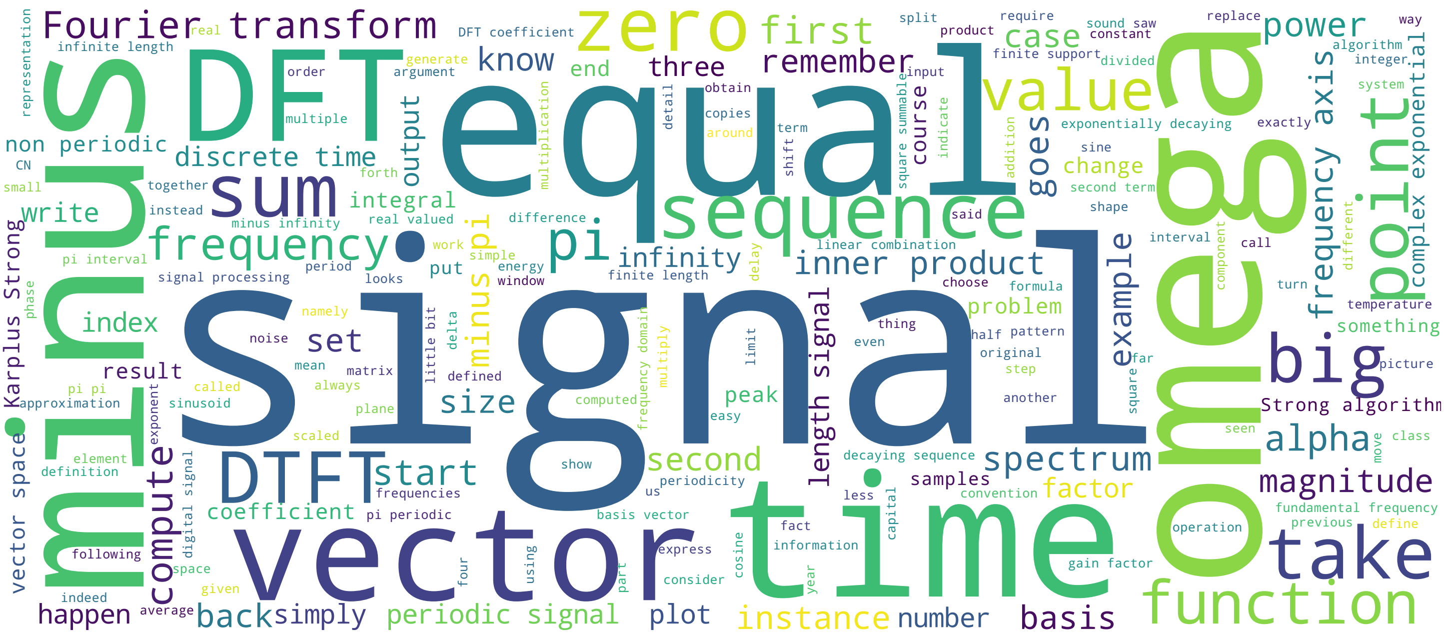
The situation so far

Fourier representation for signal classes:

- N-point finite-length: DFT
- N-point periodic: DFS
- infinite length: ?



Search MOOC

Video

Karplus-Strong revisited, part 2

- consider now $\alpha < 1$
- generated signal is infinite-length but not periodic:

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{1^{st}\text{ period}}, \underbrace{\alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M-1]}_{2^{nd}\text{ period}}, \underbrace{\alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \ldots}_{\ldots}$$

- what is a good spectral representation?

The situation so far is the following: for N-point finite-length signals we will use the DFT. For N-point periodic signals, we will use the DFS. One category is missing and that is the category of infinite length non-periodic signals. We can use the Karplus-Strong Algorithm to generate one such signal simply by putting the gain factor alpha to less than one. In this case, we will have an infinite length signal that is non-periodic, because for every repetition of the n-point sequence, the gain factor will decrease. The first period will be unaffected. The second period will be scaled by alpha. The third period will be scaled by alpha squared and so on, so forth. What is a good spectral representation for the signal?

Notes

Summary

0m 00s

DFT of increasingly long signals

- Start with the DFT. What happens when $N \to \infty$?
- $(2\pi/N)k$ becomes denser in $[0, 2\pi]$...
- In the limit $(2\pi/N)k \to \omega$:

$$\sum_n x[n]\, e^{-j\omega n} \qquad \omega \in \mathbb{R}$$

We can start with the DFT and we can see what happens when the length of the signal goes towards infinity. Intuitively, the fundamental frequency in CN, remember, the fundamental frequency of the Fourier base is in CN, is omega equal to two pi over N. As N goes to infinity, omega becomes smaller and smaller and the set of frequencies in the zero two pi range becomes denser and denser. In the limit, the set of multiples of the fundamental frequency two pi over N, will become so dense that we will try to replace this by a real valued variable omega. This real valid variable will span the zero two pi interval. If we place that in the formulation for the DFT, we get a sum now over all the points in the signal. And instead of having multiples of the fundamental frequency, we have a real variable.

