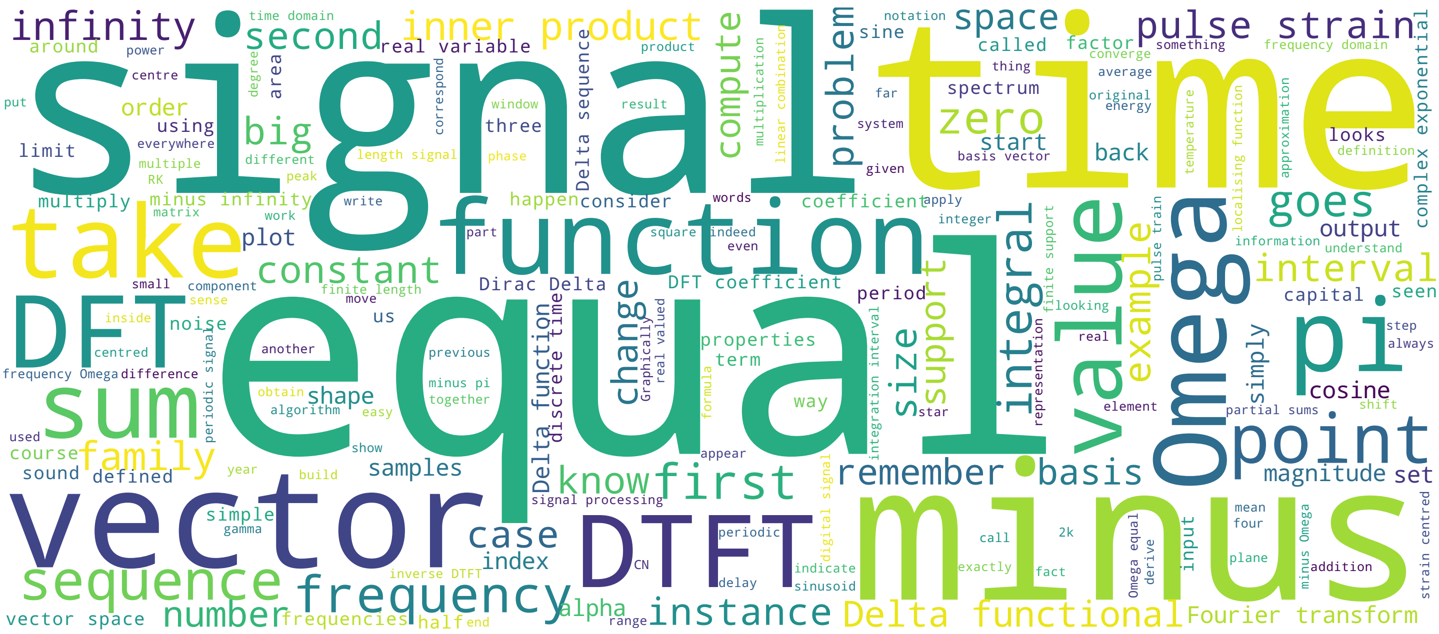
DTFT as basis expansion

Some things are OK:

- $\mathrm{DFT}\{\delta[n]\} = 1$
- $\mathrm{DTFT}\{\delta[n]\} = \langle e^{j\omega n}, \delta[n]\rangle = 1$



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DTFT as basis expansion

Some things aren't:

- $\text{DFT}\{1\} = N\delta[k]$
- $\text{DTFT}\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = ?$
- problem: too many interesting sequences are *not* square summable!

By looking at the properties of the DTFT that we have seen so far, it would appear that the DTFT is a change of bases from the space of square assemble sequences to space of square integral periodic functions. In other words, it looks like we could look at the DTFT as an extension of the DFT formalism. Indeed some things would lead us to believe so. For instance, if you take the DfT of the Delta sequence, you remember you have the constant 1 and similarly the DTFT of the Delta sequence expressed as the inner product between the pseudobasis function and Delta sequence is again 1. However, some things are not okay at all. The DfT of the constant 1 is very well defined and it's equal to n times the Delta sequence in frequency. But the DTFT of 1 is by definition the sum from n goes from minus infinity to plus infinity of e to the minus j Omega n. Now, to see that there is a problem with the sum, just put Omega equal to 0 and you see that the sum diverges. The problem is that there are too many interesting sequences that are not square sumable and of course the constant 1 is 1 of them. In order to be able to keep using the change of bases paradigm even for sequences that are not square sumable, we have to introduce a little mathematical trick called the Dirac Delta function.

