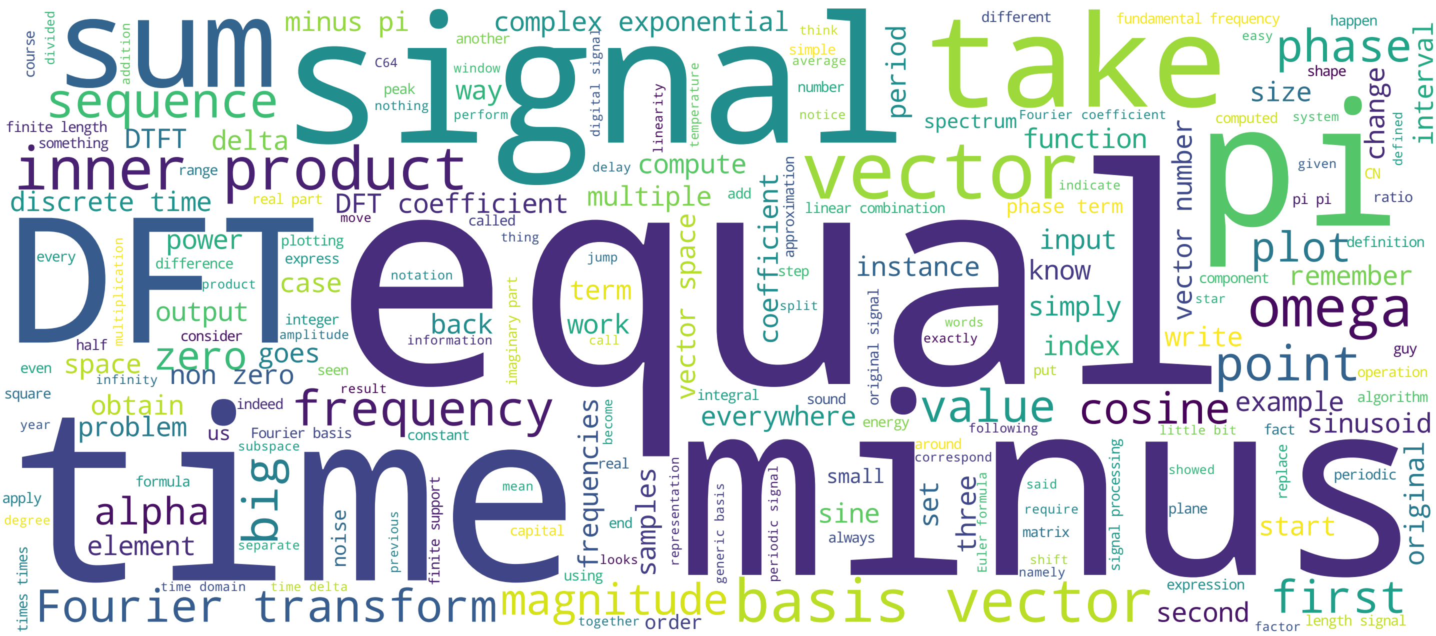
DFT is obviously linear

$$\text{DFT}\{\alpha\, x[n] + \beta\, y[n]\} = \alpha\, \text{DFT}\{x[n]\} + \beta\, \text{DFT}\{y[n]\}$$



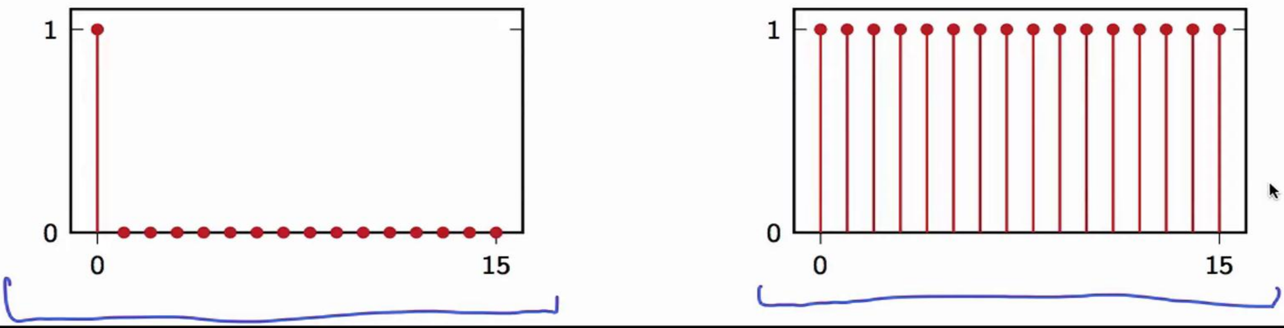
Search MOOC

Video

DFT of $x[n] = \delta[n], \quad x[n] \in \mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} \delta[n]\, e^{-j\frac{2\pi}{N}nk}$$

