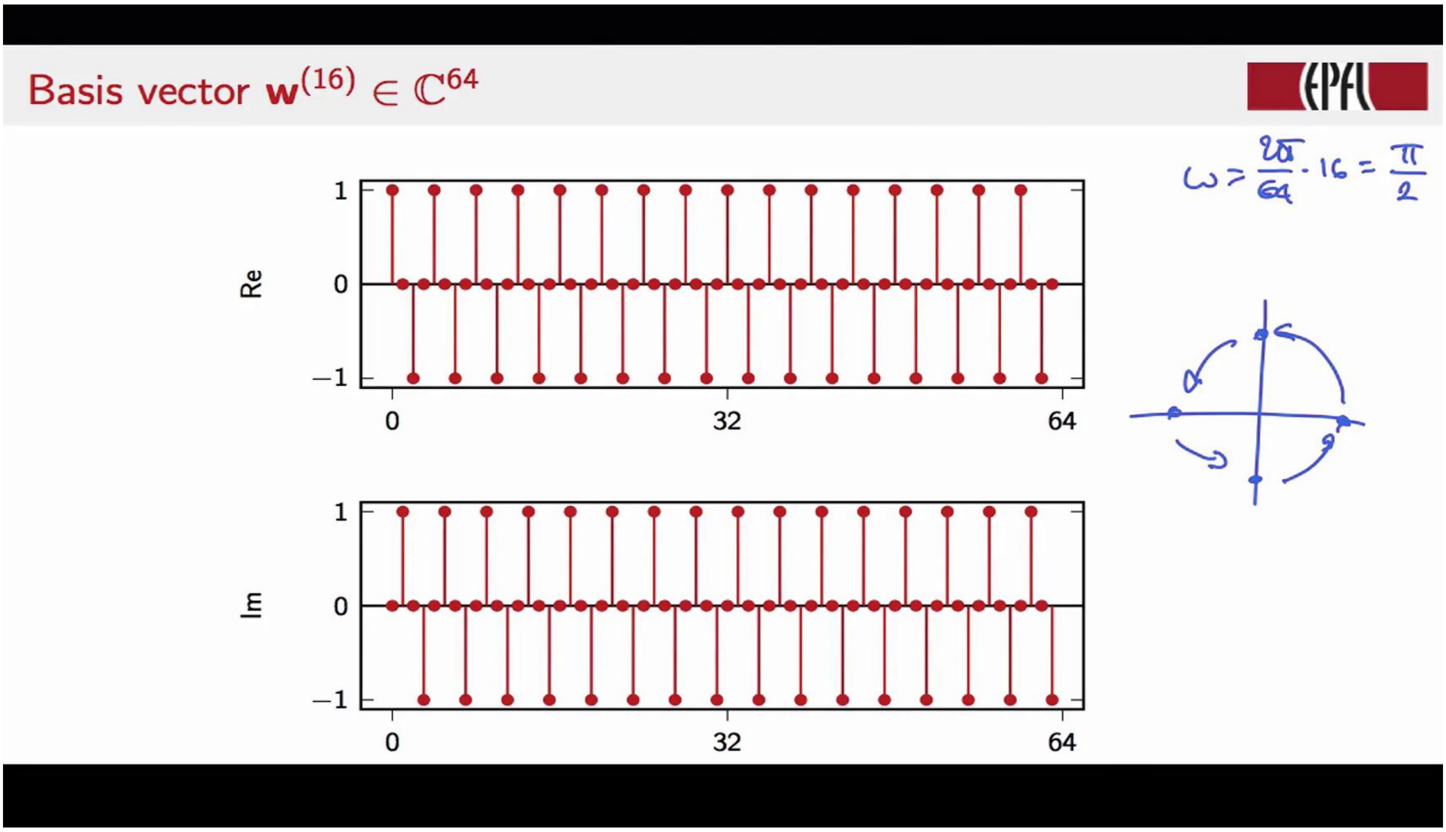


And as we go up with the basis index, the vectors will start moving faster and faster. Because of the discrete nature here, it will be hard at one point to recognise that we have sinusoids.

