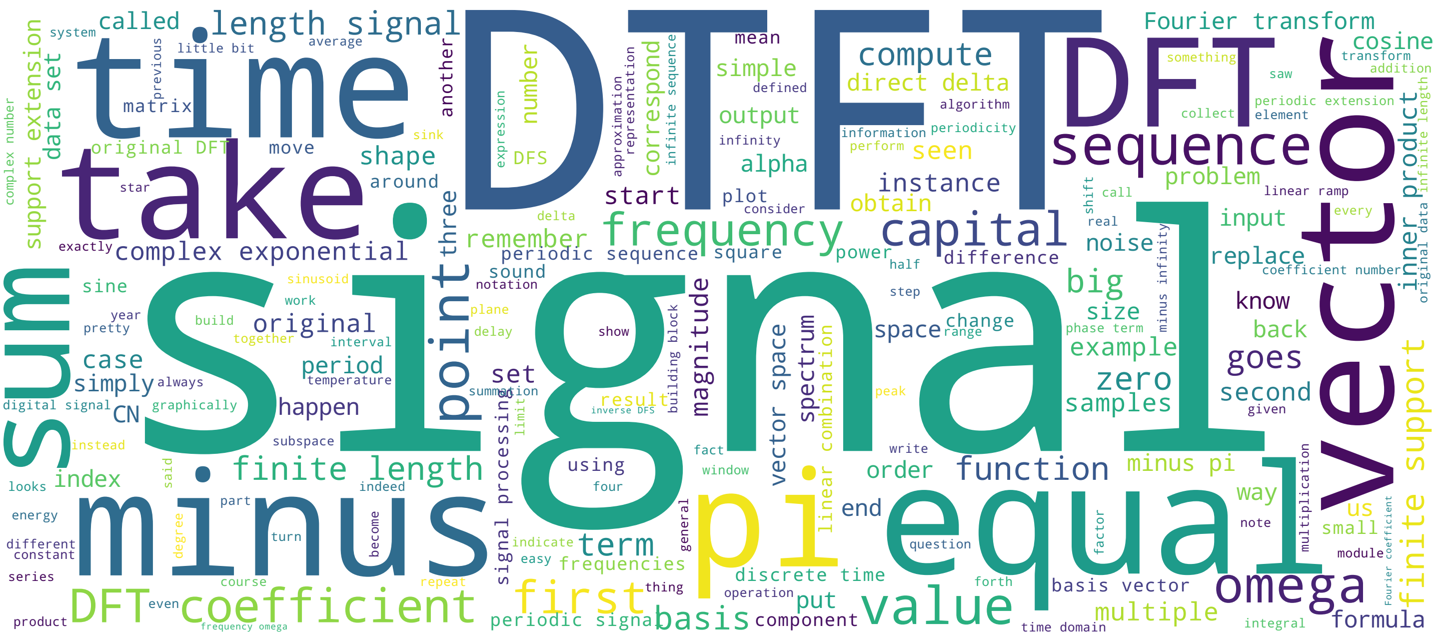
relationships between transforms



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Video

Overview:

$\mathbb{C}^N$

- DFT, DFS, DTFT
- DTFT of periodic sequences
- DTFT of finite-support sequences
- Zero padding

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We will now look at the relationships between the different Fourier representations that we have seen so far. Remember, we have seen the DFT that works for finite length signals that live as CN, such a simple change of basis; the DFS, which is pretty much the same thing, except that we take the periodicity of the underlying signals into account explicitly; and finally the DTFT, which is the most general tool. Since the DTFT is so general and we use it for formal proofs, the question is, can we take the DTFT of the two previous classes of signals, namely, can we take the DTFT of periodic sequences, or can we take the DTFT of finite length signals that we will have to embed into finite support sequences to turn them into sequences.