$$= 1$$



Let's look at some examples, and one property that we will use in the following, which is obvious from the definition of inner products, is that the DFT, —the discreet Fourier transform— is a linear operator. If you have the DFT of the sum of two vectors, this will be equal to the sum of DFTs, and the same goes if you have a scalar multiplication. The first example is the DFT of the discrete time delta. You can see the delta here in the time domain. It is 1 and 0, and 0 everywhere else over the support of the signal. Now, if you work out the inner product of the discrete time delta with a Fourier basis vector, you will see that the delta will isolate the zeroth component of each basis vector, so the product will be 1 for all values of the index k. And so the Fourier transform of the discrete time delta is the constant 1. In other words, the delta contains all frequencies over the possible range of frequencies.

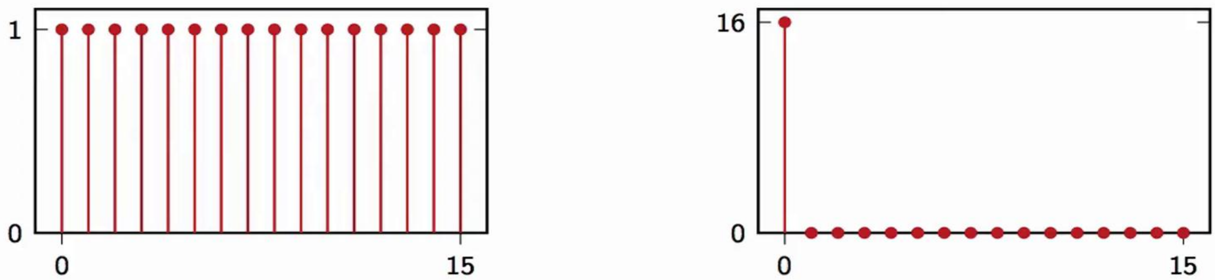
Notes

Summary

0m 01s

DFT of $x[n] = 1, \quad x[n] \in \mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk}$$
$$= N\delta[k]$$



The dual example of that is the Fourier transform of the unit function, so you have 1 for all values of the time index, and if you work out the inner product, you will see that the Fourier transform corresponds to a delta. This is quite intuitive because this is nothing else than the orthogonality formula that we showed before, so it will be equal to big N in 0, and 0 everywhere else.

Notes

Summary

1m 00s

DFT of $x[n] = 3\cos(2\pi/16\, n)$, $x[n] \in \mathbb{C}^{64}$

$N = 64$

$\omega = \frac{2\pi}{64}$

$$x[n] = 3\cos\left(\frac{2\pi}{16}\, n\right)$$

$$= 3\cos\left(\frac{2\pi}{64}\, 4\, n\right)$$

$$= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n}\right]$$

$$= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n}\right]$$

$$= \frac{3}{2}(w_4[n] + w_{60}[n])$$

Now let's try to compute the DFT of a sinusoidal function. We take x of n to be 3 times the cosine of 2 pi over 16 times n. We are in C64, so N, the dimension of our space, is 64 and the fundamental frequency of our Fourier transform will be omega equal to 2 pi over 64. All frequencies in the Fourier basis will be a multiple of this.

Notes

Summary

1m 24s

DFT of $x[n] = 3\cos(2\pi/16\, n), \quad x[n] \in \mathbb{C}^{64}$

$$
\begin{aligned}
x[n] &= 3\cos\left(\frac{2\pi}{16}\, n\right) \\
&= 3\cos\left(\frac{2\pi}{64}\, 4\, n\right) \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n}\right] \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n}\right] \\
&= \frac{3}{2}\left(w_4[n] + w_{60}[n]\right)
\end{aligned}
$$

$$\cos\omega = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

$$-j\frac{2\pi}{64}4n + j2\pi n$$

With this in mind, we can start by expressing the frequency of our sinusoid as a multiple of the fundamental frequency for the space. So 2 pi over 16 will be equal to 2 pi over 64 times 4. With this now, we can use Euler's formula that says that cosine of omega is equal to e to the j omega plus e to the minus j omega divided by two, and express the original cosine as the sum of two complex exponentials, namely e to the j 2 pi over 64 times 4 times N plus E to the minus j 2 pi over 64 times 4 times N. Now, we don't like this minus here, so what we're going to do is exploit the fact that we can always add an integer multiple of 2 pi to the exponent of a complex exponential and the point will not change on the complex plane. So minus j 2 pi over 64 times 4 times n plus j 2 pi n will be equal to e to the j 2 pi over 64 times 60 times n, because n is an integer.