Notes

Summary

0m 54s

Discrete-Time Fourier Transform (DTFT)

Formal definition:

- $x[n] \in \ell_2(\mathbb{Z})$
- define the *function* of $\omega \in \mathbb{R}$

$$F(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- inversion (when $F(\omega)$ exists):

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

Does this make sense mathematically at least? Let's try and give a formal definition of what we call the discrete time Fourier transform. A Fourier transform geared at infinite length, non-periodic signals. We require that our infinite signal is square summable. That means finite energy. It's a reasonable requirement to prevent summations from blowing up. We define now a function of a real valued variable omega. Remember, this is a function, not a sequence. The definition goes like so. F of omega is the sum for N that goes from minus infinity to plus infinity of x of n times e to the minus j omega n. When this value exists, we can actually inverted and retrieve our sequence values by taking the integral from minus pi to pi of F of omega, e to the j omega N in the omega, and normalising the integral by one over two pi. It's easy to verify that this is so if you replace f omega inside here by this definition and invoke the fact that x event n is square summable.

Notes

Summary

1m 47s

DTFT periodicity and notation

DFT
DTFT

- $F(\omega)$ is 2π-periodic
- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

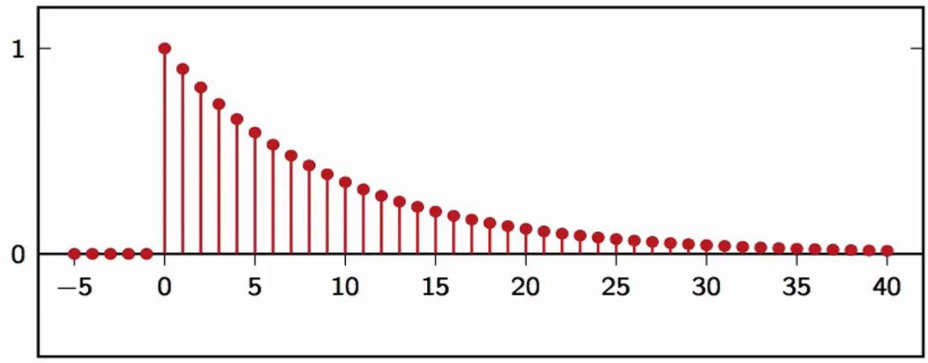
Now F of omega is two pi periodic. We can see that because it depends on e to the minus j omega n and this animal is two pi periodic in omega. To stress the periodicity of the DTFT, instead of writing F of omega for a sequence x of n, we will write the DTFT as x of e to the j omega. Although the free variable is omega, we will write e to the j omega as the argument of the function. This will remind us that this is a Fourier transform and that it is two pi periodic no matter what happens because the argument of the function is two pi periodic. Another difference with respect to DFT, is that for the DFT we choose the minus pi-pi interval as the representative interval for the transform. These are conventions of historic nature, and although they're a little bit confusing, it's particularly unfortunate, for instance, that the difference between DFT and DTFT is just a T here. We will preserve the nomenclature and the conventions in order to be compatible or congruent with the DSP literature out there.

Notes

Summary

3m 03s

$x[n] = \alpha^n u[n], \quad |\alpha| < 1$



So let's look at an example of DTFT. Let's take a simple sequence, an exponentially decaying sequence. We know the sequence from the introduction to this class, and we know that it is a non-periodic infinite support sequence. When alpha is less than one, we also know that this sequence has finite energy, so we can compute the TTFT.

Notes

Summary

4m 18s

DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

We do that. We write the formal definition. This is the sum from minus infinity, plus infinity of x of n times e to the minus j omega n. Here we use the values for the sequence. The sequence is one sided. There's the unit step here. That means that the index of the sum will start in zero. For n equal to zero or larger, the value of xn will be alpha to the power of n. We can collect the two powers under the same exponent, and we can see this is simply a geometric series with ratio alpha e to the minus j omega. The result will be in the end one over one minus alpha e to the minus j omega.

Notes

Summary

4m 39s

DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$|X(e^{j\omega})|^2 = \frac{1}{1 + \alpha^2 - 2\alpha\cos\omega}$$

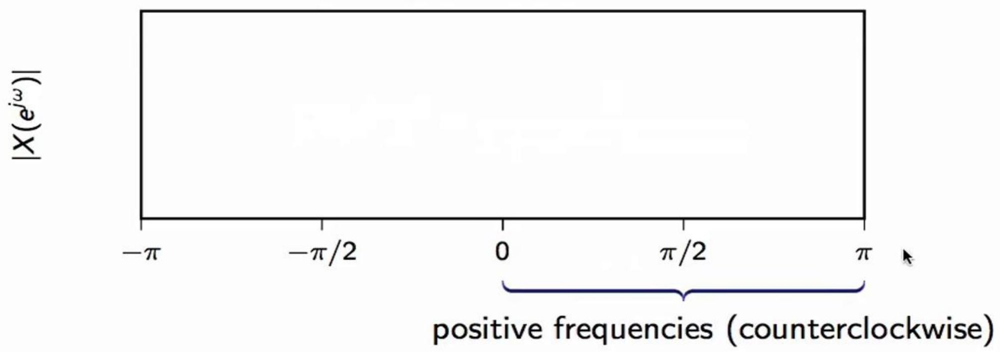
The magnitude of the Fourier transform is easy to compute. It will be one over one plus alpha squared minus two times alpha cosine of omega.