Notes

Summary

0m 00s

Transforms

- DFT, DFS: change of basis in $\mathbb{C}^N$
- DTFT: "formal" change of basis in $\ell_2(\mathbb{Z})$
- basis vectors are "building blocks" for any signal

- DFT: numerical algorithm (computable)
- DTFT: mathematical tool (proofs)

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We have seen that the DFT and the DFS are simple change of basis in CN, and so is the DTFT, although we call it a formal change of basis because some mathematical details are a little bit more delicate. But in all cases, we use the Fourier transform to decompose a signal into a series of fundamental building blocks, namely oscillations. Now the DFT is a numerical algorithm. We can input to the DFT any set of complex numbers and get the Fourier coefficients that correspond to their frequency representation. The DTFT is more of a mathematical tool, as I've said, that we can use in formal proofs that involve arbitrary signals.

Notes

Summary

0m 47s

Embedding finite-length signals

- N-tap signal $x[n]$
- natural spectral representation: DFT $X[k]$
- two ways to embed $x[n]$ into an infinite sequence:
  - periodic extension: $\tilde{x}[n] = x[n \mod N]$
  - finite-support extension: $\bar{x}[n] = \begin{cases} x[n] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$
- how does $X[k]$ relate to the DTFT of the embedded signals?

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Let's start by considering a finite length signal of length big N. You will remember from one of our early lessons that there are two ways to transform this finite data set into an infinite sequence. The first way is to build what is called a periodic extension, so we will repeat the same N samples an infinite number of times. The second way is to build what is called a finite support extension for the finite length signals, which is put in this sample, this N sample from 0 to capital N minus 1, and then put in zeros everywhere else. Now how does the natural representation for a signal, —which is the DFT— how does this relate to the DTFT of either of these extensions of the data set to an infinite sequence?

Notes

Summary

1m 28s

DTFT of periodic signals

$$\tilde{x}[n] = x[n \mod N]$$

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\omega n}$$

DFS

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} \right)$$

30

The first way we will tackle the problem is to consider the periodic extension of our finite data set. We use the notation tilde of x, and from the original finite length signal, we can obtain tilde of x, this infinite sequence, by taking x of N module of capital N. This is a periodic sequence with period capital N. The DTFT of this sequence, well, let's compute it, x of e to the j omega, is simply the DTFT sum here. And if we replace the value of tilde of x N with the inverse DFS, because the natural representation for this signal is a DFS, we obtain this expression here. Now we have an outer sum that corresponds to the DTFT, and an inner sum which corresponds to the inverse DFS that gives us the value tilde x of N.

Notes

Summary

2m 21s

DTFT of periodic signals

$$\tilde{x}[n] = x[n \mod N]$$

$$\begin{aligned}
\tilde{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \left(\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} \right)
\end{aligned}$$

30

Now when we have two sums, the first thing we do is exchange the order of summation and we can do this because the inner sum has a finite number of terms. And with this expression, we can collect the complex exponentials and have this formula here. Let's concentrate first on this term here in parentheses.

Notes

Summary

3m 17s

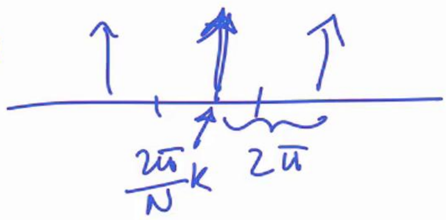
We've seen this before

$$\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{N}nk} e^{-j\omega n} = \text{DTFT}\left\{e^{j\frac{2\pi}{N}nk}\right\}$$

$$= \underbrace{\tilde{\delta}(\omega - \frac{2\pi}{N}k)}$$

$\omega = \frac{2\pi}{N}k$

DTFT



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Now this, if we look at it, is the sum over all values for the index N of e to the j 2 pi over N nk times e to the minus j omega N. So this is the DTFT of a complex exponential at frequency 2 pi over N times k. We have seen that the DTFT of a complex exponential over the entire integer axis is a periodic direct delta centered at 2 pi over N times k. This is, again to repeat, this is a complex exponential whose frequency is equal to 2 pi over N times k, and when we perform the DTFT, we get a series of impulses that are spaces 2 pi apart, and the frequency in the minus pi-pi interval where the first delta appears is 2 pi over N times k.