Notes

Summary

1m 50s

DFT of $x[n] = 3\cos(2\pi/16\, n), \quad x[n] \in \mathbb{C}^{64}$

$$
\begin{aligned}
x[n] &= 3\cos\left(\frac{2\pi}{16}\, n\right) \\
&= 3\cos\left(\frac{2\pi}{64}\, 4\, n\right) \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n}\right] \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n}\right] \\
&= \frac{3}{2}(w_4[n] + w_{60}[n])
\end{aligned}
$$

Now we have expressed the original signal as a sum of two of the Fourier basis vectors. So x of n will be equal to 3 over 2 times basis vector number 4 plus basis vector number 60. At this point, we can apply the DFT as an inner product with each vector in the basis, and we can exploit the linearity of the operator.

Notes

Summary

3m 00s

DFT of $x[n] = 3\cos(2\pi/16\, n), \quad x[n] \in \mathbb{C}^{64}$

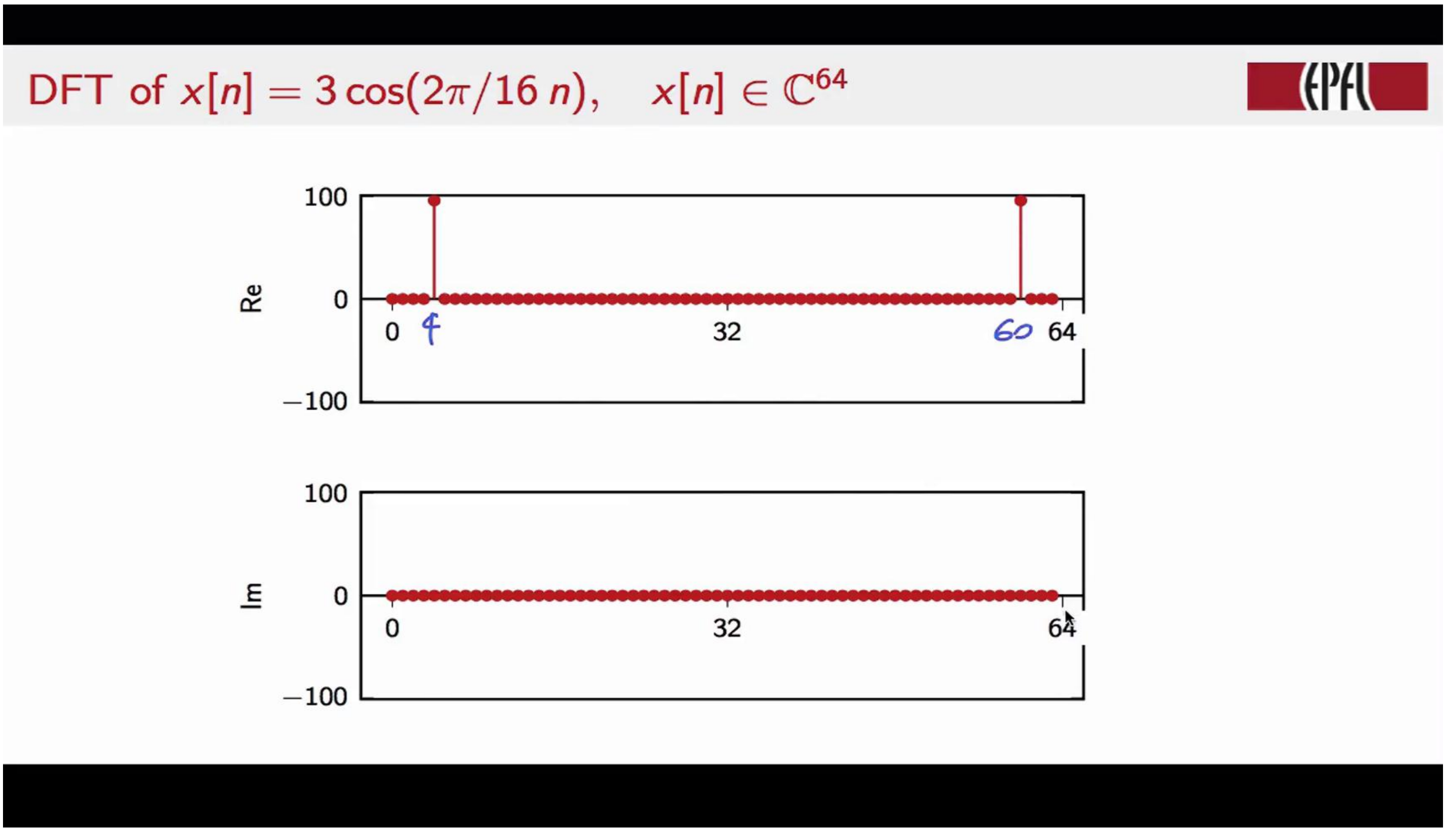
$$\begin{aligned} X[k] &= \langle w_k[n], x[n] \rangle \\ &= \langle w_k[n], \frac{3}{2}(w_4[n] + w_{60}[n]) \rangle \\ &= \frac{3}{2}\langle w_k[n], w_4[n] \rangle + \frac{3}{2}\langle w_k[n], w_{60}[n] \rangle \\ &= \begin{cases} 96 & \text{for } k = 4,\ 60 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

