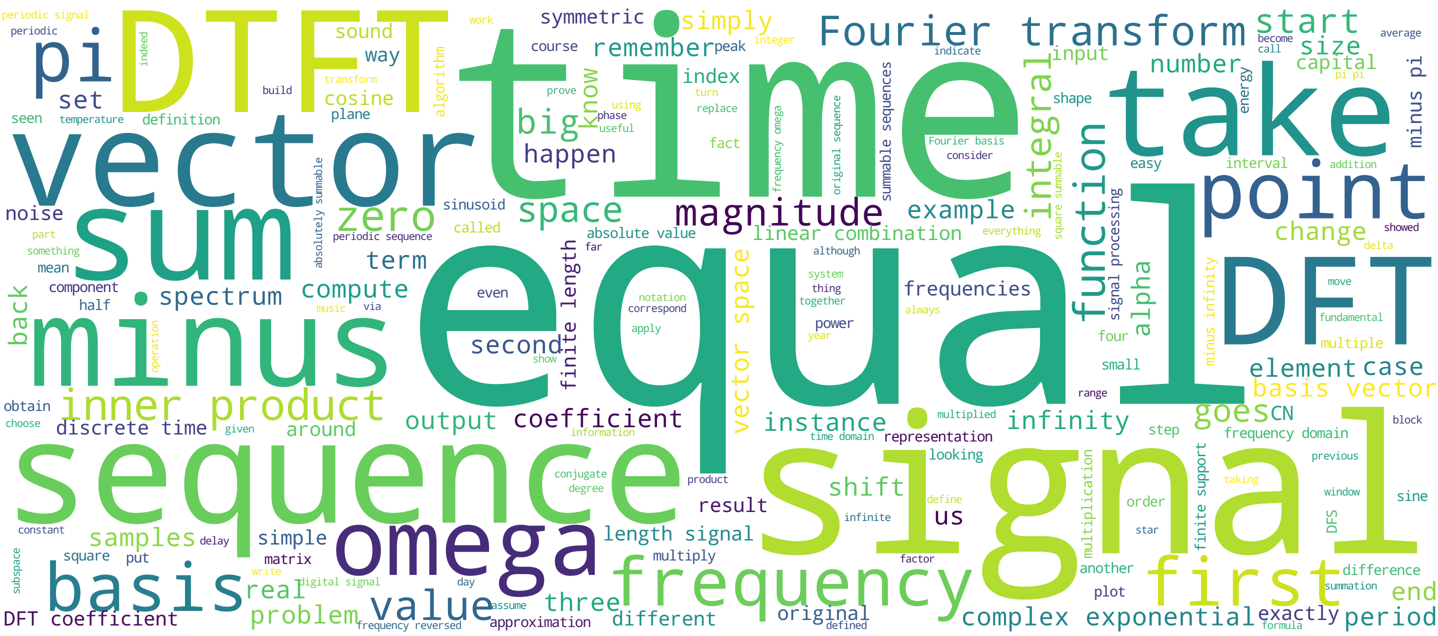
Existence easy for absolutely summable sequences

$$
\begin{aligned}
|X(e^{j\omega})| &= \left| \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \right| \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]\, e^{-j\omega n}| \\
&= \sum_{n=-\infty}^{\infty} |x[n]| \\
&< \infty
\end{aligned}
$$



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Video

Inversion easy for absolutely summable sequences

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right) e^{j\omega n} d\omega$$

$$= \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} \frac{e^{j\omega(n-k)}}{2\pi} d\omega$$

$$= x[n]$$

Existence means simply that the sum that defines a DTFT does not blow up. This is easy to prove for absolutely summable sequences. If you take the magnitude of the DTFT at any point omega, this is equal to the sum for n that goes from minus infinity to plus infinity of x of n times e to the minus j omega n in magnitude. Now, every time you have the absolute value of the sum, you know that this is maximised by the sum of the absolute values of the elements of the sum. We do this, and because the magnitude of the complex exponential is 1, this is actually equal to the sum of the absolute values of the sequence. Since our initial hypothesis was that the sequence was absolutely summable, this is less than infinity and therefore the DTFT exists for all values of omega. Similarly, we can invert the DTFT very easily if we assume absolute summability of the underlying sequence. As we showed in the previous module, the inversion formula is 1 over 2 pi times the integral between minus pi and pi of the DTFT times e to the j omega n in the omega. We replace x of e to the j omega by the definition of the DTFT in here, and because of the absolute summability of the sequence, we can invert the summation and the integral.

