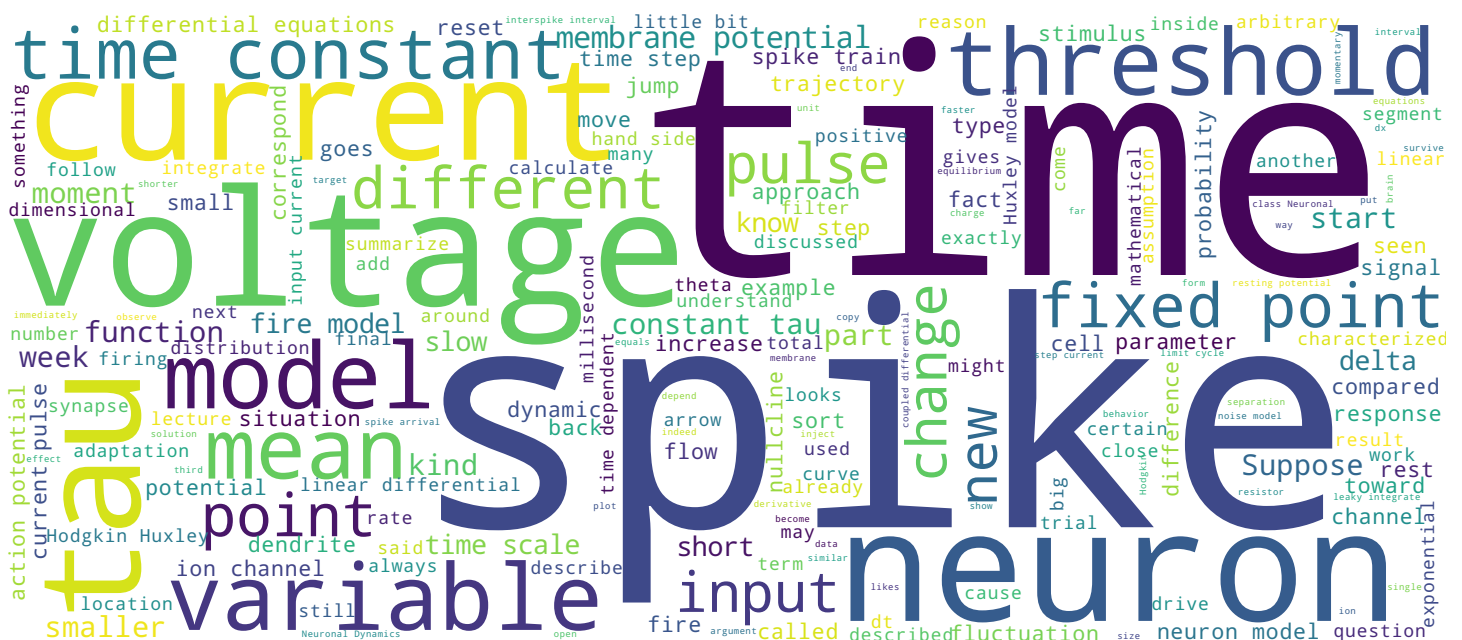


ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

EPFL, Lausanne, Switzerland

- from two to one dimension



EPFL



Week 4 – part 1 : Separation of time scales



4.1 From Hodgkin-Huxley to 2D

- ✓ - Overview: From 4 to 2 dimensions
- MathDetour 1: Separation of time scales
- MathDetour 2: Exploiting similarities

4.2 Phase Plane Analysis

- role of nullclines

4.3 Analysis of a 2D Neuron Model

- MathDetour 3: Stability of fixed points

4.4 Type I and II Neuron Models

- where is the firing threshold?
- MathDetour 4: separation of time scales

4.5. Nonlinear Integrate-and-fire

- from two to one dimension



So welcome back to the class, Neuronal Dynamics. This mathematical detour lecture is about the problem of separation of time scales. Those who have had no problem with the quiz of part 1 of Week 4 can skip this lecture, and go immediately to the next lecture. For the others, I have added here some details that may help you understand the notion of separation of time scales.