So Xk, the kth Fourier coefficient, will be equal to the inner product between basis vector number k and r signal. If we replace the signal with what we just computed before, we have this formula here. Because of linearity, we can split this into two distinct inner products, and we have 3 over 2 times the inner product of the generic basis vector times basis vector number 4 plus 3 over 2 times the inner product of the generic basis vector with basis vector number 60. But now we know that this guy here will be non-zero only for k equal to 4, and this guy here will be non-zero only for k equal to 60. When k is equal to 4, this will be equal to big N, namely 64, and when k is equal to 60, this will be equal again to 64. If we work out the arithmetic, we get that the DFT of our original signal is equal to 96 if k is equal to 4, or is equal to 60, and 0 otherwise.

Notes

Summary

3m 23s



We can plot the DFT of our cosine, and it will be 0 everywhere in the real part except in 4 and in 60, whereas the imaginary part will be 0 everywhere.

Notes

Summary

4m 34s

DFT of $x[n] = 3\cos(2\pi/16\, n + \pi/3)$, $\quad x[n] \in \mathbb{C}^{64}$

$$
\begin{aligned}
x[n] &= 3\cos\left(\frac{2\pi}{16}\, n + \frac{\pi}{3}\right) \\
&= 3\cos\left(\frac{2\pi}{64}\, 4\, n + \frac{\pi}{3}\right) \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} e^{j\frac{\pi}{3}} + e^{-j\frac{2\pi}{64}4n} e^{-j\frac{\pi}{3}}\right] \\
&= \frac{3}{2}\left(e^{j\frac{\pi}{3}}\, w_4[n] + e^{-j\frac{\pi}{3}}\, w_{60}[n]\right)
\end{aligned}
$$

Okay, so let's try another example. Here we take the same cosine as before, but we add an initial phase. We take 3 times the cosine of 2 pi over 16 times n plus pi over 3. The first thing we do is to express the frequency of the cosine as a multiple of the fundamental frequency for the space, —we're still in C64— then we use Euler's formula and we decompose the cosine as the sum of two conjugate complex exponentials. We can express the phase term. We can take it out of the complex exponential as a complex scalar multiplicative factor. Then we can use the same trick as before to bring this negative frequency into the 0-63 range, and we obtain that our signal is simply 3 over 2 times the sum of basis vector number 4 multiplied by e to the j pi over 3 plus basis vector number 60 multiplied by e to the minus j pi over three.

Notes

Summary

4m 48s

DFT of $x[n] = 3\cos(2\pi/16\, n + \pi/3), \quad x[n] \in \mathbb{C}^{64}$

$$X[k] = \langle w_k[n], x[n] \rangle$$
$$= \begin{cases} 96 e^{j\frac{\pi}{3}} & \text{for } k = 4 \\ 96 e^{-j\frac{\pi}{3}} & \text{for } k = 60 \\ 0 & \text{otherwise} \end{cases}$$

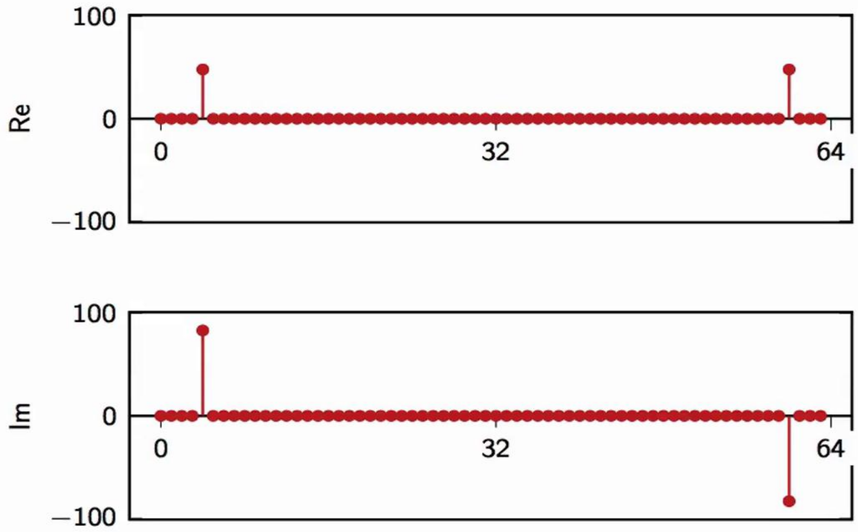
And when we compute the DFT, we take the inner products of that with the generic basis vector for the Fourier basis, and just like before, we obtain just two non-zero values. When k is equal to 4, the first term will survive, giving us a DFT value of 96 times e to the j pi over 3, and when k is equal to 60, we will get 96 e to the minus j pi over 3, and 0 everywhere else.