Notes

Summary

0m 00s

Inversion easy for absolutely summable sequences

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right) e^{j\omega n} d\omega$$

$$= \sum_{k=-\infty}^{\infty} x[k] \int_{-\pi}^{\pi} \frac{e^{j\omega(n-k)}}{2\pi} d\omega$$

$$= x[n]$$

When we do that, we have the sum for k that goes from minus infinity to plus infinity of x of k time this integral here. Each element of the sequence in the sum is multiplied by this integral, which depends on k. But now look at the numerator of this fraction here. This is a complex exponential, and if n is different than k, this will span an integer number of periods in the minus pi-pi interval, and therefore the integral will be zero. What that means is that this integral is zero unless n is equal to k, at which point all the elements in the sum will be killed except for x of n. In the end, we have the result we're looking for.

Notes

Summary

1m 26s

A formal change of basis

- formally DTFT is an inner product in $\mathbb{C}^\infty$:

$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \langle e^{j\omega n}, x[n] \rangle$$

- "basis" is an infinite, uncountable basis: $\{e^{j\omega n}\}_{\omega \in \mathbb{R}}$
- something "breaks down": we start with sequences but the transform is a function
- we used absolutely summable sequences but DTFT exists for all square-summable sequences (proof is rather technical)

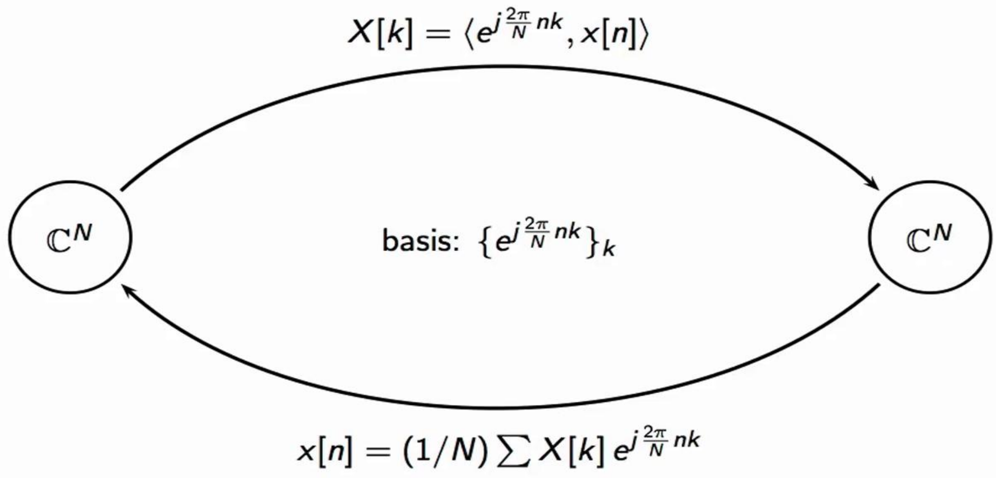
The DTFT looks exactly like an inner product in the space C infinity. If you take this inner product here between an infinite sequence and the sequence e to the j omega n, and you write definition of the inner product, you get exactly the formulation for the DTFT. The problem here is that C infinity is not really a well-defined vector space. There are sequences that do not converge in C infinity. Nonetheless, if we manage to establish a formal parallel between a change of basis and the DTFT, it will mean that everything that we discovered about the DFT, which is a well-defined entity, will apply to the DTFT as well, and all our intuition about the frequency domain will translate to the DTFT. Now the basis here that we're talking about is not really a basis, because it's an infinite and uncountable set of vectors, indexed by a real valued variable omega. Something breaks down really, we start with sequences, but we end up landing in a space of functions. On top of it all, although we just proved existence and invertibility for absolutely summable sequences, in reality the DTFT exists for all square summable sequences, which is a larger set of sequences. But in that case, the proofs we just gave become much more technical, and so we will skip them here.