Notes

Summary

5m 34s



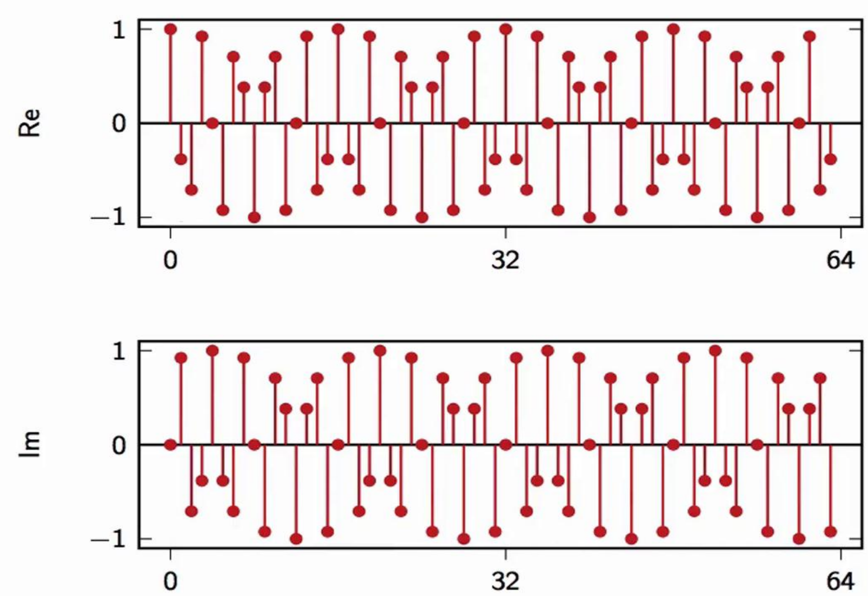
Here for instance, you have w16 that moves very quickly, but you have to imagine that the movement captures a rotation that goes very fast around the unit circle. Here for instance, we have only four points because the angle of the complex exponential turns out to be two Pi over 64 times 16, which is equal to Pi over two. And so here the point will move between these four points, and if you look at the coordinates in the real and imaginary part, you see exactly this behaviour.

Notes

Summary

5m 46s

Basis vector $\mathbf{w}^{(20)} \in \mathbb{C}^{64}$



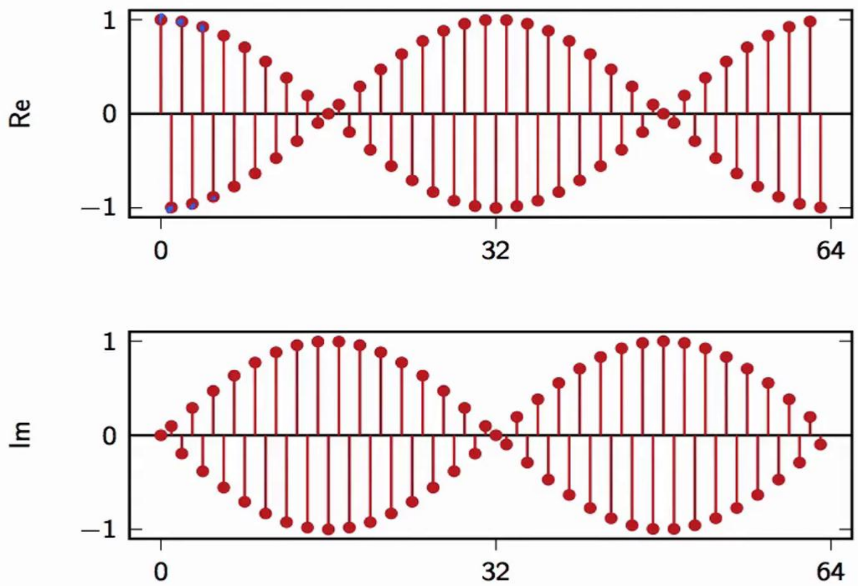
As we increase the index of our Fourier vector, the underlying frequency increases as well. Now, the discrete nature of the vector makes it a little bit hard to understand the sinusodial shape of the signal.