Notes

Summary

5m 51s

DFT of $x[n] = 3\cos(2\pi/16\,n + \pi/3)$, $\quad x[n] \in \mathbb{C}^{64}$



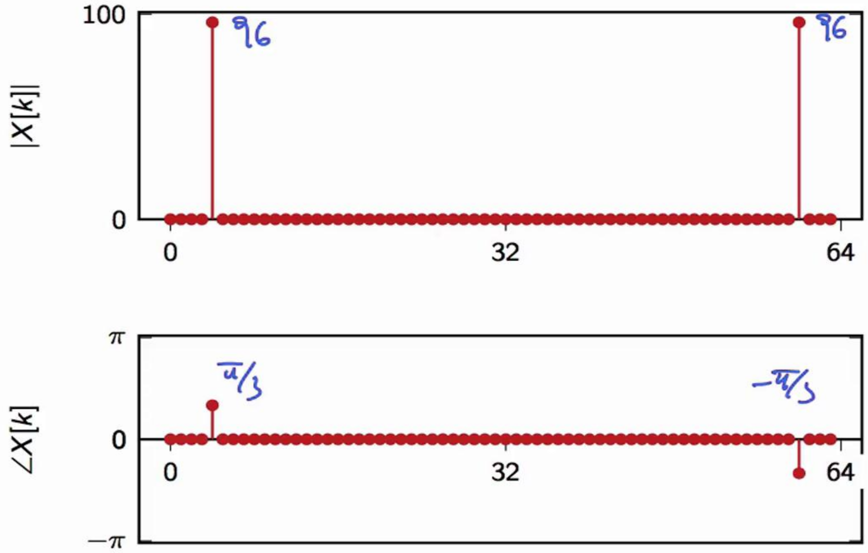
We can plot the Fourier transform like before, and we will have —in this case— non-zero values both in the real and in the imaginary part. Probably a clearer way of plotting this Fourier transform, however, is to plot both the magnitude and the phase of the Fourier coefficients.

Notes

Summary

6m 20s

DFT of $x[n] = 3\cos(2\pi/16\, n + \pi/3)$, $\quad x[n] \in \mathbb{C}^{64}$



If you remember what we just computed, the magnitude will be equal to 96 both for k equal to 4 and for k equal to 60, and the phase will be equal to pi over 3 for k equal to 4 and minus pi over 3 for k equal to 60. This way of plotting the data shows us that we can use the Fourier transform to separate the amplitude of a sinusoid from its phase. I'm sure that by now, most of you will be saying, "Okay, but what about sinusoids whose frequency is not a multiple of the fundamental frequency for the space?

Notes

Summary

6m 36s

DFT of $x[n] = 3\cos(2\pi/10\, n), \quad x[n] \in \mathbb{C}^{64}$

$$\frac{2\pi}{64}6 < \frac{2\pi}{10} < \frac{2\pi}{64}7$$

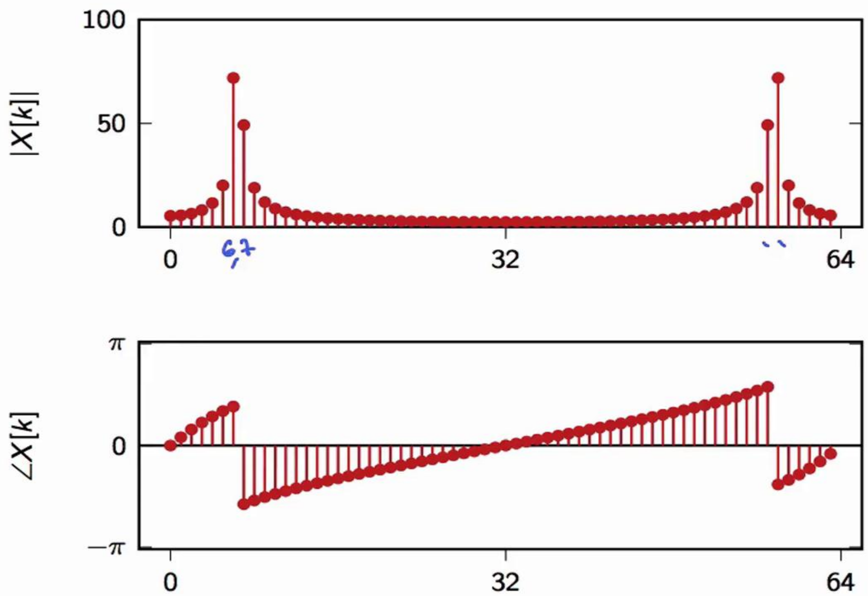
What do we do there?" And indeed, let's say we take a sinusoid with frequency 2 pi over 10, so 3 times the cosine of 2 pi over 10 n. Well, this cannot be reduced to a multiple of 2 pi over 64. It is a frequency that actually falls between 2 pi over 64 times 6 and 2 pi over 64 times 7. What do we do there? Well, we simply compute DFT numerically. Remember, the DFT is an algorithm that requires a finite number of operations and that can be computed with the desired level of precision in numerical packages such as MATLAB, or you can even write your own C code routine to do so. We will see later in module four-nine that very, very efficient algorithms exist to compute the DFT, so this is not a problem.