Notes

Summary

2m 09s

Review: DFT

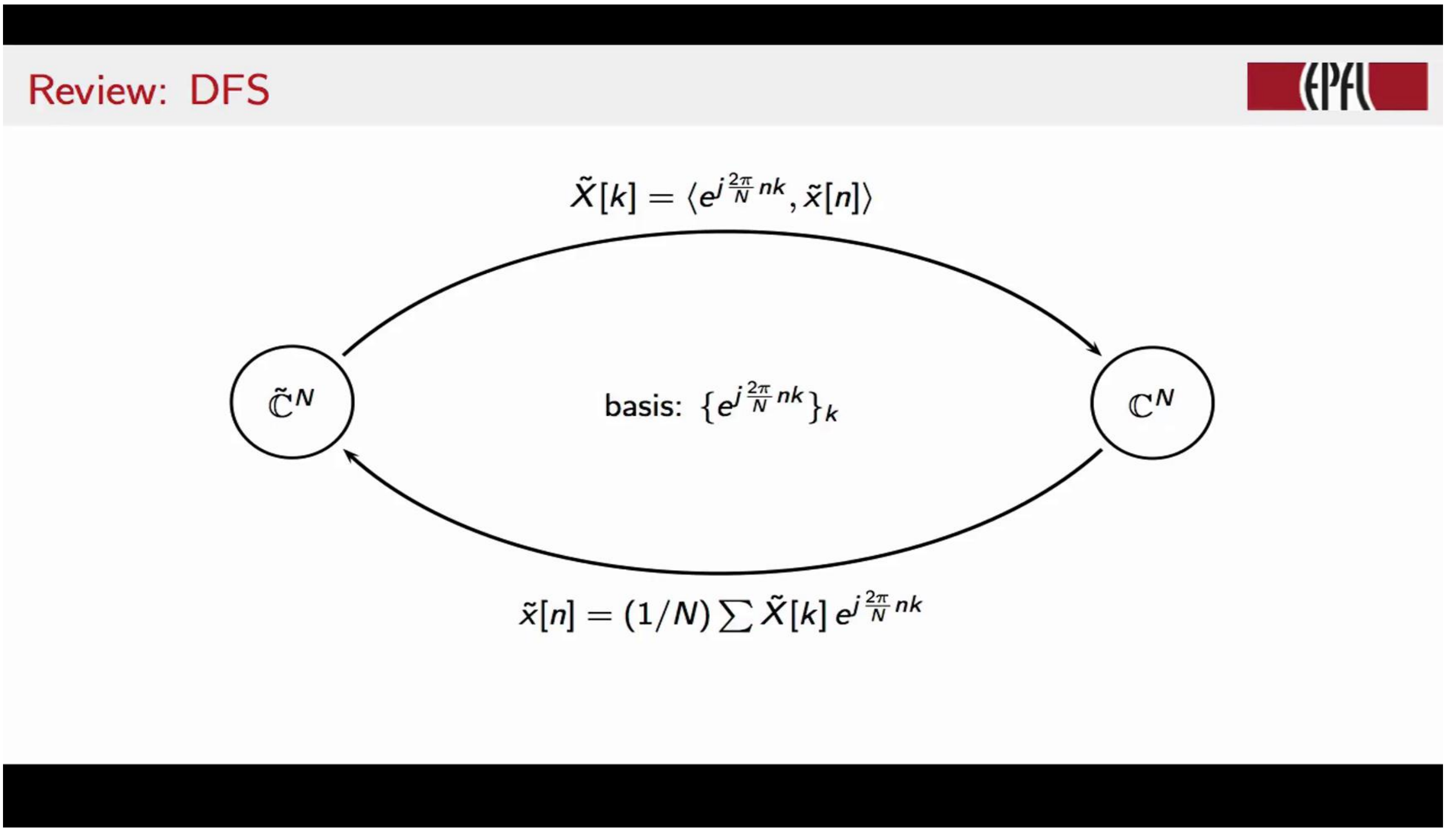


Let's sum up the situation so far. For finite length signals, we start in CN, and via a change of basis, we compute their representation, the frequency domain, which as well lives in CN. We can go back to the original sequence via the inversion formula, and the basis that allows us to go from the time domain to the frequency domain is the DFT basis, the Fourier basis for the DFT, which is a countable set of n Fourier basis vectors.