Notes

Summary

3m 37s

DTFT of periodic signals

$$\tilde{X}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} \underbrace{X[k]}_{\text{DFT}}\tilde{\delta}(\omega - \frac{2\pi}{N}k)$$

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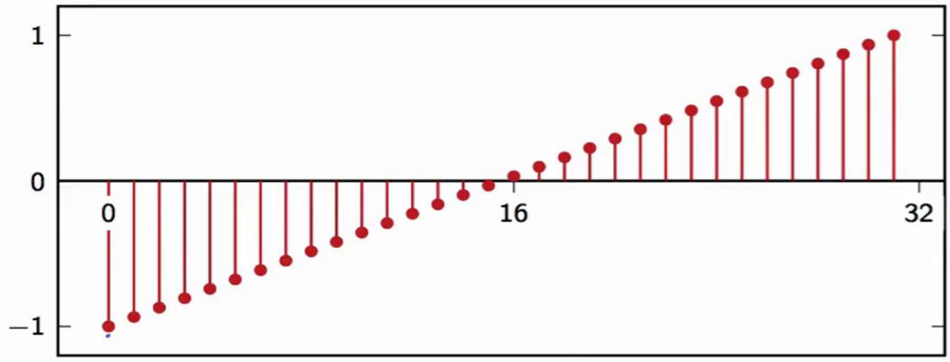
We replace this into the summation, and we find that the DTFT of a periodic sequence is given by the sum of capital N direct deltas centered at multiples of 2 pi over N, and each delta is scaled by the DFT coefficient number k of the original endpoint data sequence. Let's see this with a simple example.

Notes

Summary

4m 35s

32-tap sawtooth



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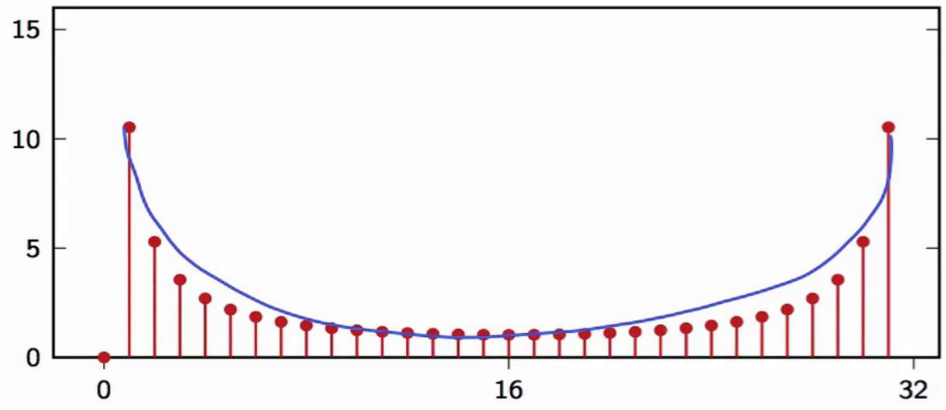
If we take a very simple signal that moves linearly from minus 1 to 1 over say 32 points, we can compute the DFT of this 32-point signal either analytically or numerically.

Notes

Summary

5m 04s

DFT of 32-tap sawtooth



34

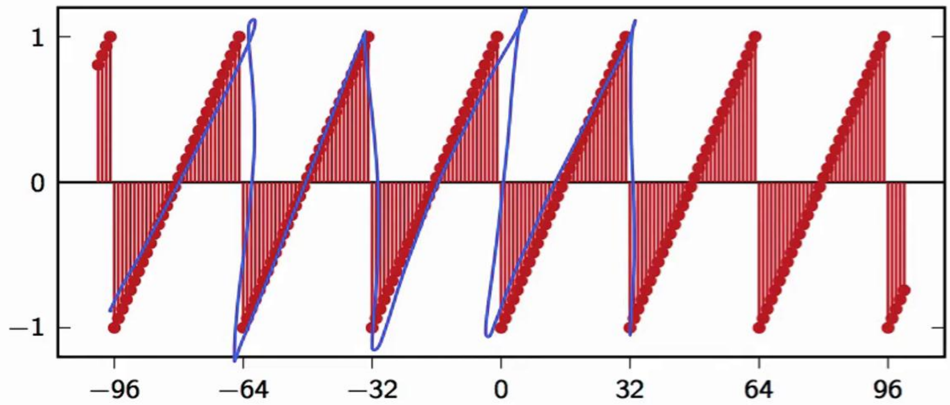
We get this set of DFT coefficients. The coefficient number 0 is 0 because the signal is 0 mean, and then you have this hyperbolic curve for the magnitude.