Notes

Summary

5m 20s

Plotting the DTFT



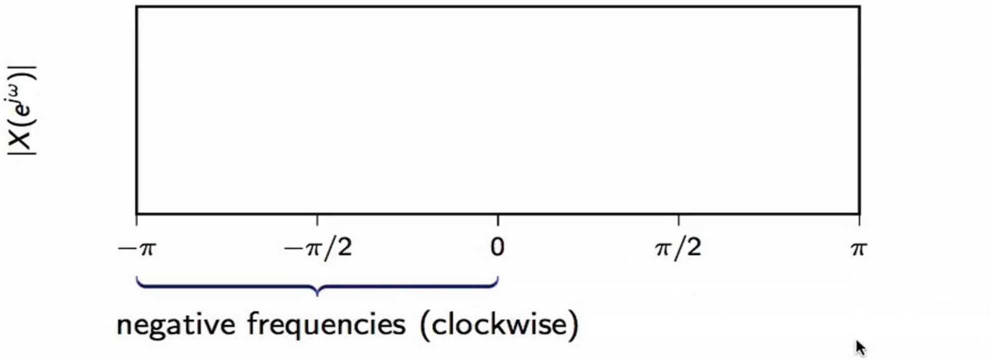
When we plot this, remember now the convention is that we're plotting frequencies from minus pi to pi rather than from zero to pi. So the positive frequencies will be on the right hand side of the frequency axis.

Notes

Summary

5m 31s

Plotting the DTFT

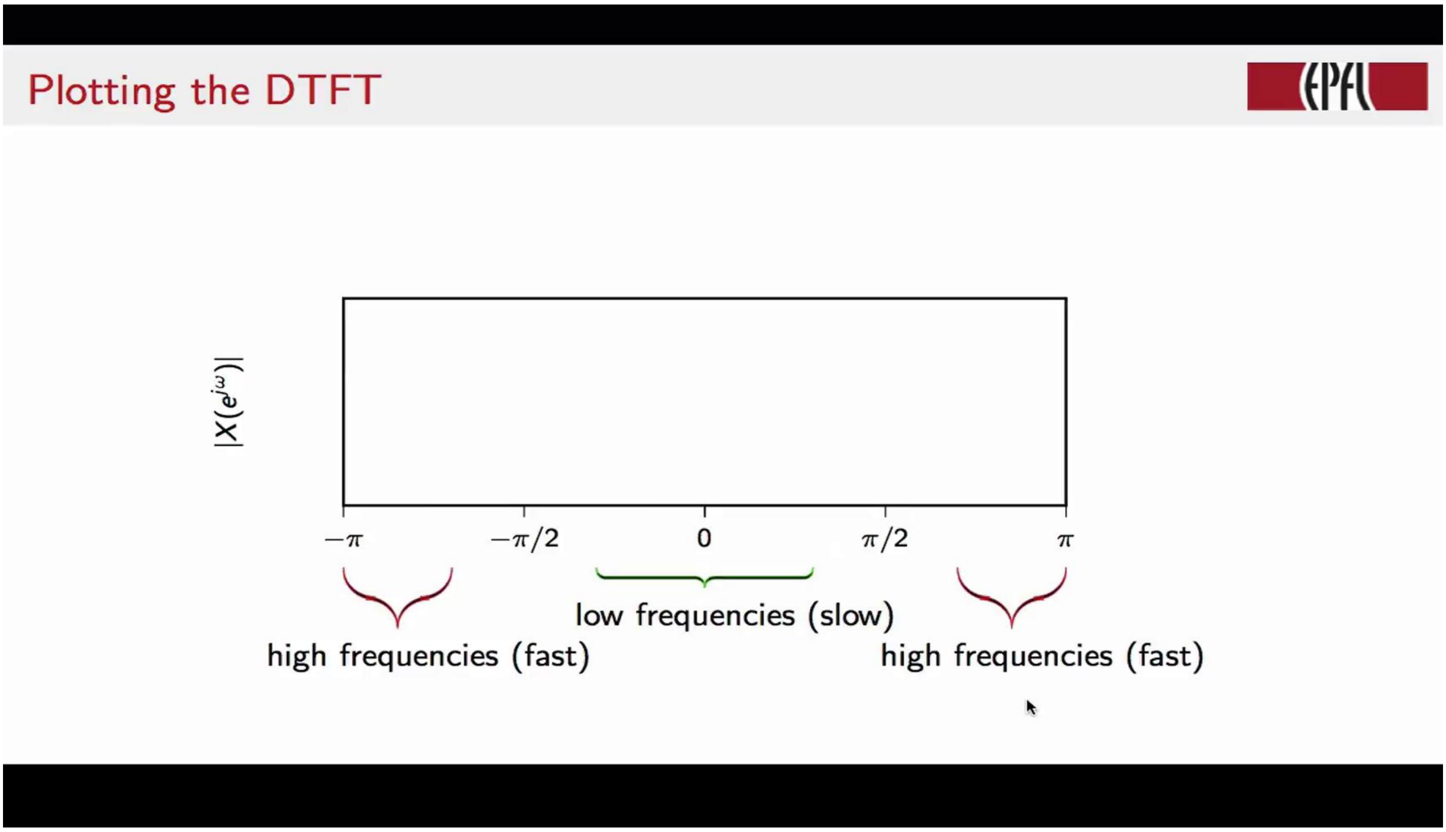


The negative frequencies, the one that turn clockwise will be on the negative side of the frequency axis.