Notes

Summary

0m 00s

The Dirac delta functional

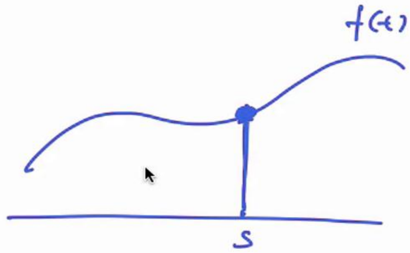
$\delta(t)$

Defined by the "sifting" property:

$$\int_{-\infty}^{\infty} \delta(t-s)f(t)dt = f(s)$$

for all functions of $t \in \mathbb{R}$.

$s \in \mathbb{R}$



This little animal here, which is usually indicated by the symbol Delta, but now Delta of a real variable t, not the Delta sequence, is defined by the shift in property that looks like this. If we take a Delta function, we centre it in s, where s is a variable in R, and then we multiply this Delta function by any function of a real variable t, and then we take the integral from minus infinity to plus infinity. Then what we get is the value of f in the point s. Graphically, we usually represent the Delta functional as an upwards point and arrow. We centre this in S. We multiply this by any function of a real variable T and then we integrate this from minus infinity to plus infinity and we get the value of the function in S.

Notes

Summary

1m 30s

Intuition

- family of *localizing* functions $r_k(t)$ with $k \in \mathbb{N}$ and $t \in \mathbb{R}$
- support inversely proportional to k
- constant area

In order to develop some intuition for the properties of the Dirac Delta functional, let's consider a family of so called localising functions RK of t, where k is an integer index andTis a real valued variable. The properties of this family of functions are two. The support of each function is inversely proportional to the index k, but regardless of the index k, the area of each function, the integral from minus infinity plus infinity of each function is constant.

Notes

Summary

2m 25s

Intuition

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Consider the localizing family $r_k(t) = k\,\mathrm{rect}(kt)$:

- nonzero over $[-1/2k, 1/2k]$, i.e. support is $1/k$
- area is 1

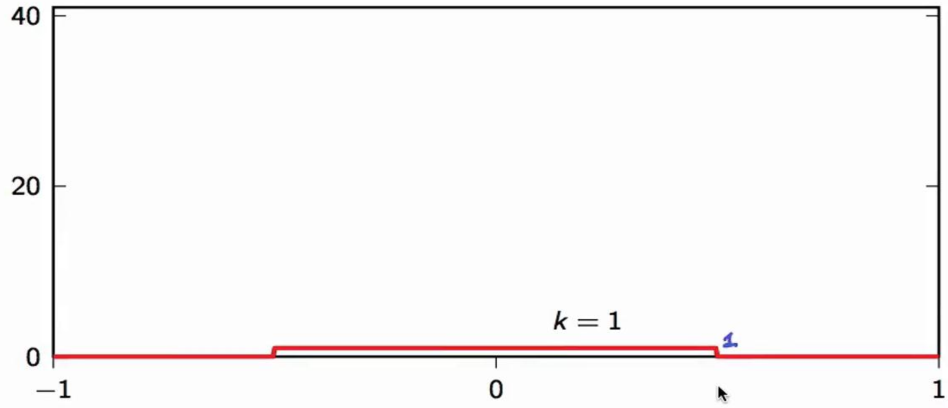
As an example, take the rect function. The rect function is a classic indicator function that is equal to 1 from minus 1 half to 1 half and 0 everywhere else. So this function has a support of 1 and an area of 1. We can use this function to build a family of localising functions. Like so we multiply the rect by a factor k and we shrink the support of the rect by a factor k. RK of T in this case will have a support that goes from minus 1 over 2k to 1 over 2k. The support is 1 over k and the area is 1.