Notes

Summary

7m 10s

DFT of $x[n] = 3\cos(2\pi/10\, n)$, $x[n] \in \mathbb{C}^{64}$



We run the DFT in a numerical package and we obtain the magnitude and the phase. Now, if you see, a high level of magnitude indicates a similarity of the input with the corresponding basis vector. It's a definition of inner product. So here, unsurprisingly, the input is particularly close to basis vectors number six and number seven, as we showed before, and by extension, to 64 minus 6 and 64 minus 7. But in order to obtain the exact 2 pi over 10 frequency, we need contributions from all the DFT coefficients in the 0-63 range. And similarly, the phase is non-zero for the whole range as well. This is actually a general result. Unless you have an input that is a linear combination of basic vectors, most of your DFT coefficients will be non-zero.

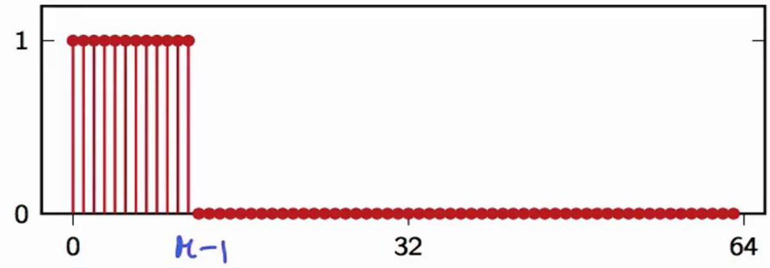
Notes

Summary

7m 59s

DFT of length-M step in $\mathbb{C}^N$

$$x[n] = \sum_{h=0}^{M-1} \delta[n-h], \qquad n = 0, 1, \ldots, N-1$$



Let's, for instance, consider an example that we can compute exactly. Let's consider the DFT of the step vector in CN. We're in CN, and we have a signal that is 1 for n equal to 0 to n minus 1, and then is 0 everywhere else.

Notes

Summary

8m 59s

DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \underbrace{\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}} = \sum_{n=0}^{M-1} e^{-j\frac{2\pi}{N}nk}$$

$$\sum_{n=0}^{M-1} a^n = \frac{1-a^M}{1-a}$$

$$= \frac{1 - e^{-j\frac{2\pi}{N}kM}}{1 - e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{e^{-j\frac{\pi}{N}kM}\left[e^{j\frac{\pi}{N}kM} - e^{-j\frac{\pi}{N}kM}\right]}{e^{-j\frac{\pi}{N}k}\left[e^{j\frac{\pi}{N}k} - e^{-j\frac{\pi}{N}k}\right]}$$

$$= \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

If we compute a DFT, we can just write the inner product explicitly and replace the values for the input signal. Now x of N will be equal to 1 for small n that goes from 0 to big N minus 1, and 0 everywhere else. This simply reduces the sum that originally went from 0 to big N minus 1 to 0, to big M minus 1, and the coefficient for each term will be 1. This is a geometric sum. We have seen that before, right? Sum for n that goes from 0 to big M minus 1 of a to the power of N is equal to 1 minus A to the power of big N divided by 1 minus a.

Notes

Summary

9m 17s

DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{M-1} e^{-j\frac{2\pi}{N}nk}$$

$$= \frac{1 - e^{-j\frac{2\pi}{N}kM}}{1 - e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{e^{-j\frac{\pi}{N}kM}\left[e^{j\frac{\pi}{N}kM} - e^{-j\frac{\pi}{N}kM}\right]}{e^{-j\frac{\pi}{N}k}\left[e^{j\frac{\pi}{N}k} - e^{-j\frac{\pi}{N}k}\right]}$$

$$= \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

$$1 - e^{-j\alpha}$$

$$e^{-j\alpha/2}\left(\underbrace{e^{j\alpha/2} - e^{-j\alpha/2}}_{\downarrow}\right)$$

$$\underbrace{2j\sin\frac{\alpha}{2}}$$

We apply that, we obtain this ratio, and now we're going to perform a little trick. Let me write it here on the side. If we have an expression of the form 1 minus e to the minus j alpha, we can always collect half the phase of this complex exponential and write e to the minus j alpha over 2 that multiplies e to the j alpha over 2 minus e to the minus j alpha over 2. Why do we do this? Well, because this guy here is nothing else than 2 j sine of alpha over 2. We can separate this element into a real valued component and into a phase term.