Notes

Summary

5m 44s

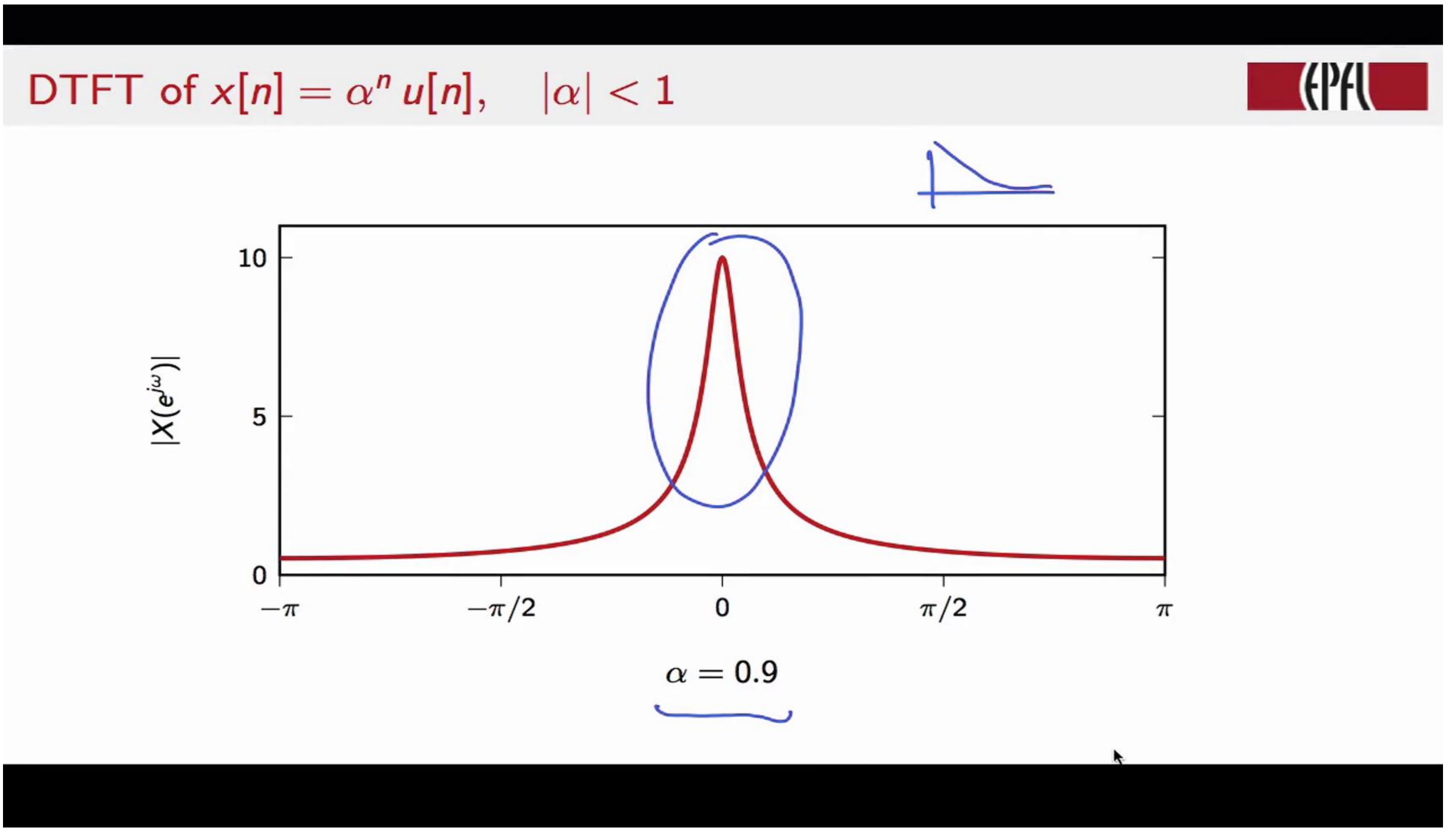


The low frequencies will be now centred around zero, and the high frequencies will be on the extremes of the band.

Notes

Summary

5m 50s



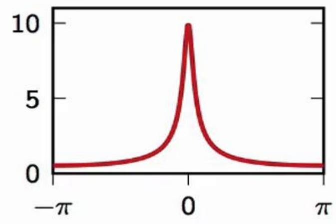
With this conversion in mind, we can look at the spectrum of the exponentially decaying sequence. And it looks like this for say, alpha equal to 0.9. It's a signal that contains energy in the frequency domain, mostly around the low frequencies. Of course, because it moves rather slowly to zero.

Notes

Summary

5m 58s

Remember the periodicity!



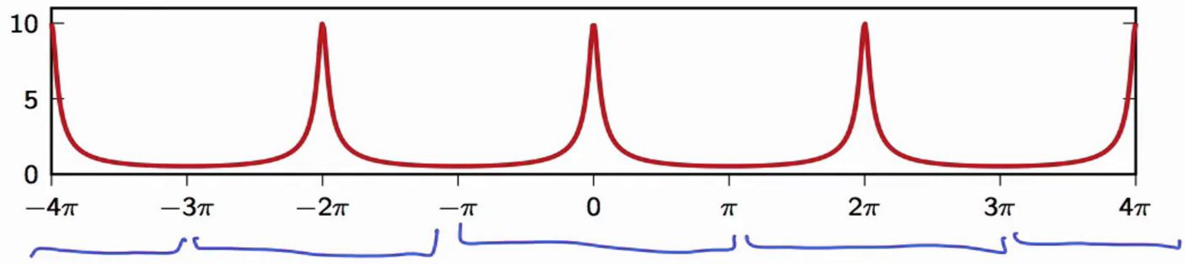
However, never forget the inherent periodicity of the Fourier transform. If we expanded the frequency axis outside of the minus pi-pi interval, and we can certainly do that because omega is a real valid variable, what we find is that the shape of the spectrum repeats outside with a periodicity of two pi.

Notes

Summary

6m 18s

Remember the periodicity!