Notes

Summary

2m 57s

The family $r_k(t) = k\,\mathrm{rect}(kt)$



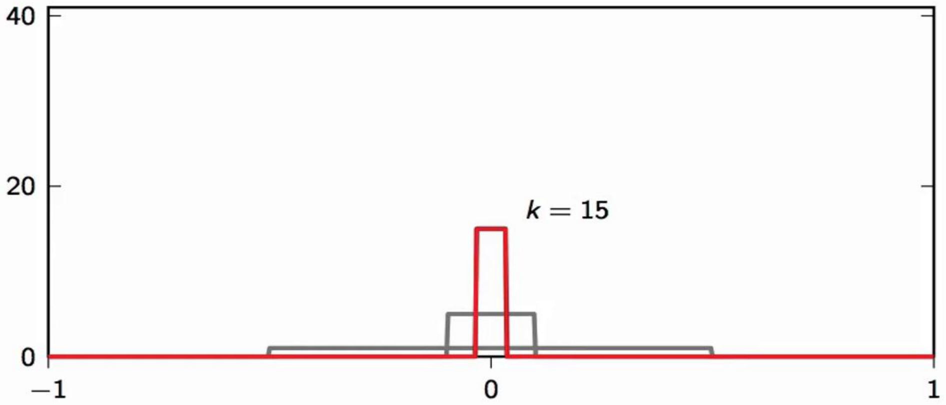
If we plot some functions in this family, this is what we get for k equal to 1, the value here is 1.

Notes

Summary

3m 38s

The family $r_k(t) = k\,\mathrm{rect}(kt)$



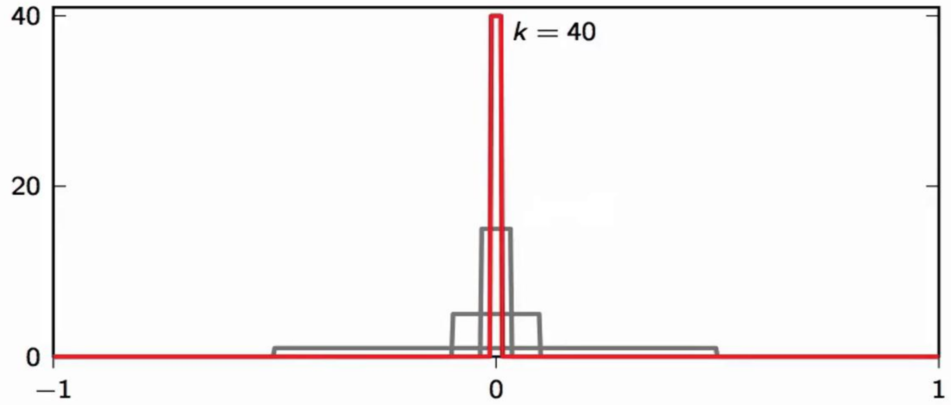
For k equal to 5, the support has shrunk to 1 over 5, and the area is still 1 because the value here is 5.

Notes

Summary

3m 44s

The family $r_k(t) = k\,\mathrm{rect}(kt)$



We go to 15, it will look like this, and to 40 it will go like this and we go on to infinity.

Notes

Summary

3m 53s

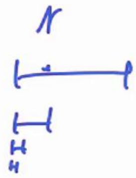
Extracting a point value

By the Mean Value theorem:

$$\int_{-\infty}^{\infty} r_k(t)f(t)dt = k \int_{-1/2k}^{1/2k} f(t)dt$$
$$= f(\gamma)|_{\gamma\in[-1/2k,1/2k]}$$

and so:

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} r_k(t)f(t)dt = f(0)$$



Now consider the integral between minus infinity and plus infinity of the product between RK of T and any function f of a real variable T. RK of T is known 0 only between minus 1 over 2k and 1 over 2k. These are the new integration limits of this product. The value of RK ofT over the integration interval is k so we can bring this outside of the interval, and inside we have simply the integral of the function over this interval. Now we invoke the mean value theorem. You can go back to your calculus textbook to revise its proof. The mean value theorem says that the value of this integral here will be equal to f of gamma for some point gamma within the integration interval. Now, we don't know where gamma is inside this interval, but we do know that it exists. Now, as k goes to infinity, the support of the indicator function becomes smaller and smaller and so gamma, which is somewhere in the integration interval, will be sandwiched between interval limits that grow closer and closer. In the limit f of gamma will be f of 0, because the width of the interval has shrunk down to an infinitesimal width.

Notes

Summary

3m 59s

The Dirac delta functional

The delta functional is a shorthand. Instead of writing

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} r_k(t-s)f(t)dt$$

we write

$$\int_{-\infty}^{\infty} \delta(t-s)f(t)dt.$$

as if $\lim_{k\to\infty} r_k(t) = \delta(t)$,

The Delta functional is really a shorthand for this limiting operation. Instead of writing the limit of the integral for a family of localising function, we just use the Delta notation. What is interesting is that the shape of the base function that we use to build a family of localising functionis not really critical. We can use pretty much any shape and as long as the two properties of shrinking support and constant area are satisfied, the limit will convert to the point wise value of the function.

