

Week 3 – part 3B: Derivation of the Cable Equation



Neuronal Dynamics: Computational Neuroscience of Single Neurons

Week 3 – Adding Detail: Dendrites and Synapses

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3.1 Synapses

3.2 Short-term plasticity

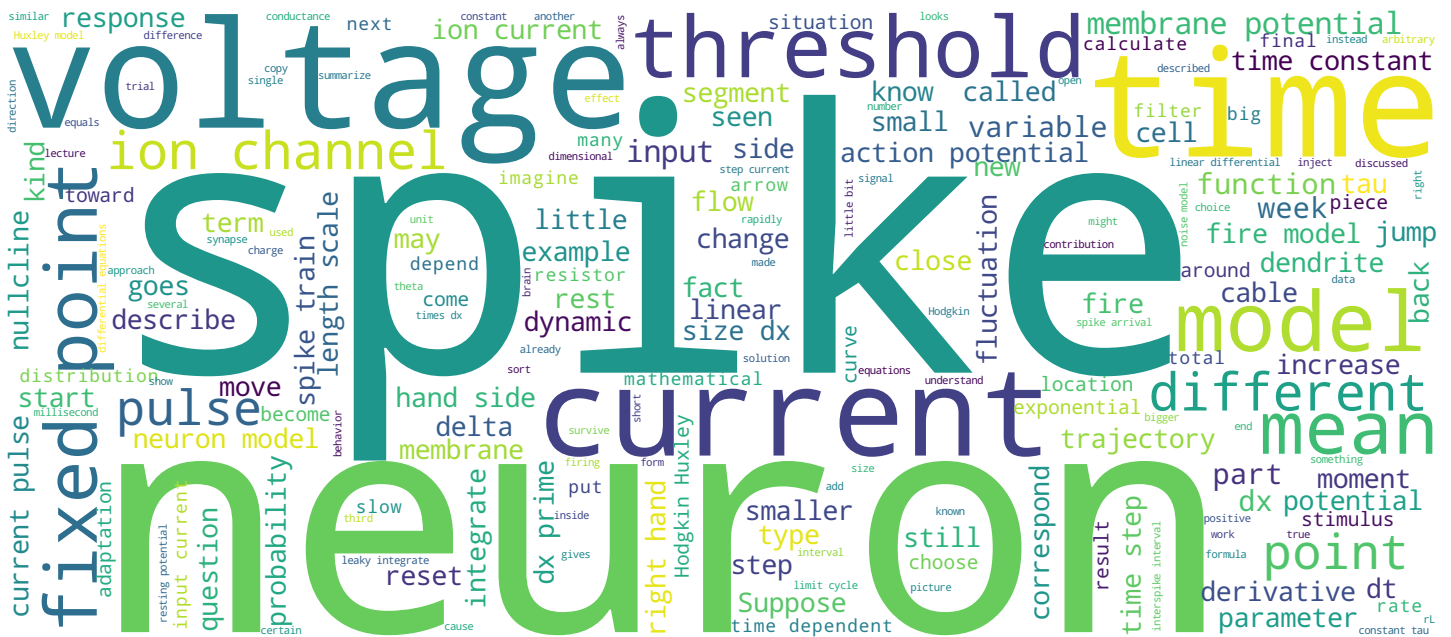
3.3 Dendrite as a Cable

Derivation of cable equation

3.4 Cable equation

3.5 Compartmental Models

- active dendrites



EPFL

Video



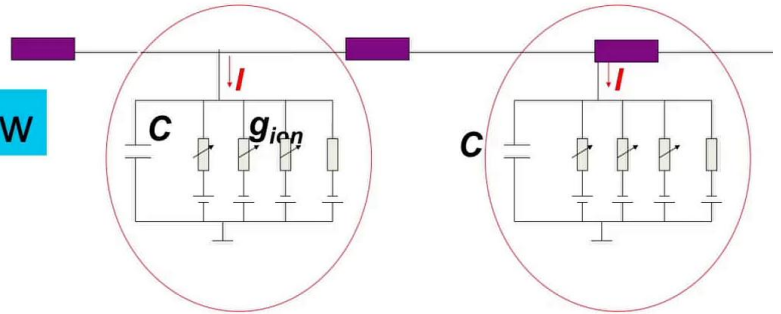
Neuronal Dynamics – 3.3b Derivation of Cable Equation