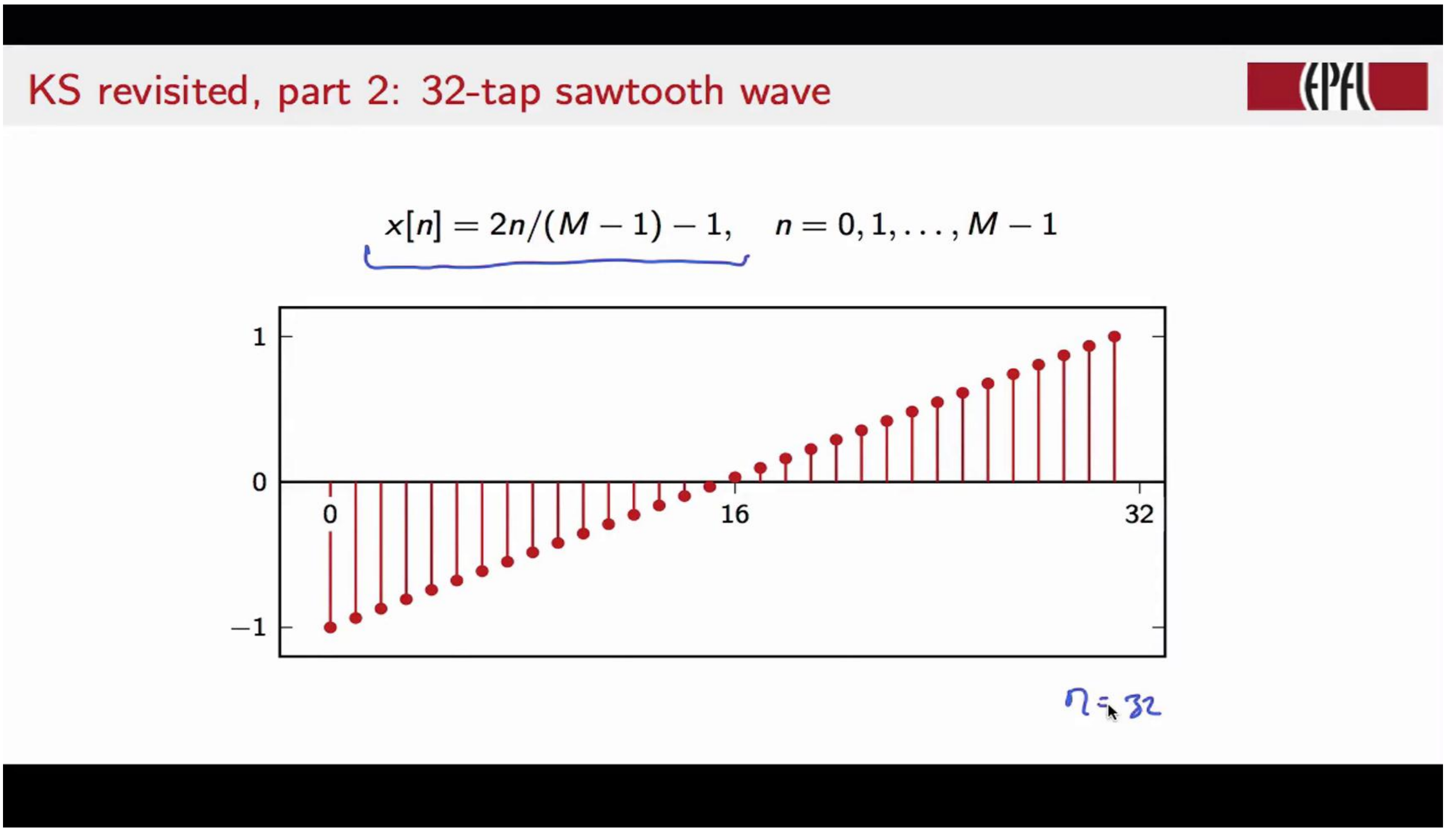


The fundamental interval is this, but as we plot outside, we get further copies of the spectrum. Again, fundamental interval, first, the repetition, second repetition in the positive axis and same thing on the other side.

Notes

Summary

6m 40s



Let's go back to the Karplus-Strong algorithm. Try to generate a non-periodic infinite support signal and then compute its spectrum according to the new DTFT paradigm. We start with the same shape as before for the initial data for this Karplus-Strong algorithm. Since we're going to be precise in the derivation of the spectrum, it is useful at this point to look at the analytical formulation for this shape. So x of n between 0 and 31, is equal to 2n divided by big M minus 1, minus 1. Where big M in this case is 32.