Notes

Summary

5m 27s

The "pulse train"

$$\tilde{\delta}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

just a technicality to use the Dirac delta in the space of 2π-periodic functions

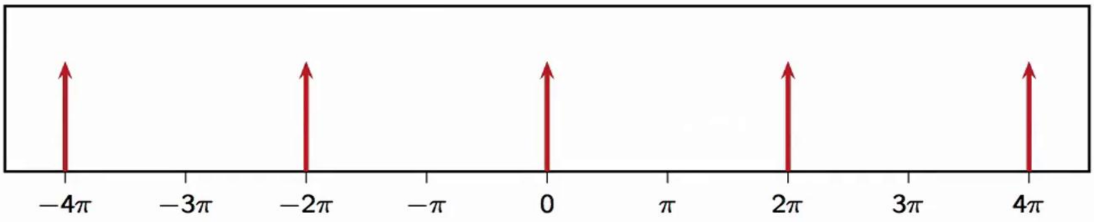
Okay, so now a last technicality before we understand why we're doing all this. We will be using the Dirac Delta functional in the frequency domain. Now, we know that all DTFT spectra are 2 pi piperiotic, so if we want to use this tool in the frequency domain, we have to pariahize it. The periodic version of the Dirac Delta functional is called the pulse strain and it is built by placing copies of the Delta function every 2 pi and by scaling the whole signal by 2 pi.

Notes

Summary

5m 56s

Graphical representation



If you want to represent that in the frequency domain with the usual upward arrow notation, we see that there will be a pulse every 2 pi.

Notes

Summary

6m 26s

Now let the show begin!

$$\text{IDTFT}\left\{\tilde{\delta}(\omega)\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\delta}(\omega)e^{j\omega n}d\omega$$
$$= \int_{-\pi}^{\pi}\delta(\omega)e^{j\omega n}d\omega$$
$$= e^{j\omega n}|_{\omega=0}$$
$$= 1$$

$$\text{IDFT}\{N\delta[k]\} = 1$$

Now we can let the show begin. Let's consider the inverse DTFT of the pulse train. Well, we apply the definition and we have 1 over 2 pi times the integral between minus pi and pi of the pulse strain times e to the j Omega n in the Omega. The first thing to remark is that since, the paradise Delta is scaled by a factor of 2pi, this cancels the normalisation factor in front of the integral. Then we are integrating only between minus pi and pi. In this integral we only have 1 pulse so we can remove the implicit periodisation and then because of the shifting property of the Delta functional, this integral will be just the value of this function of the real valued variable Omega in 0, because this Delta is centred in 0. The value of e to the j Omega n for Omega equal to 0 is equal to 1. This is formally similar to the fact that the inverse DfT of n Delta of k is actually equal to 1. By using the Delta functional, we have established another formal parallel between the DfT and DTFT.

Notes

Summary

6m 36s

In other words

$$\text{DTFT}\{1\} = \tilde{\delta}(\omega)$$

If the inverse DTFT of the Delta function is 1, then it means that the direct DTFT, the forward Fourier transform of the constant 1, is formally equal to the pulse train.

Notes

Summary

7m 53s

Does it make sense?

Partial DTFT sum:

$$S_k(\omega) = \sum_{n=-k}^{k} e^{-j\omega n}$$

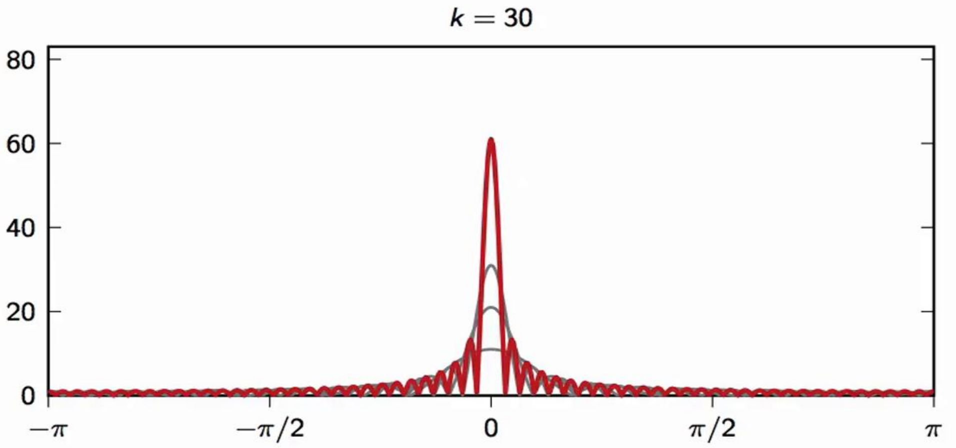
Does this make sense? Well, we could try to compute numerically the partial sums that are involved in the computation of the DTFT of the constant 1. Define SK of Omega as the sum that goes from minus k to k of e to the minus j Omega n. As k goes to infinity SK of Omega should converge to the DTFT of the constant 1. If we plot this partial sums in magnitude for increasing values of k, we have something like this.