$$\frac{u(t, x - dx) - 2u(t, x) + u(t, x + dx))}{R_L} = C \frac{d}{dt} u(t, x) + \sum_{ion} I_{ion}(t, x) - I^{ext}(t, x) \quad (*)$$

Let's get rid of segment size dx

mathematical derivation, now

coupled segments



$$\frac{d^2}{dx^2} u(t, x) = cr_L \frac{d}{dt} u(t, x) + r_L \sum_{ion} i_{ion}(t, x) - r_L i^{ext}(t, x) \quad \text{(cable equation)}$$

Notes

Summary

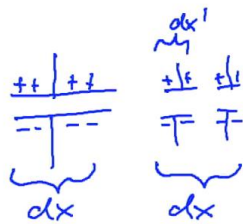


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Neuronal Dynamics – 3.3 Modeling the Dendrite

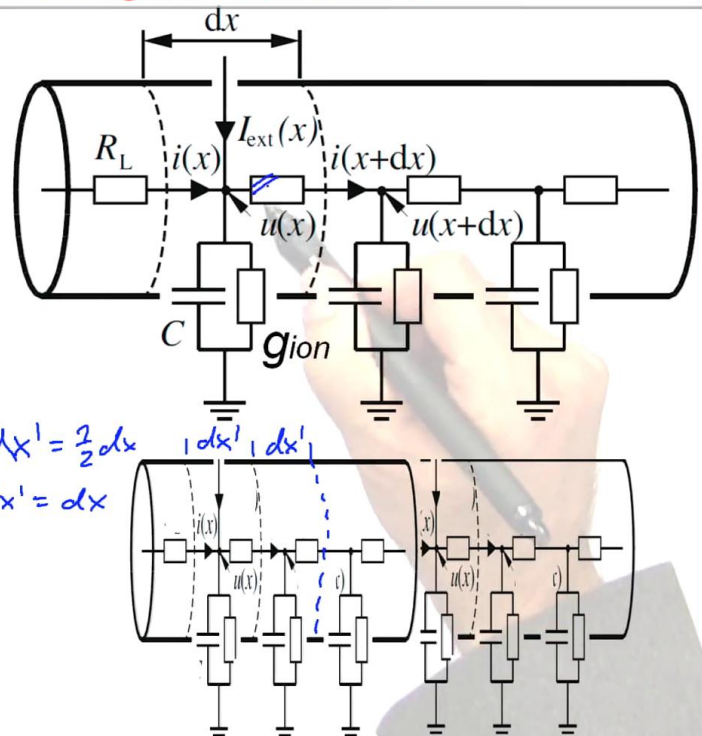
$$R_L = r_L dx$$

$$C = c dx$$



$$dx' = \frac{1}{2} dx$$

$$2 dx' = dx$$



Now here I've redrawn the same kind of picture but embedded in a circular table that you can imagine a little bit the structure of the dendrite as a cable. Now our segments have a certain size, they have a certain length scale. And this length scale is denoted here by dx . A dx is my choice. I've chosen a certain segment size. And with the same right, I could choose a smaller segment. I have a new segment which is now dx prime, which is just $1/2 dx$. So this is dx prime. It's smaller. Now what can we say? Well, on this scale that we had before there are now two segments. This is another dx prime. $2 dx$ prime gives dx . So the total capacitor that we have here is now split into two separate capacitors. So imagine I have a big capacitor, and now I split it in two smaller ones. So if this was the size dx , then this would be the size dx now. And dx prime is smaller. So, the total capacity that I have on a piece of membrane depends on my chosen length scale dx . If I choose a smaller length scale then the capacity will be smaller. The little c is the specific capacity, a unit of cable length. Now a similar argument can be made for the longitudinal resistor. So here's my longitudinal resistor in a segment of size dx .