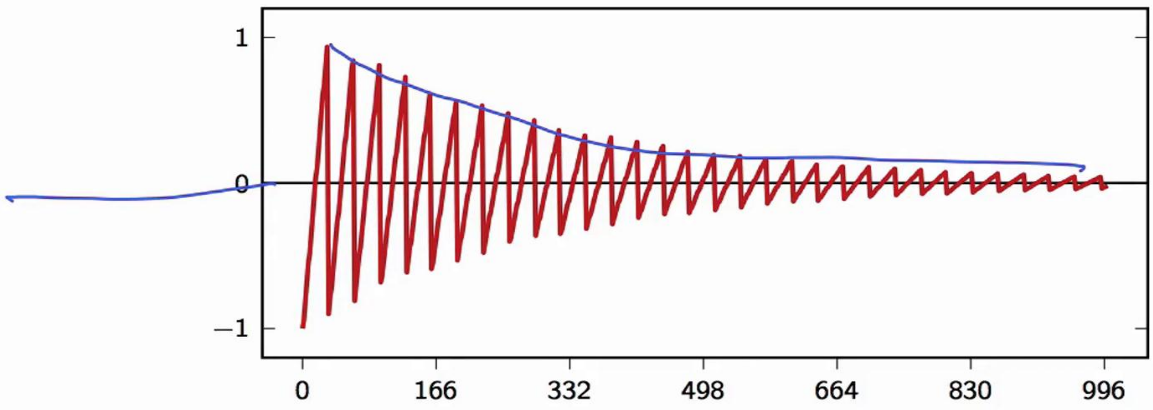
Notes

Summary

6m 55s

KS revisited, part 2: decay $\alpha = 0.9$

$$y[n] = \alpha^{\lfloor n/M \rfloor}\,\bar{x}[n \mod M]\,u[n]$$



If we put a decay factor in the Karplus-Strong algorithm equal to 0.9, you see that the repetitions generated by the karplus-Strong algorithm have an exponential in decaying envelope. And we can express the output as alpha to the power of the floor of n divided by big M, that multiplies the periodic signal obtained by putting copies of the pattern we saw in the previous picture, one after the other. And all of this is multiplied by the units step because we assume that the signal starts at n equal to zero. So zero everywhere here and exponentially decaying sawtooth wave for n greater than zero.

Notes

Summary

7m 35s

DTFT of KS signal

$$
\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n} \\
&= \sum_{p=0}^{\infty} \sum_{n=0}^{M-1} \alpha^{p} \bar{x}[n]\, e^{-j\omega(pM+n)} \\
&= \underbrace{\sum_{p=0}^{\infty} \alpha^{p} e^{-j\omega M p}} \sum_{n=0}^{M-1} \bar{x}[n] e^{-j\omega n} \\
&= A(e^{j\omega M}) \bar{X}(e^{j\omega})
\end{aligned}
$$

Analytically, the Fourier transform of this signal can be computed like so. Remember the definition. So Y of e to the j omega, the DTFT of the signal we just showed in the picture, is equal to the sum for n that goes from minus infinity to plus infinity of the value of the signal in n times e to the minus j omega n. Just like we did before, we split the sum into two parts, an outer sum that spans every quote unquote, repetition of the basic shape. Although, of course, the stand, the signal is not periodic, so each repetition will be scaled by a different factor and an inner sum that goes inside each repetition. Inside each repetition we have the values for the basic pattern and alpha to the power of p and exponentially decreasing gain factor that creates the decaying envelope. We can split the exponent in the complex exponential again as p big M plus n. By doing so, we can split the sum into the product of twosome. The first term is something that looks like a DTFT. Notice that the index is P and we have alpha to the power of P, that multiplies e to the minus j omega big Mp. If it wasn't for this big M here, this would be a discrete time Fourier transform of a sequence that starts in zero and goes to infinity and whose samples take the value alpha to the power of P.

Notes

Summary

8m 20s

DTFT of KS signal

$$
\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n} \\
&= \sum_{p=0}^{\infty} \sum_{n=0}^{M-1} \alpha^p \bar{x}[n]\, e^{-j\omega(pM+n)} \\
&= \sum_{p=0}^{\infty} \alpha^p e^{-j\omega Mp} \sum_{n=0}^{M-1} \bar{x}[n] e^{-j\omega n} \\
&= \underbrace{A(e^{j\omega M})}\underbrace{\bar{X}(e^{j\omega})}
\end{aligned}
$$

Namely, this is the DTFT of an exponentially decaying sequence, except for this factor here. We'll see how to take care of this in a second. The second factor is equal to the DTFT of a signal that is equal to the basic pattern from zero to big N minus one and zero everywhere else. We can write this as the product of two DTFTs. The first one is the DTFT of the exponentially decaying sequence that we saw before. The factor of M in the exponent propagates to the argument of the DTFT, and simply means that we are contracting the frequency axis by a factor of M. Is just the rescaling of the frequency axis. The second term we'll compute in just a second.

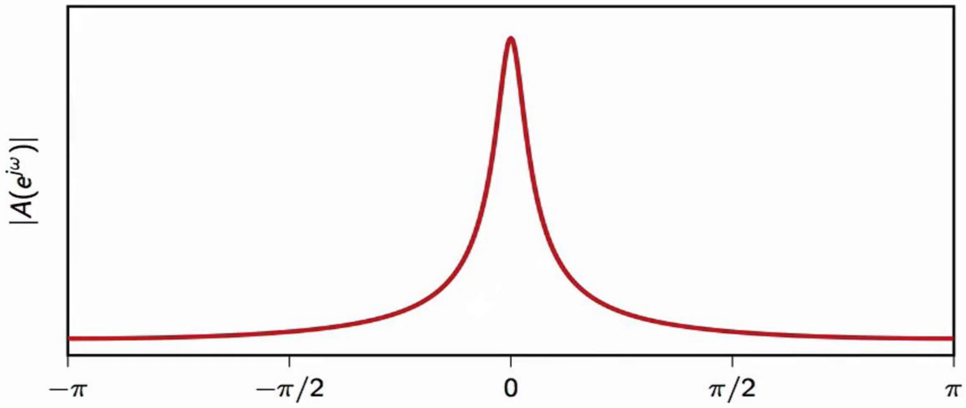
Notes

Summary

9m 51s

We know the first term

$$A(e^{j\omega}) = \text{DTFT}\{\alpha^n u[n]\} = \frac{1}{1-\alpha e^{-j\omega}}$$



So let's look back at what the Fourier transform of the exponential decay looks like.