Notes

Summary

3m 33s

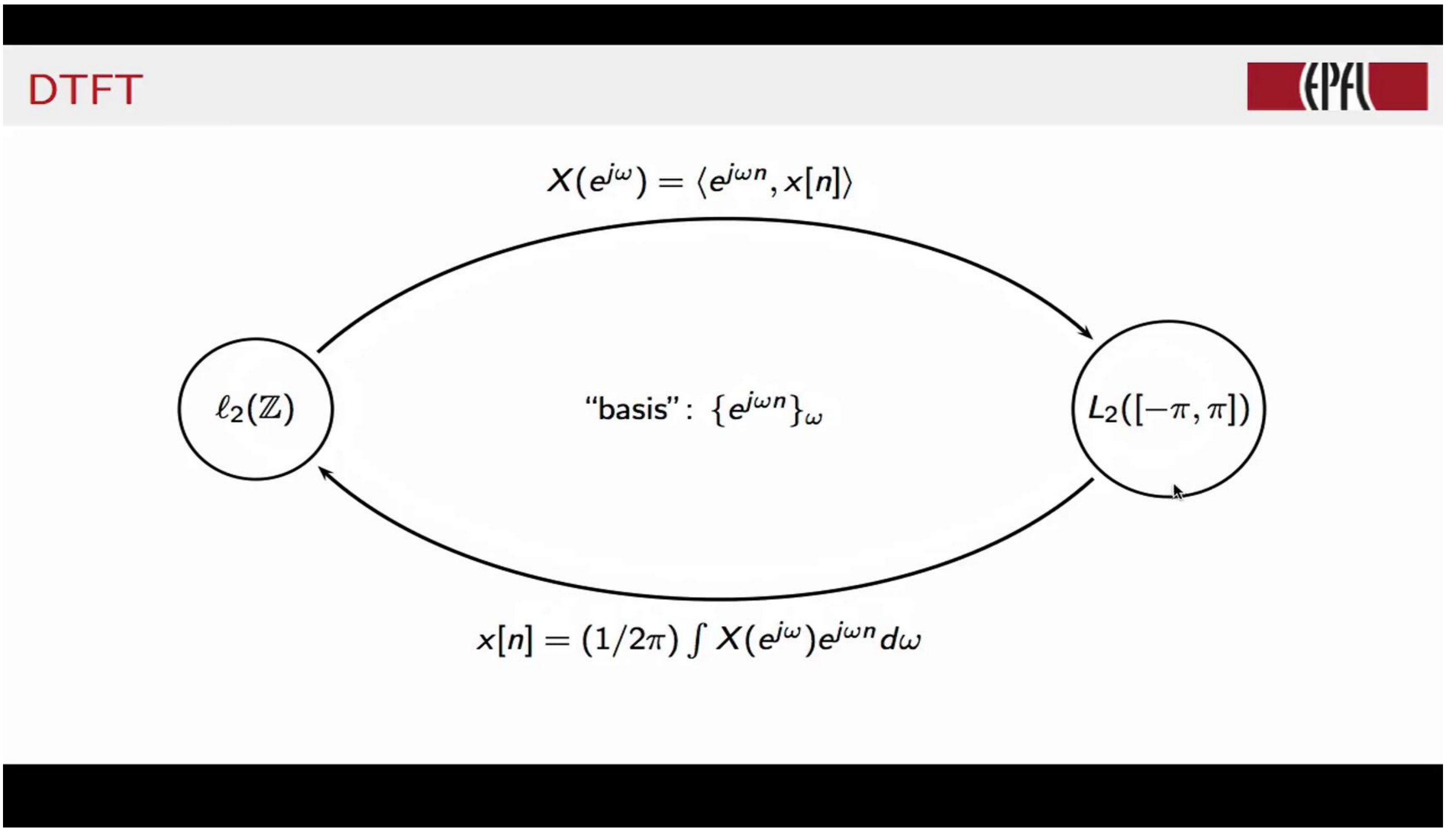


The DFS is exactly the same. The expansion and reconstruction formulas are the same, except that in this case, we assume that everything is periodic underneath.

Notes

Summary

4m 08s



Now we have the DTFT. We start from the space of square summable sequences, and via a formal change of basis, so basis here is in quote, we end up in the space of square-integrable functions on the interval minus pi-pi.

Notes

Summary

4m 20s

DTFT properties

- linearity

$$\text{DTFT}\{\alpha x[n] + \beta y[n]\} = \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$

- time shift

$$\text{DTFT}\{x[n-M]\} = e^{-j\omega M} X(e^{j\omega})$$

- modulation (dual)

$$\text{DTFT}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)})$$

By looking at the DTFT as a formal basis expansion, the linearity property follows easily from the linearity of the inner product. The DTFT of a linear combination of two sequences will be the linear combination of the DTFTs. A second property that is easy to prove from the definition of the DTFT is the time shift property. If we take a sequence and we shift it in time by big M samples, the DTFT of this shifted sequence is equal to the DTFT of the original sequence times a delay factor, e to the minus j omega big M, which is very similar to what we obtained in the case of the DFS when we took the shift of a periodic sequence. The dual of this property is the modulation property of the DTFT. If we take a sequence and we multiply this by a complex exponential at frequency omega 0, what happens in frequency is that we have a shift of the spectrum by omega 0.

Notes

Summary

4m 35s

DTFT properties

- time reversal

$$\text{DTFT}\{x[-n]\} = X(e^{-j\omega})$$

- conjugation