Notes

Summary

6m 19s

Basis vector $\mathbf{w}^{(31)} \in \mathbb{C}^{64}$



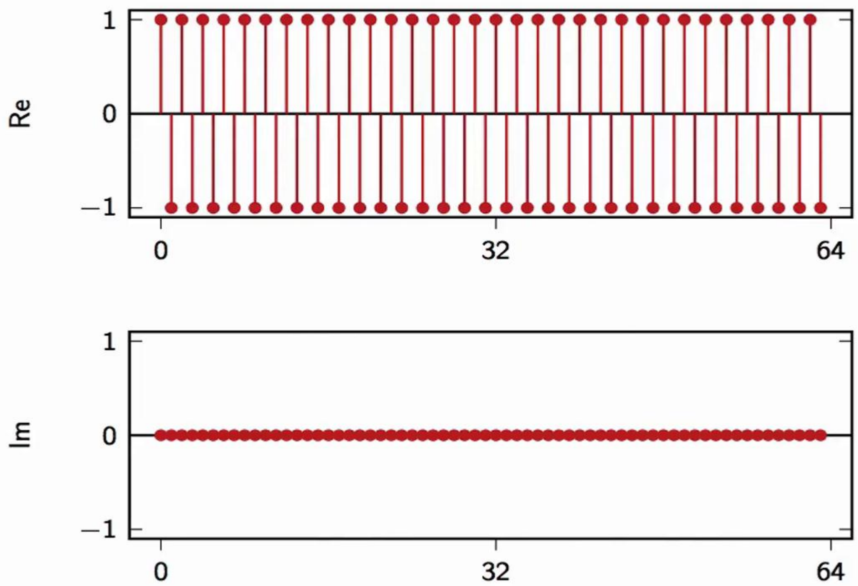
But you can see, for instance here that we are in the high frequency range because the sign of the samples changes at every step. So the sign alternation is a telltale sign of a high-frequency sign zone.

Notes

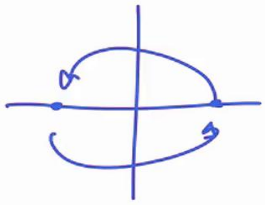
Summary

6m 32s

Basis vector $\mathbf{w}^{(32)} \in \mathbb{C}^{64}$



$\omega = \frac{2\pi}{64} 32 = \pi$



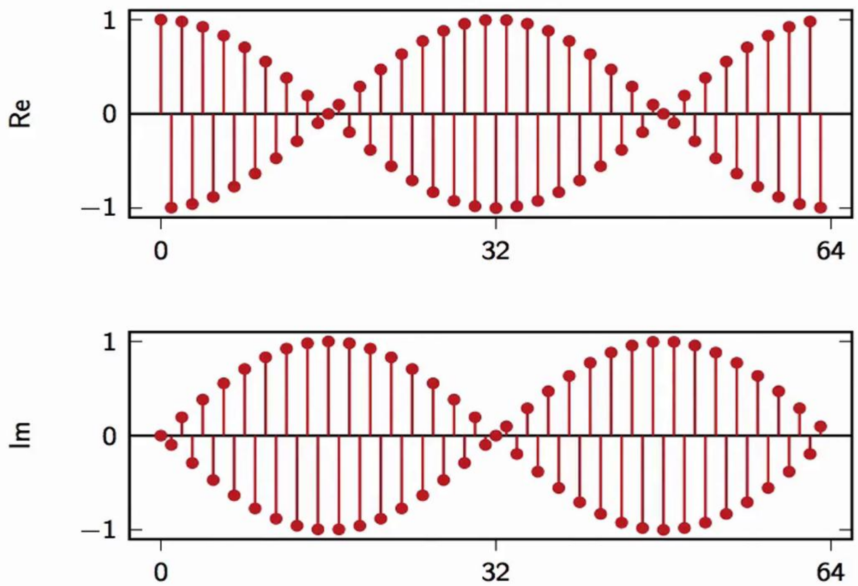
Finally, we reach the highest frequency that discrete time complex exponential can have. So here it happens for k equal to 32, where our frequency becomes two Pi over 64 times 32, which is equal to Pi. At this point we will be moving from this position to this position in the complex plane, and this is clearly indicated by the real and imaginary components of the vector, which are alternating plus and minus one in the real part and identical is zero in the imaginary part.