Notes

Summary

8m 09s

Plotting $|S_k(\omega)|$



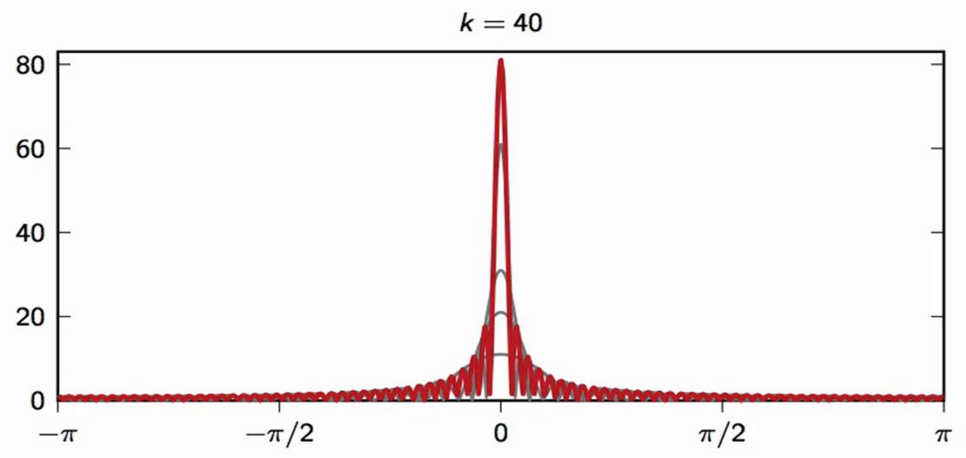
For k equal to 5 we have this shape and as we increase the index we see that this family of partial sums looks like a family of localising functions.

Notes

Summary

8m 39s

Plotting $|S_k(\omega)|$



The support gets narrower and the area stays constant. It really makes sense to say that in the limit this partial sums will converge to the Dirac Delta function. With this fundamental result in our pocket, we can now proceed to derive some other interesting DTFT pairs for nonswear sumable sequences.

Notes

Summary

8m 52s

Using the same technique

$$\text{IDTFT}\left\{\tilde{\delta}(\omega-\omega_0)\right\} = e^{j\omega_0 n}$$

So:

- $\text{DTFT}\{1\} = \tilde{\delta}(\omega)$
- $\text{DTFT}\{e^{j\omega_0 n}\} = \tilde{\delta}(\omega-\omega_0)$
- $\text{DTFT}\{\cos\omega_0 n\} = [\tilde{\delta}(\omega-\omega_0) + \tilde{\delta}(\omega+\omega_0)]/2$
- $\text{DTFT}\{\sin\omega_0 n\} = -j[\tilde{\delta}(\omega-\omega_0) - \tilde{\delta}(\omega+\omega_0)]/2$

With the same technique we used before, we can show that the inverse DTFT of a shifted pulse strain, a pulse strain shifted by a frequency Omega 0 gives a complex expential of frequency Omega 0 in the time domain. If the DTFT of 1 is the pulse strain centred in 0, the DTFT of an arbitrary complex exp1ntial of frequency Omega 0 is the pulse train shifted by Omega 0. By using Euler's relation and the linearity of the DTFT, we can derive the DTFT of the cosine of Omega 0 n, this is just 1 half times the sum of two pulse strains, 1 centred in Omega 0 and the other 1 centred in minus Omega 0. The DTFT of the sin of Omega 0 n which is minus j over two times a pulse strain centred in Omega 0 and another pulse strain centred in minus Omega 0.

Notes

Summary

9m 14s