$$\text{DTFT}\{x^*[n]\} = X^*(e^{-j\omega})$$

The time reversal property tells us that the Fourier transform of a time-reversed sequence, a sequence where we flip the values across the origin, will be equal to a frequency-reversed Fourier transform. The conjugation property says that if we conjugate every value of the sequence, the Fourier transform will be both conjugated and frequency-reversed.

Notes

Summary

5m 37s

Some particular cases:

- if $x[n]$ is symmetric, the DTFT is symmetric:

$$x[n] = x[-n] \Longleftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$$

- if $x[n]$ is real, the DTFT is Hermitian-symmetric:

$$x[n] = x^*[n] \Longleftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

- special case: if $x[n]$ is real, the magnitude of the DTFT is symmetric:

$$x[n] \in \mathbb{R} \Longrightarrow |X(e^{j\omega})| = |X(e^{-j\omega})|$$

- more special case: if $x[n]$ is real and symmetric, $X(e^{j\omega})$ is also real and symmetric

Now some particular cases that are very useful to remember because they appear often. First of all, if the sequence is symmetric, then the DTFT is symmetric as well. If the sequence is real, then the DTFT is Hermitian-symmetric. Reality of the sequence can be expressed mathematically by saying that x of n is equal to the conjugate of x of n. This infrequency implies that the Fourier transform of the sequence is equal to the conjugate and frequency-reversed version of the Fourier transform. A simple corollary of this property is the fact that if x of n is real, then the magnitude of the DTFT is symmetric. You can verify this simply by taking the magnitude of both terms of this equation. Even more special case states that if x of n is real and symmetric, then X of j omega is also real and symmetric.

Notes

Summary

6m 02s