Notes

Summary

6m 45s

Basis vector $\mathbf{w}^{(33)} \in \mathbb{C}^{64}$



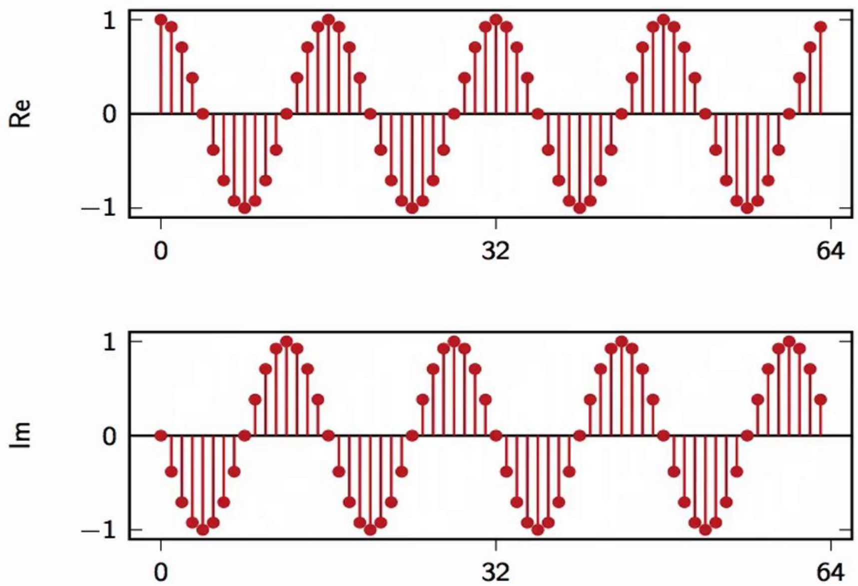
As we go forward the index, as we saw in model two point two, the apparent speed of the point decreases, and the direction of the rotation changes from counterclockwise to clockwise.

