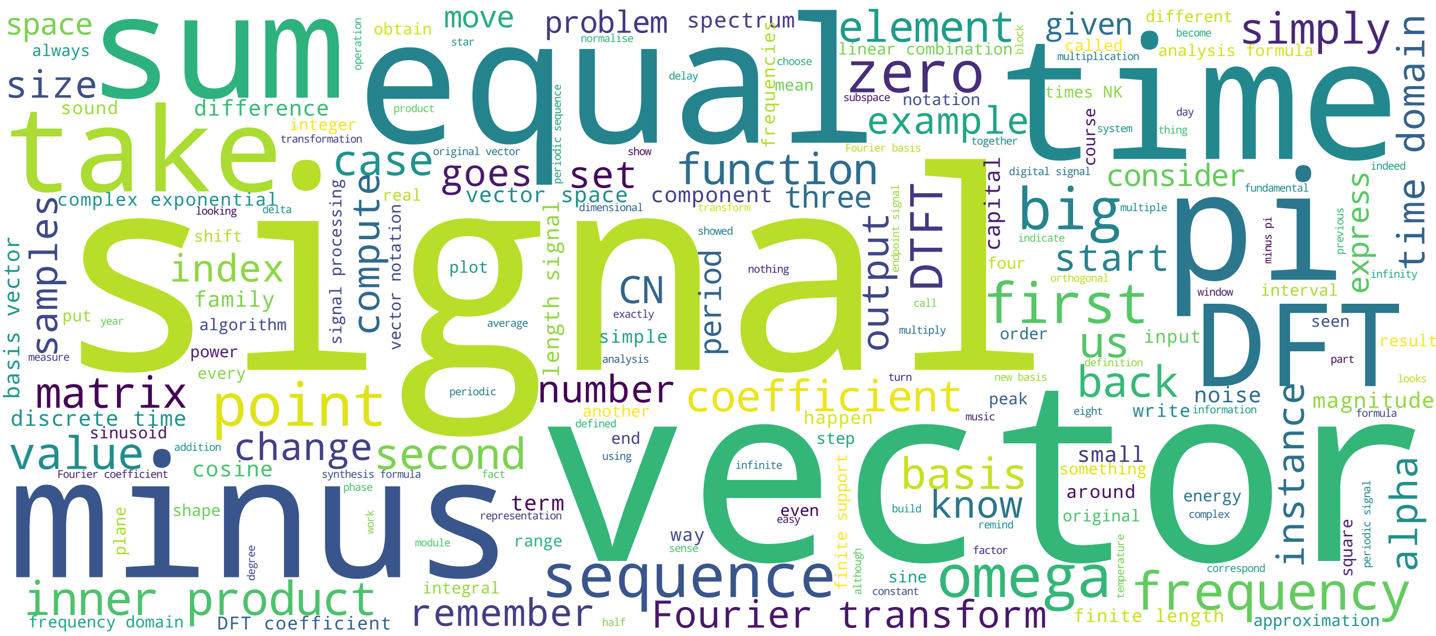
The Fourier Basis for $\mathbb{C}^N$

- in "signal" notation: $w_k[n] = e^{j\frac{2\pi}{N}nk}, \qquad n, k = 0, 1, \ldots, N-1$
- in vector notation: $\{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,N-1}$ with $w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$



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Video

The Fourier Basis for $\mathbb{C}^N$

- N orthogonal vectors $\longrightarrow$ basis for $\mathbb{C}^N$
- vectors are not ortho*normal*. Normalization factor would be $1/\sqrt{N}$
- will keep normalization factor explicit in DFT formulas

In the previous lesson, we introduced a Fourier basis for CN. We showed that it has a dual nature. You can interpret it as a family of signals, finite length signals of length n. This family is indexed by an index k and each element of each signal is given by E to the j two Pi over big N times NK, and both n and K will move from zero to big N minus one. Another way to look at this family of signals is to write them up in vector notation, in which case you will have a family of vectors indexed by K and each element of the vector will be again E to the j to Pi over N times NK. So once again, it's a complete isomorphism between the sigma notation and the vector notation. We showed that the vectors, these nNvectors are orthogonal and therefore they constitute a basis for CN. They're not orthnormal, and therefore we will have to normalise the inner products explicitly.