Notes

Summary

5m 17s

32-periodic sawtooth



35

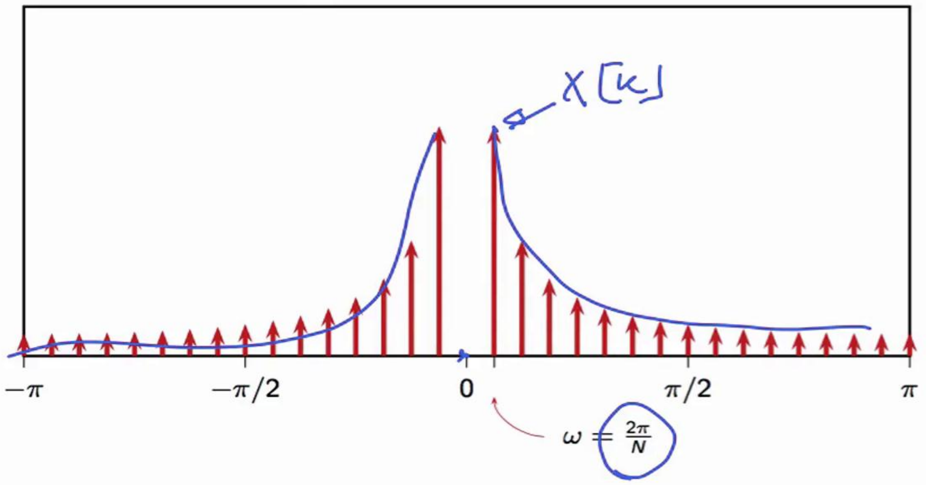
If we now use the periodic extension of the original linear slope, we obtain a so-called sawtooth signal because you see the signal moves this way in time. What is the DTFT of this periodic signal? Well, according to the formula we saw before, it will look like this.

Notes

Summary

5m 28s

DTFT of periodic extension



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You will have a direct delta at every multiple of the frequency 2 pi over N, and the weight of each direct delta will be equal to the DFT coefficient number k. So you see in 0 there will be no delta because the DFT coefficient is equal to 0, and then this is the same hyperbolic shape that you saw in the DFT. So when we move from the DFT of a period to the DTFT of the whole periodic signal, we are basically converting the DFT coefficients into direct deltas that are spaced 2 pi over capital N apart.

Notes

Summary

5m 48s

DTFT of finite-support signals

$$\bar{x}[n] = \begin{cases} x[n] & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\bar{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \bar{x}[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} \right) e^{-j\omega n} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} \right)
\end{aligned}$$

37

Now let's look at what happens if instead of using a periodic extension, we use a finite support extension. We take our original endpoint signal and we transform it into a sequence by using the original sample values for N that goes from 0 to capital N minus 1 and we put 0 everywhere else. If we take the DTFT, well, we can write the DTFT formula here, and this finite support extension only affects the limits of the DTFT summation. Instead of going from minus infinity to plus infinity, we go from 0 to capital N minus 1 because all the other terms in this sum will be 0. Now we perform exactly the same trick that we did before. We replace the value x of N by the inverse DFT of the finite length signal, and so we replace x of N by this expression here, and then we interchange the order of summation, and we obtain something very similar to what we had before. We have the original DFT coefficients here in the outer sum, and here we have a term in parentheses that we have to analyze.

Notes

Summary

6m 29s

DTFT of finite-support signals

$$\sum_{n=0}^{N-1} e^{-j\sigma n}$$

$$\sum_{n=0}^{N-1} e^{-j\overbrace{(\omega - \frac{2\pi}{N}k)}^{\sigma} n} = \underbrace{\bar{R}(e^{j(\omega - \frac{2\pi}{N}k)})}$$

where $\bar{R}(e^{j\omega})$ is the DTFT of $\bar{r}[n]$, the rectangular signal:

$$\bar{r}[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

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Remember, before this term turned out to be the DTFT of a complex exponential, in this case, the term will turn out to be the shifted DTFT of a rectangular signal. Let's see. We have the sum for N that goes from 0 to big N minus 1 of e to the minus j omega minus 2 pi over Nk that multiplies N. So if you call this, I don't know, sigma, this would be the sum for N that goes from 0 to big N minus 1 of e to the minus j sigma N. This is the DTFT of a signal that is 1 from 0 to capital N minus 1. By replacing back sigma to its original value, you see that this is R of e to the j omega minus 2 pi over N times k, where R of e to the j omega is the DTFT of the rectangular signal that goes from 0 to N minus 1, and graphically it looks like so.