Notes

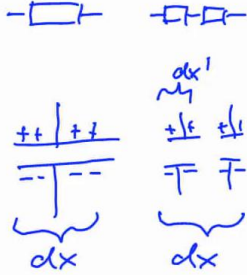
Summary



Neuronal Dynamics – 3.3 Modeling the Dendrite

$$R_L = r_L dx$$

$$C = c dx$$



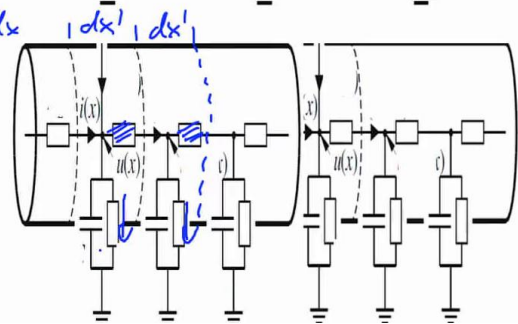
$$I_{ion} = i_{ion} dx$$

$$I^{ext} = i^{ext} dx$$



$$dx' = \frac{1}{2} dx$$

$$2 dx' = dx$$



Now, within the same segment dx , I now have two of these smaller resistors. So instead of one big resistor, I now have two smaller ones, two resistors in [INAUDIBLE]. The bigger I choose my segment, the bigger the resistor. And this is expressed by this formula here. r_L is the specific resistance per unit length of the dendritic cable. A similar argument can be made for the ion currents. Here I have my ion currents flowing through my piece of membrane. [INAUDIBLE], and in this piece of membrane, I have now two of these ion currents. So imagine that my membrane of a certain size dx , has a certain number of ion channels [INAUDIBLE] through which the ions can flow. Now if I replace this membrane by a smaller membrane, then where I had six ion channels before, I now only have three ion channels per segment. So the conductance will be proportional to my length dx . And, therefore, the total ion current that flows through will depend on my length scale dx of the segments. So dx is an arbitrary choice. But I can get rid of my choice dx by considering smaller and smaller segments. Here I move from a segment of size dx to a new segment, dx' , which is just half the size. I could iterate these steps and reduce my length scale even further.

Notes

Summary



Neuronal Dynamics – 3.3 Derivation of cable equation

$$\frac{u(t, x-dx) - 2u(t, x) + u(t, x+dx)}{R_L} = C \frac{d}{dt} u(t, x) + \sum_{ion} I_{ion}(t, x) - I^{ext}(t, x)$$

$$R_L = r_L dx //$$

$$\frac{u(t, x-dx) - 2u(t, x) + u(t, x+dx)}{r_L \cdot dx} = c \cdot \frac{dx}{dt} \frac{d}{dt} u(t, x) + \sum_{ion} i_{ion}(t, x) \cdot dx - \dots \quad C = c dx$$

$$I_{ion} = i_{ion} dx$$

$$\frac{u(t, x-dx) - 2u(t, x) + u(t, x+dx)}{r_L (dx)^2} = c \cdot \frac{d}{dt} u(t, x) + \sum_{ion} i_{ion}(t, x) - i^{ext}(t, x) \quad I^{ext} = i^{ext} dx$$

✎ ✎ ✎ ✎

Now let's try to do that. So I've repeated here the equation from the previous slides that's the discrete space equation that reflects the conservation of currents in a piece of dendrite. Let's now use our expressions that we just discussed before. So I copy the numerator x minus dx minus $2dx$ plus tx plus dx , and I write iL as the specific longitudinal resistance times dx from this formula here. Similarly, the big C becomes the small c times dx . And I have a d/dt of u of tnx . And the big ion current becomes the small ion current. i_{ion} tx times dx . And the same is true for the external current. Now I see that, in each term on the right hand side, I have the dx which I can bring over to the other side. So I copy again the numerator, x minus dx $2u$ tx plus u tx plus dx . Now, in the denominator, I have the dx squared because I brought this dx to the other side, I divided by dx . On the right hand side I now have the little c d/dt u plus sum of all ion types and all ion channel types q of x -- the dx has gone-- and I copied the external current-- i external of t and x . Let's now take the limit that dx becomes very small.