Notes

Summary

0m 01s

Basis expansion

$\bar{x} = \sum x_n \bar{e}^{(k)}$

$\delta[n-k]$

Analysis formula:

$$X_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)}$$

Okay, so we're in CN and given an arbitrary element of CN called it X, we want to express X in this new Fourier basis. Well, so we know that the analysis formula will give us an new coefficient for the vector in the new basis. And each coefficient which we call big X of K will be simply the inner product of X with each vector of the new basis. If we want to go back to the original representation for X, we simply have to sum a scaled version of all the basis functions. And the scaling factor is none else than the coefficient that we found in analysis formula. Please note here that since the basis is not orthnormal, we will have to normalise somewhere. And we choose to normalise in the synthesis formula by dividing the sum by big N. Well, so this is the Fourier transform wall. Think about it. We usually assume that a signal X lives in the time domain. In vector notation that means that we can express X as the sum of each component of the vector X of K times the Kth canonical vector Ek. And Ek is nothing but a delta in n minus K. So each canonical vector does nothing but extract one precise time instant. Here we're saying that we can express the same vector X as a linear combination not of delta functions, but of sinusoidal functions.

Notes

Summary

1m 00s

Basis expansion

$\bar{x} = \sum x_n \bar{e}^{(k)}$

$\delta[n-k]$

Analysis formula:

$$X_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)}$$

What will change are the coefficient of this expansion and these coefficients are given to us by the analysis formula. So we're moving from the time domain to the frequency domai.

Notes

Summary

2m 29s

Change of basis in matrix form

Define $W_N = e^{-j\frac{2\pi}{N}}$
(or simply W when N is evident from the context)

Change of basis matrix $\mathbf{W}$ with $\mathbf{W}[n,m] = W_N^{nm}$:

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W^1 & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{2(N-1)} \\ & & & \dots & & \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \dots & W^{(N-1)^2} \end{bmatrix}$$

To stress that this is just a linear algebra operation. We can express the change of basis in matrix notation, as we always can in the case of Euclidean spaces. So if we define big W of N as E to the minus j two Pi over n, or simply W when big N is evident from the context, then we can write this big matrix where we put basically the conjugate of each basis vector in each row.

Notes

Summary

2m 39s

Change of basis in matrix form

Analysis formula:

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

Synthesis formula:

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$$

And with this notation, the analysis formula will give us a vector of Fourier coefficients as the product of the matrix W times the signal vector x. Vice versa the synthesis formula will allow us to retrieve the original vector in the canonical basis simply by applying the permision operator to the matrix W. And I remind you that the permision operator is a combination of transposition and conjugation of each element of the matrix. So we multiply the Fourier coefficient vector by this matrix and normalised by one over N, and we get back the original vector.

Notes

Summary

3m 08s

Basis expansion (signal notation)

Analysis formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}, \qquad k = 0, 1, \ldots, N-1$$

N-point signal in the *frequency domain*

Synthesis formula:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}, \qquad n = 0, 1, \ldots, N-1$$

N-point signal in the *"time" domain*

A third way of looking at the discrete Fourier transform is to consider explicitly the operations involved in the transformation. In this case, we will consider signals explicitly. And this is a notation that is particularly useful if we want to consider the algorithmic nature of the transform. So here we will consider that the frequency domain signal that we obtain after the transformation is obtained as the sum for small n that goes from zero to big n minus one of each element of the signal vector times E to the minus j to Pi over big N times NK. So what we're performing here is the inner product in an explicit fashion. We move from an endpoint signal in the time domain to an end point signal in the frequency domain. The synthesis formula is the dualof that with the only difference that here we have to remember to put the normalisation coefficient in front. And so from an endpoint signal in the frequency domain, we go back to an endpoint signal in the time domain. Maybe this is a good point to remind you that although we talk about time domain, in the discrete time world, time is really a dimensional. So it won't be time in the sense of seconds or any other physical unit of measure. It will be just an index that moves over the range of integers.

Notes

Summary

3m 48s