Notes

Summary

10m 01s

DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{M-1} e^{-j\frac{2\pi}{N}nk}$$

$$= \frac{1 - e^{-j\frac{2\pi}{N}kM}}{1 - e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{e^{-j\frac{\pi}{N}kM}\left[e^{j\frac{\pi}{N}kM} - e^{-j\frac{\pi}{N}kM}\right]}{e^{-j\frac{\pi}{N}k}\left[e^{j\frac{\pi}{N}k} - e^{-j\frac{\pi}{N}k}\right]}$$

$$= \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

We do this for this expression and we collect half the phase here and half the phase here. If we work out the algebra, we get that the DFT coefficients are equal to the ratio of two sine functions time a pure phase term here. Now, I'm not going to read out loud this formula, but we're going to try and plot it, so we're going to try and get some intuition on how this is going to look.

Notes

Summary

10m 49s

DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \boxed{\frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)}}\underbrace{e^{-j\frac{\pi}{N}(M-1)k}}_{1\cdot 1=1}$$

- $X[0] = M$, from the definition of the sum
- $X[k] = 0$ if Mk/N integer $(0 \leq k < N)$
- $\angle X[k]$ linear in k (except at sign changes for the real part)

First of all, if we think about the magnitude of this coefficients, note that this phase term will always be equal to 1 in magnitude because it's a pure phase term, so the magnitude will be determined by the ratio of these two trigonometric functions. In 0, this ratio is ill-defined, but if you go back to the expression for the DFT sum, you can see that if you put k equal to 0, it becomes simply the sum of big M once, so the DFT in 0 is equal to M.

Notes

Summary

11m 18s

DFT of length-M step in $\mathbb{C}^N$

$$X[k] = \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

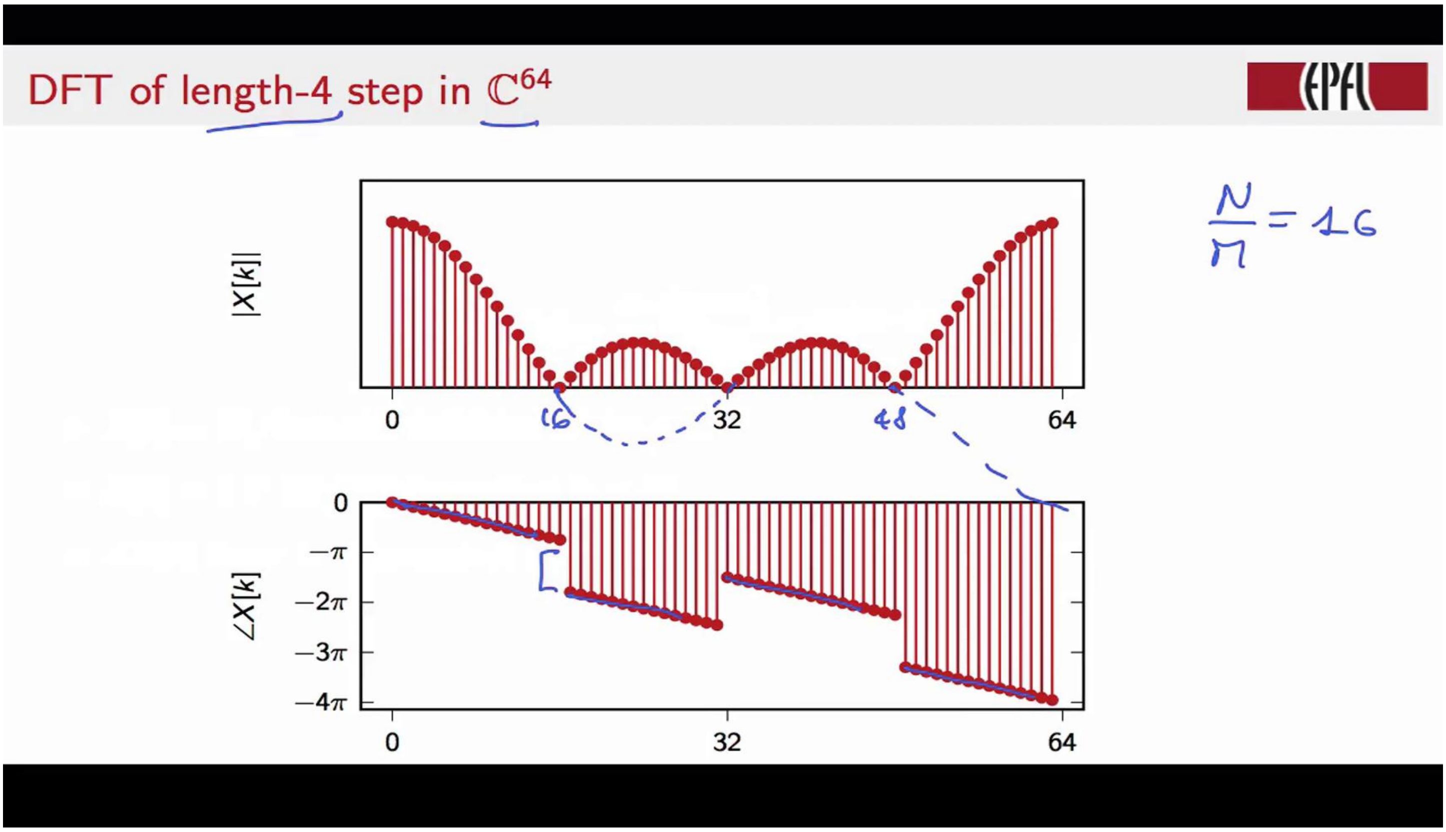
- $X[0] = M$, from the definition of the sum
- $X[k] = 0$ if Mk/N integer $(0 \leq k < N)$
- $\angle X[k]$ linear in k (except at sign changes for the real part)