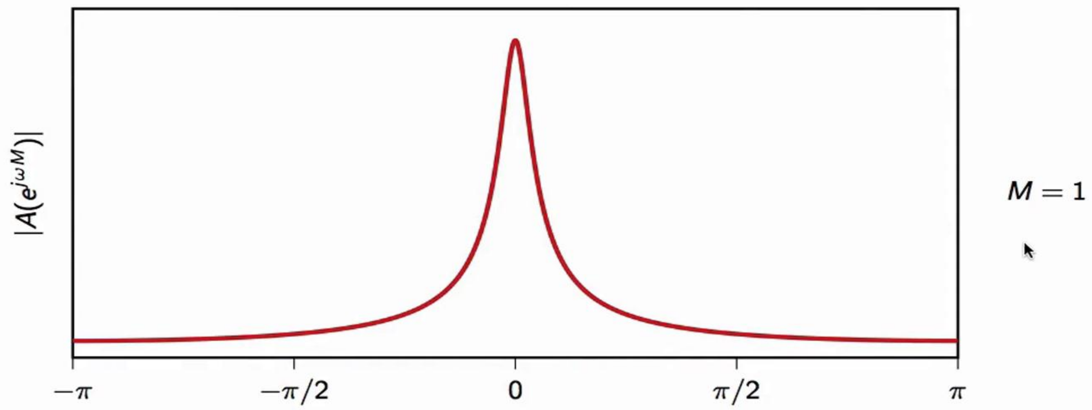
Notes

Summary

10m 35s

We know the first term

$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



The fact that there is a factor of M in the exponent simply implies the rescaling of the frequency axis. So we're contracting the frequency axis by a factor of big M. But be careful because this contraction we'll actually have to take periodicity into account. So when big M is equal to one, we have the standard spectrum of decaying exponential.

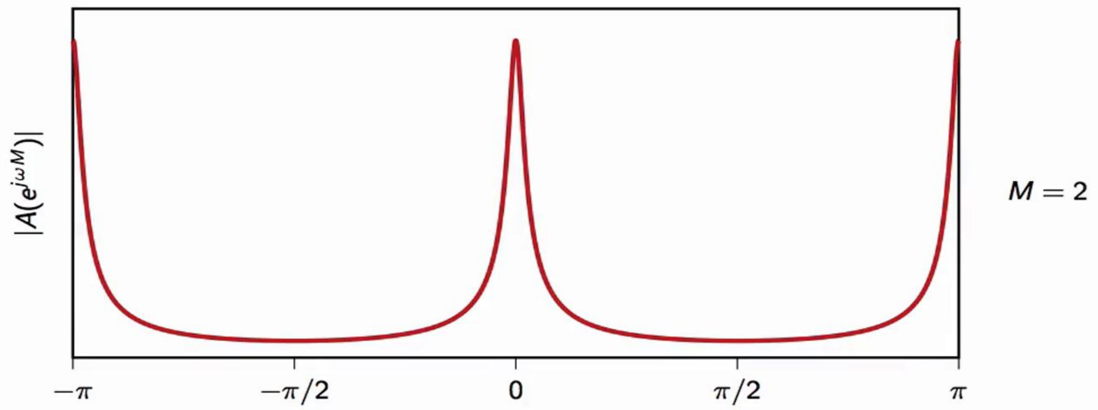
Notes

Summary

10m 42s

We know the first term

$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



When M is equal to two, we're bringing in two copies of the spectrum inside the minus pi-pi interval. What before was periodic over two pi, now becomes periodic over pi.

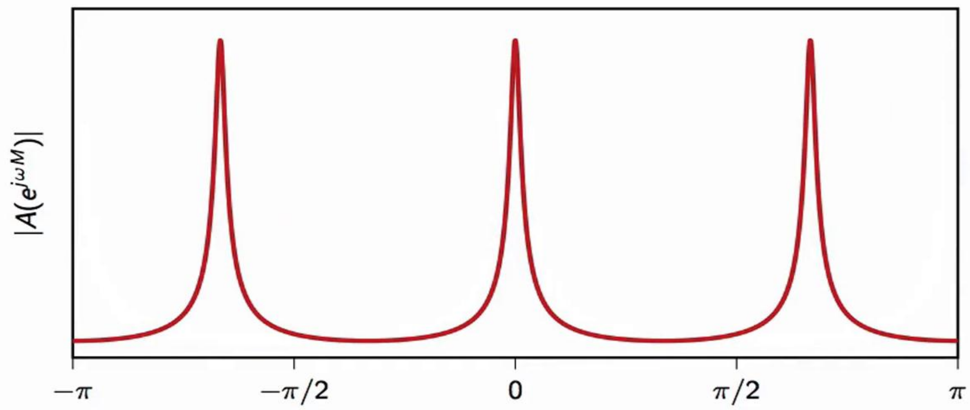
Notes

Summary

11m 02s

We know the first term

$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



$M = 3$

When M is equal to three, you will have three copies of the basic spectrum that fall into the minus pi-pi interval and so on and so forth.