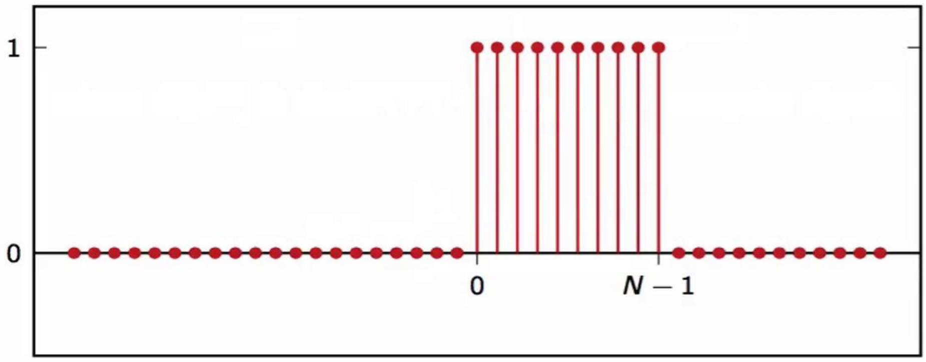
Notes

Summary

7m 36s

Rectangular step signal

$$\bar{r}[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$



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Now we have to compute the DTFT of this infinite length signal that is non-zero only between 0 and capital N minus 1. How do we do that?

Notes

Summary

8m 36s

DTFT of rectangular step signal

$$
\begin{aligned}
\bar{R}(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\frac{\omega N}{2}} \left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}} \right]}{e^{-j\frac{\omega}{2}} \left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right]} \\
&= \frac{\sin\left(\frac{\omega}{2} N\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{\omega}{2}(N-1)}
\end{aligned}
$$

40

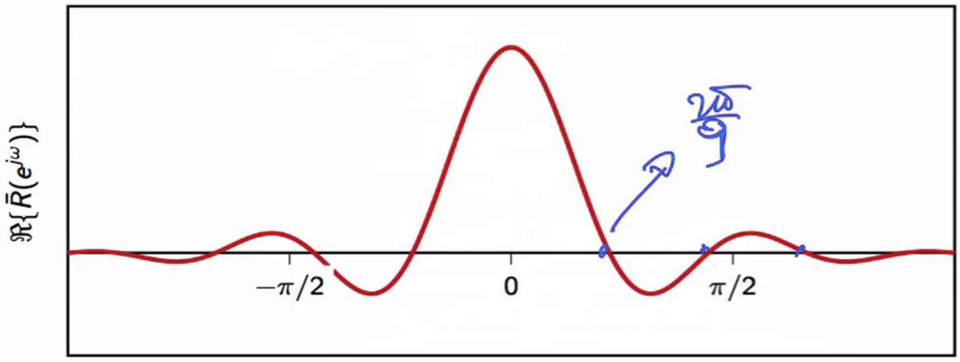
Well we compute the standard geometric sum. We know how to do this. We collect the half phase term at numerator and denominator, and in the end, we have a real term that will determine the magnitude of the DTFT times a pure phase term. This DTFT looks like so.

Notes

Summary

8m 45s

DTFT of interval signal ($N = 9$)



note that $\bar{R}(e^{j\omega}) = 0$ for $\omega = 2\pi k/N$, $k \in \mathbb{Z}/\{0\}$

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It's a sink-like shape and note that the DTFT will be equal to 0 every time the frequency omega hits a multiple of 2 pi over N. Here for instance, we have capital N equal to 9. At all multiples of 2 pi over 9, the DTFT in magnitude will be equal to 0. If we now go back to our DTFT of a finite support extension, remember we had the linear combination of shifted copies of this DTFT of a rectangular function weighed by the DFT coefficients of the original data set. What we're having here is a smooth interpolation of the original DFT values where the interpolator is this sink-like shape that we have just seen.