Notes

Summary

7m 18s

Basis vector $\mathbf{w}^{(60)} \in \mathbb{C}^{64}$

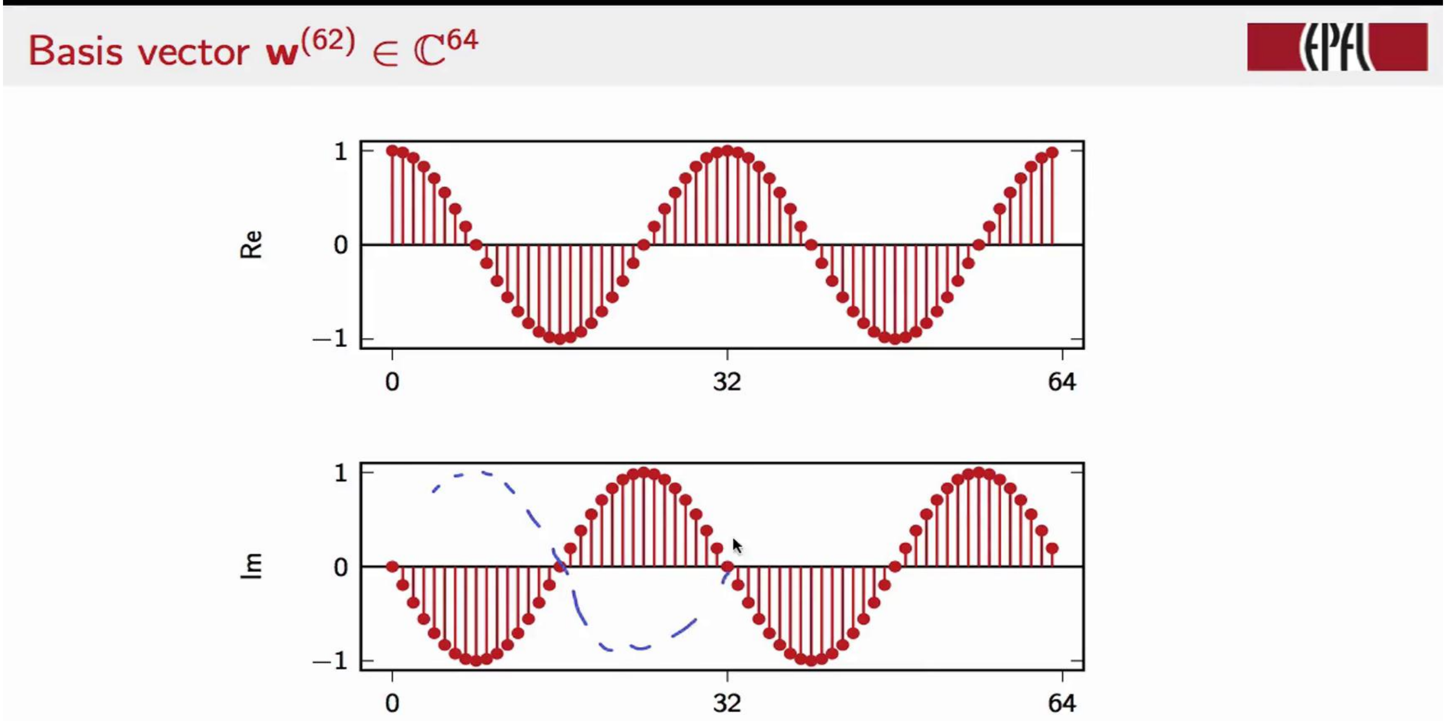


So here, as the index increases from 32 up to 63, we see that the apparent speed of the sinusoid decreases back to zero.