Notes

Summary



5m 06s

Neuronal Dynamics – 3.3 Derivation of cable equation

$$\frac{u(t, x - dx) - 2u(t, x) + u(t, x + dx)}{R_L} = C \frac{d}{dt} u(t, x) + \sum_{ion} I_{ion}(t, x) - I^{ext}(t, x)$$

$$R_L = r_L dx \quad //$$

$$\frac{u(t, x - dx) - 2u(t, x) + u(t, x + dx)}{r_L \cdot dx} = c \cdot \frac{dx}{dt} \frac{d}{dt} u(t, x) + \sum_{ion} i_{ion}(t, x) \cdot dx - \dots \quad C = c dx$$

$$I_{ion} = i_{ion} dx$$

$$I^{ext} = i^{ext} dx$$

$$\lim_{dx \rightarrow 0} \frac{u(t, x - dx) - 2u(t, x) + u(t, x + dx)}{dx^2} = c \cdot \frac{d}{dt} u(t, x) + \sum_{ion} i_{ion}(t, x) - i^{ext}(t, x)$$

$$// \quad \frac{d^2}{dx^2} u(t, x) = c r_L \frac{d}{dt} u(t, x) + r_L \sum_{ion} i_{ion}(t, x) - r_L i^{ext}(t, x) \quad // \quad \text{cable equation}$$

In fact, people who have done Taylor series expansions, calculus, people who have done numerical analysis, will see, rapidly, that this just is the second derivative. And so I have a final equation that has here, on the left hand side, the second derivative with respect to space of the voltage. On the right hand side it has the first derivative with respect to time and the different ion currents. Between this step and the final line the r_L has been moved to the other side, we have multiplied by r_L . Now, this equation is known as the cable equation. What you see here is, on the right hand side, I still have all the different types of ion channels.

Notes

Summary



Neuronal Dynamics – 3.3 Dendrite as a cable



$$\frac{d^2}{dx^2}u(t,x) = cr_L \frac{d}{dt}u(t,x) + r_L \sum_{ion} i_{ion}(t,x) - r_L i^{ext}(t,x)$$

$$\sum_{ion} i_{ion}(t,x) = leak \quad \text{passive dendrite}$$

$$\sum_{ion} i_{ion}(t,x) = Ca, Na, \dots \quad \text{active dendrite}$$

$$\sum_{ion} i_{ion}(t,x) = Na, K, \dots \quad \text{axon}$$

In fact, you can choose and adapt the ion channel types to the dendritic segment of your dendrite, of the experimental preparation that you have under your hands. Sometimes people make the choice of a passive dendrite. But if you know what type of ions are available, what type of ion channels are around in your segment of the dendrite at location x , then you can put in these ion channels explicitly. The same type of cable equation can also describe the spread of an action potential, the traveling of an action potential, in an axon. The cable equation is, in fact, nothing special to neuroscience. The cable equation is a well known equation from electricity, and describes the spread of electric current in all sorts of cables. Natural ones, like neurons, or artificial ones like you have in your home.

Notes

Summary



8m 51s

Neuronal Dynamics – Quiz 3.3

Multiple answers possible!

Scaling of parameters.

Suppose the ionic currents through the membrane are well approximated by a simple leak current. For a dendritic segment of size dx , the leak current is characterized by a membrane resistance R . If we change the size of the segment from dx to $2dx$

- ☐ the resistance R needs to be changed from R to $2R$.
- ☐ the resistance R needs to be changed from R to $R/2$.
- ☐ R does not change.
- ☐ the membrane conductance increases by a factor of 2.

⌂ / ☐ ➡

Please take a few minutes to look at the quiz before we continue with the next part.

Notes

Summary



10m 02s