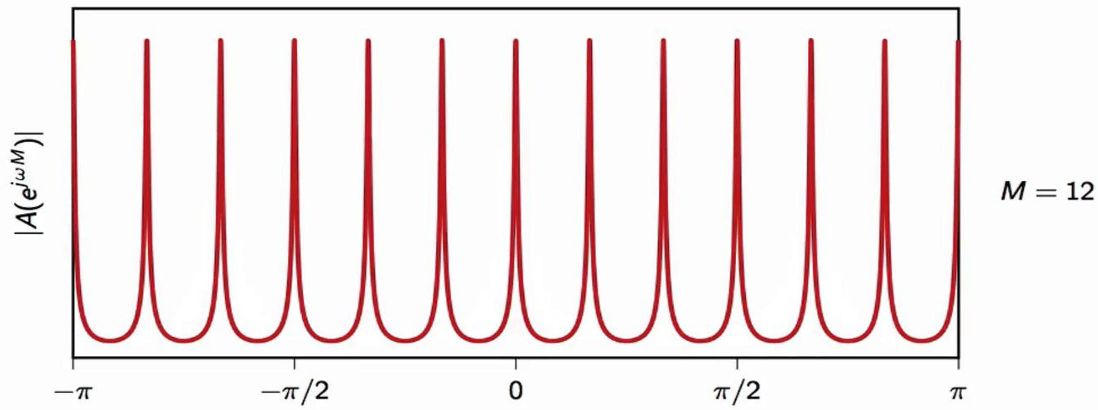
Notes

Summary

11m 16s

We know the first term

$A(e^{j\omega M})$ rescales the frequency axis: periodicity!



This is, for instance, the case for M equal to 12.

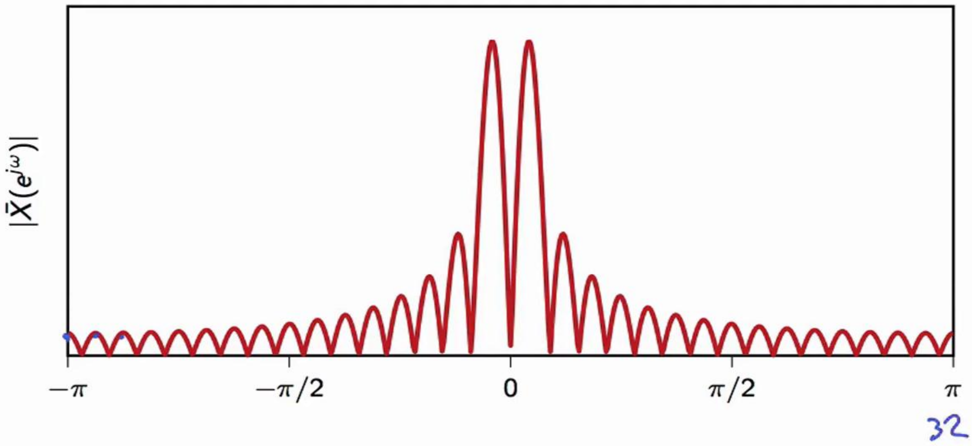
Notes

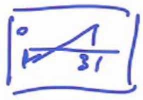
Summary

11m 23s

Second term is left as an exercise

$$\bar{X}(e^{j\omega}) = e^{-j\omega}\left(\frac{M+1}{M-1}\right)\frac{1-e^{-j(M-1)\omega}}{(1-e^{-j\omega})^2} - \frac{1-e^{-j(M+1)\omega}}{(1-e^{-j\omega})^2}$$





Now on to the second term of the product. As we said, that was the DTFT of a finite support sequence that is linear between zero and Big M minus one. The result is a little bit ugly and a little bit cumbersome to derive. So we'll leave that as an exercise and we'll put some slides in the end that detail how to compute this. But fundamentally, this is the result. If you plot the magnitude of this function, you'll have something like this. In the figure, you have the magnitude of the DFT, of the 32 point sawtooth period that we saw before, this is now 0 between 0 and 31. And indeed, if you count the local maxima here, there are 32 of them.

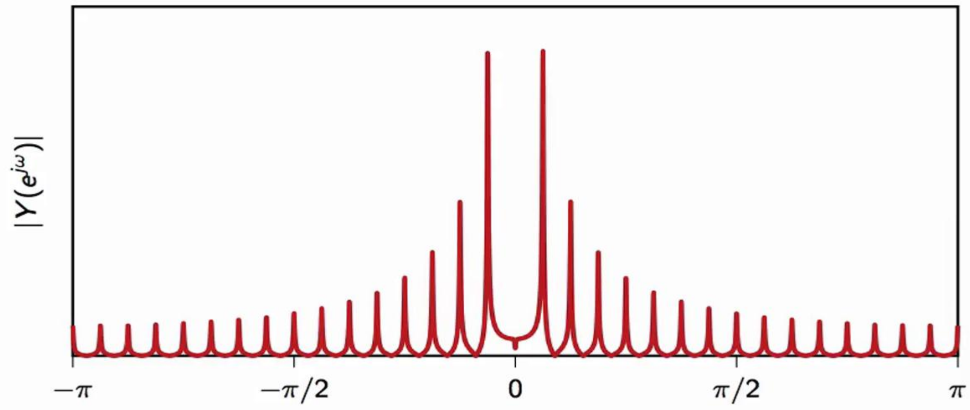
Notes

Summary

11m 27s

DTFT of KS with decay

$$Y(e^{j\omega}) = \underbrace{A(e^{j\omega M})}\underbrace{\bar{X}(e^{j\omega})}$$



Okay. To compute the final spectrum, we have to multiply the first turn, which is remember 32 repetition of the basic spectrum of a decaying exponential times the spectrum we just computed. It turns out that this peaks in a of e to the j omega m aligned with local maxima of the spectrum of the first period, so that the final result looks like this, where the peaks of the first term in the product slim down the lobes of the second term. So here we have derived the DTFT, namely the spectrum of an infinite non-periodic signal that is not a trivial signal. Now so far we have treated the DTFT as a formal operator and in the next module we will see how the DTFT relates to the concept of change of basis in an appropriate Hilbert space.

Notes

Summary

12m 12s