Notes

Summary

7m 31s

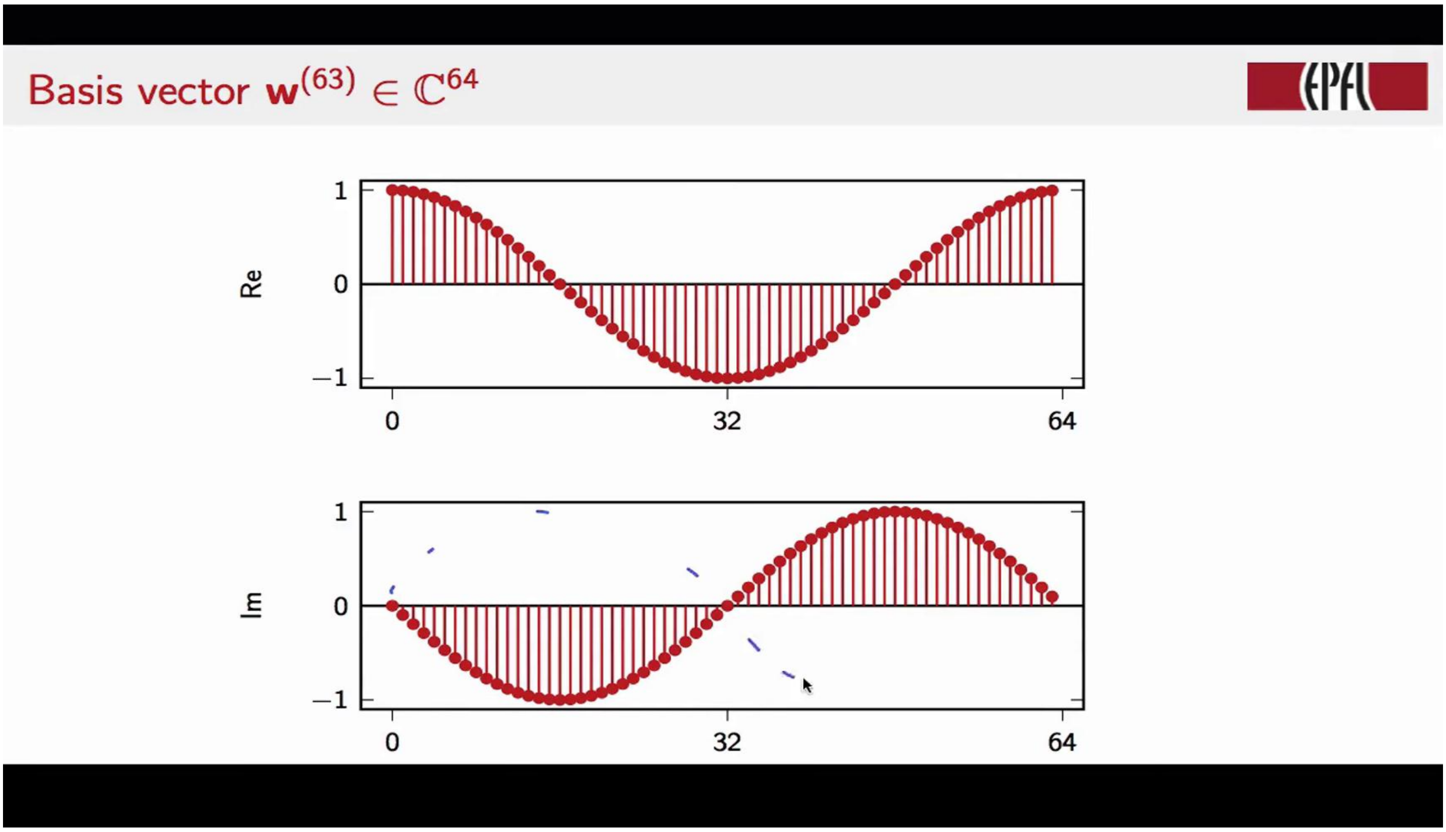


But the sign that we're past Pi as far as the angular increment at each step is concerned, comes from the phase of the imaginary part, which is inverted with respect to the equivalent frequency that we have seen for low values of their index. So here, for instance, if you compare this basis vector to w of two, you would see that the real part is the same, but the imaginary part has a sign inversion. In w of two, it would be like this.

Notes

Summary

7m 43s



Same goes for w63. Real part is equal to w of one, but imaginary part has a sign inversion.

Notes

Summary

8m 11s

Proof of orthogonality

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(h)} \rangle = \sum_{n=0}^{N-1} (e^{j\frac{2\pi}{N}nk})^* e^{j\frac{2\pi}{N}nh}$$

$h = k$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(h-k)n} = 1$$

$$= \begin{cases} N & \text{for } h = k \\ \dfrac{1 - e^{j2\pi(h-k)}}{1 - e^{j\frac{2\pi}{N}(h-k)}} = 0 & \text{otherwise} \end{cases}$$

Okay, so this is all nice and fine. We just showed a very beautiful family of sinusoidal vectors, but in order for these vectors to be a basis of CN, we have to prove that they are orthogonal to each other. Orthogonality requires that the inner product between any two vectors of the family is zero. So we write out the inner product between w of k and w of h. We write out explicitly the inner product by considering each component of the first vector, conjugate the component, and multiply it by the corresponding component of the second vector. And we obtain a sum for n. Small n goes from zero to big N minus one, of ∈ to the j two Pi over big N h minus k times n. So how do we solve this formula. Well, there are two cases. First of all, let's assume h is equal to k. So we are actually taking the inner product of a vector with itself. At this point, all these elements in the sum will be equal to one, and so the sum will converge to n.

Notes

Summary

8m 20s

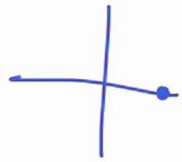
Proof of orthogonality

$$h \neq k \qquad \sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a}$$

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(h)} \rangle = \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}nk}\right)^* e^{j\frac{2\pi}{N}nh}$$

$$= \sum_{n=0}^{N-1} \underbrace{e^{j\frac{2\pi}{N}(h-k)n}}$$

$$= \begin{cases} \boxed{N} & \text{for } h = k \\ \underbrace{\dfrac{1 - e^{j2\pi(h-k)}}{1 - e^{j\frac{2\pi}{N}(h-k)}}} = 0 & \text{otherwise} \end{cases} \qquad (h-k) \in \mathbb{N}$$



In case h is not equal to k, then this is the sum of the form small n goes from zero to big n minus one, a to the n. And we know that this is equal to one minus a to big N, one minus a. So we plug the complex exponential into this formula,and we obtain this ratio. But now this guy here on top is ∈ to the j two Pi times h minus k. This guy is an integer, and therefore this is ∈ to the j times an integer multiple of two Pi. So what that means is that no matter what the value of h minus k is, we are in this point here on the complex plane. And so one minus one will be zero, and this will be zero for all h not equal to k.

Notes

Summary

9m 28s

Remarks

- N orthogonal vectors $\longrightarrow$ basis for $\mathbb{C}^N$
- vectors are not ortho*normal*. Normalization factor would be $1/\sqrt{N}$

$$\langle w^{(h)}, w^{(h)} \rangle = N$$

So the vectors are orthogonal and therefore they form a basis for CN. Now, note that the vectors are not orthonormal, because w of h times w of h is equal to N as we showed before. The normalisation factor will be one over square root of N, but you will see later that we usually keep this normalisation factor explicit rather than implicit for computational reasons.

Notes

Summary

10m 23s