Notes

Summary

9m 05s

DTFT of finite-support signals

define $\Lambda(\omega) = \frac{1}{N}\bar{R}(e^{j\omega})$

$$\bar{X}(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\Lambda(\omega - \frac{2\pi}{N}k)$$

smooth interpolation of DFT values

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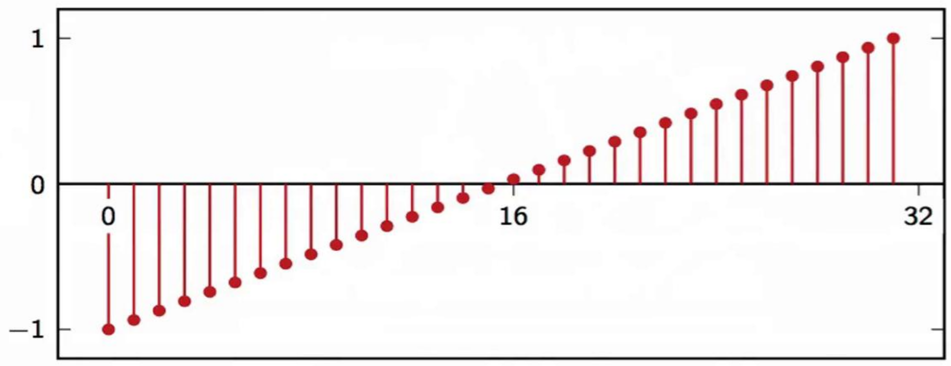
Notes

Summary

10m 00s

32-tap sawtooth



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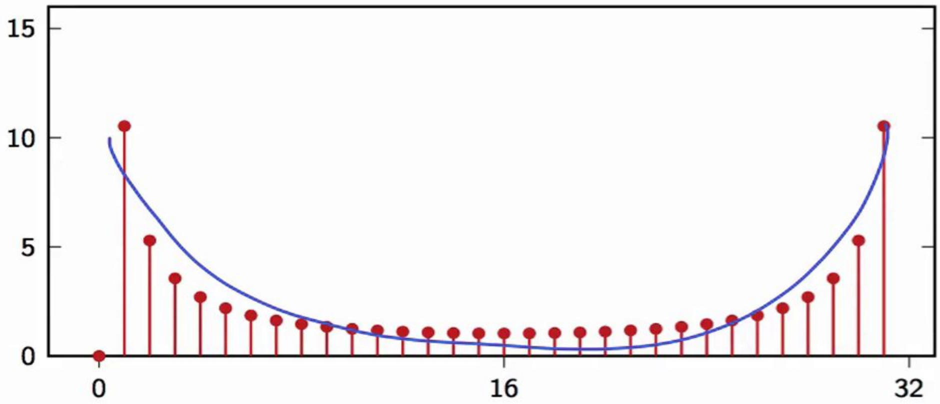
Graphically, if we want to repeat the experiment with the linear ramp from minus 1 to 1 over 32 points, we start with the signal in the time domain, finite length signal, we compute the DFT, and we have the same shape we've seen before with this hyperbolic shape.

Notes

Summary

10m 01s

DFT of 32-tap sawtooth



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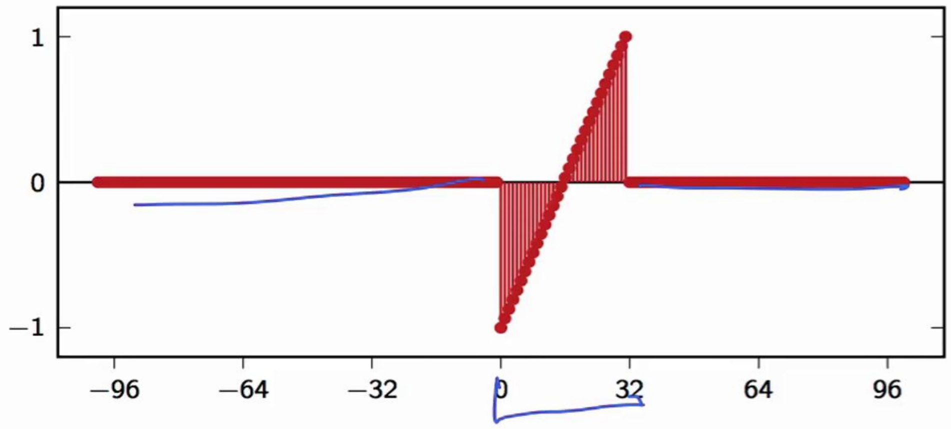
Now the finite support extension of this linear ramp will look like so.