Another point to notice is that if k is a multiple of N over M, then the numerator here will be equal to 0, and so the DFT coefficient will be equal to 0. Finally, the phase which depends only on this complex exponential here is linear in k, as you can see from the formula, except a sign changes for the real part where we will have a jump of pi.

Notes

Summary

11m 51s



If we plot this in magnitude and phase, we have that. The shape looks like so. We are using a length four-stepped in C64, so N over M will be equal to 16, and indeed, the DFT coefficient will be zero in 16, 32, and 48 in all multiples of 16. The phase is indeed linear in k, and whenever there is a discontinuity in the sine of the real part, we will have a jump of pi. Discontinuity, you can see it here. Here, we're plotting the magnitude of the real part, but in reality, if we were to plot the part with a sine, this would become negative here, go back to positive here, and then go negative here. Notice that here the phase extends from zero to minus 4 pi. This is a little bit unusual. We said that 0 to 2 pi or minus pi 2 pi is the only interval that we're interested in in discrete time, because everything maps back to that interval.

Notes

Summary

12m 16s

Wrapping the phase

Often the phase is displayed "wrapped" over the $[-\pi, \pi]$ interval.

- most numerical packages return wrapped phase
- phase can be unwrapped by adding multiples of 2π

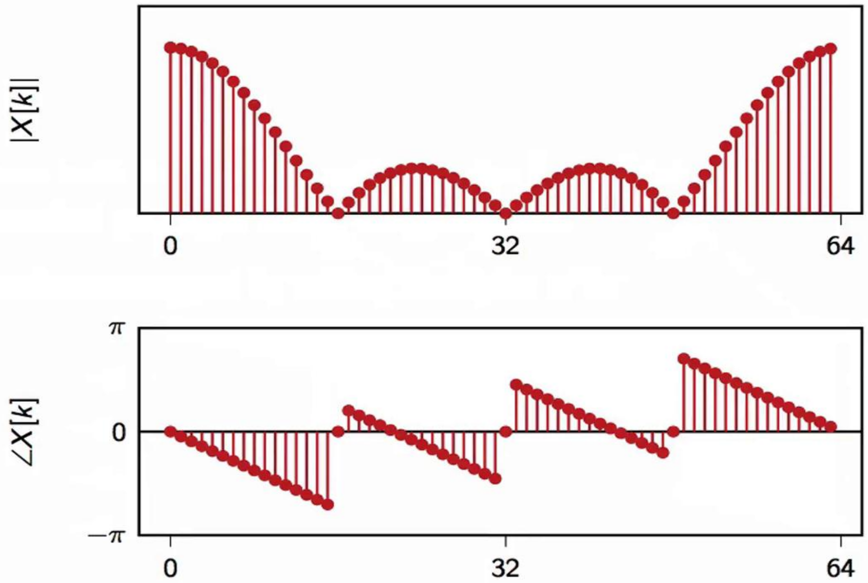
It's very common to see plots where the phase is wrapped over the minus pi-pi interval. What that means, simply, is that for every value of the phase, we add or subtract 2 pi until we fall back into the minus pi-pi integral.

Notes

Summary

13m 19s

DFT of length-4 step in $\mathbb{C}^{64}$ (phase wrapped)



Here's the same plot as before, but with the phase wrapped.

Notes

Summary

13m 33s