Notes

Summary

10m 18s

Sawtooth: finite support extension



45

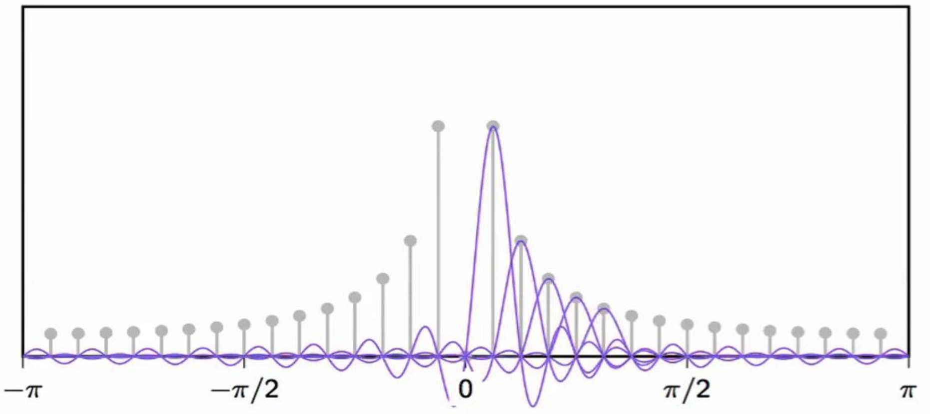
We put the original data set from 0-31, and we put 0 everywhere else.

Notes

Summary

10m 24s

DTFT of finite support extension (sketch)



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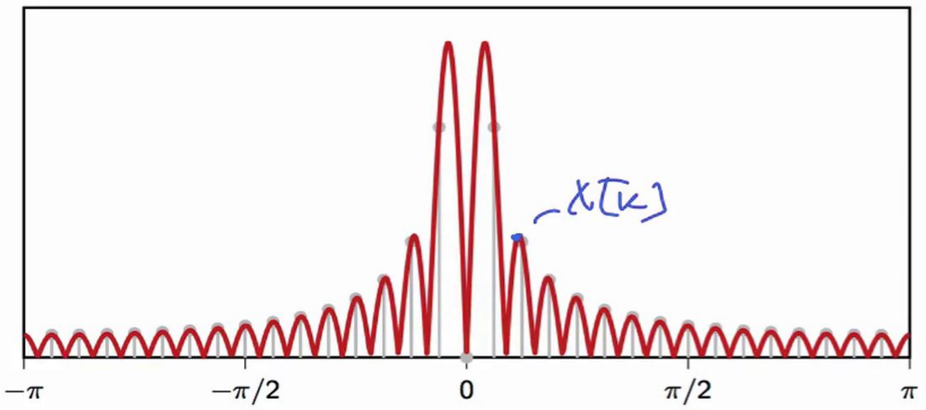
We take the DTFT of this infinite size sequence and what we have is this smooth interpolation of the original DFT coefficients. If these are the original DFT coefficients placed at multiples of 2 pi over N, like so, then we interpolate them with this sink-like shape that we derived before, and remember I said that this sink-like shape will be equal to 0 at all multiples of 2 pi over N. It is really an interpolator because it doesn't affect the original values of the signal at multiples of 2 pi over N, so the first coefficients will be interpolated by this function, the second will be interpolated by this, and so on and so forth, and then we sum all these terms together to obtain the DTFT of a finite support extension of our linear ramp.

Notes

Summary

10m 30s

DTFT of finite support extension



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You see at multiples of 2 pi over N, this interpolation will be equal to the value of X of k, and in between values, it will be a smooth curve. We're representing this here in magnitude.

Notes

Summary

11m 24s