Notes

Summary



0m 06s

Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + c(t)$$

$$\tau_2 \frac{dy}{dt} = f(y) + g(x)$$

Separation of time scales

$$\tau_1 \ll \tau_2 \quad \text{"much smaller" than}$$

Reduced 1-dimensional system

$$\tau_2 \frac{dy}{dt} = f(y) + g(c(t))$$

✎ ✎ ✎ ✎

The assumption is that we have two coupled differential equations, and the time scales of these differential equations are τ_1 and τ_2 , respectively.

Notes

Summary

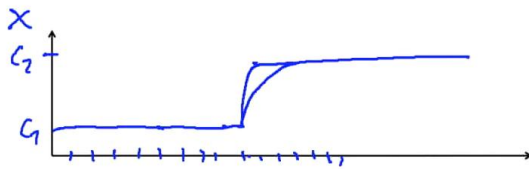


0m 42s

Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Linear differential equation $\tau_1 \frac{dx}{dt} = -x + c(t)$ || τ_1 short

$x \rightarrow c(t)$



step

✎ ✎ ✎ ✎

And the assumption is that one of the two variables is much faster, or τ_1 is much faster than τ_2 . So let's look at an example. We start off with a single linear differential equation characterized by a time scale τ_1 . x likes to go towards c . c is a constant, but the value of the constant may change, so you may think about this as some kind of input. First it has a constant value, say, c_1 , and at some point, it changes to a new value, c_2 . Then we know that x will approach the value c with some time constant τ_1 . So suppose it was already at the equilibrium, so I plot here x . x was at equilibrium. At this point here, we have to change, and x will follow, and approach the new value, c_2 , exponentially with some time constant τ_1 . Now suppose that τ_1 is short. That means it's sort of goes very quickly up to the new value. Now if I don't look at a very fine time scale, but if I sort of step back a little bit, and I look at some steps, then it looks as if the variable x has jumped. It has responded immediately. Here, it was still below, and now suddenly, it's up here. So x approaches the value c of t rapidly. "Rapidly" means there's a time constant τ_1 , but τ_1 is short.

Notes

Summary

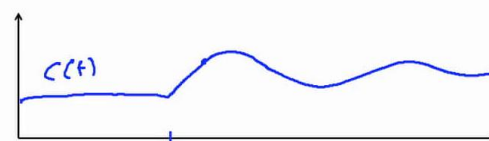
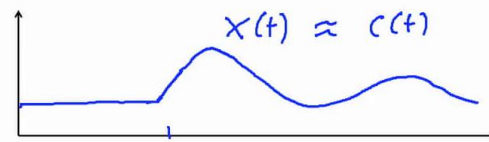
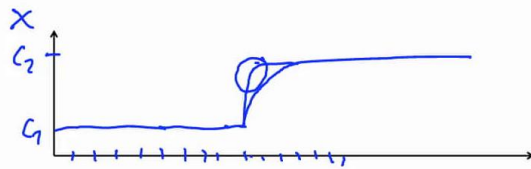


0m 55s

Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Linear differential equation $\tau_1 \frac{dx}{dt} = -x + c(t)$ || τ_1 short

$\tau_1 \rightarrow 0: x \rightarrow c(t)$



step

'slow drive'

Now if I look more closely, if I increase my resolution, I will always find that, well, indeed, there's this phase of exponential approach. And the reason is that a step signal is a signal that makes a rapid change. Now let's consider a situation where the change is slower. Suppose that at this point here, my signal, c of t — which drives my differential equation, changes to some new value, or it might continue to change all the time. Now each moment of time, x will approach the present value c of t , but takes some time τ . However, if the time constraint τ_1 is short, then it will sort of approach immediately its value, and it will copy the momentary value, c of t . There's a tiny, tiny little delay between the two, but it's not visible. So it's a very good approximation that x is equal to its target value, c of t . In fact, this becomes true in the limit that τ_1 goes to 0. Then, x will immediately take its new value, c of t . So far for single differential equation.

Notes

Summary



3m 10s

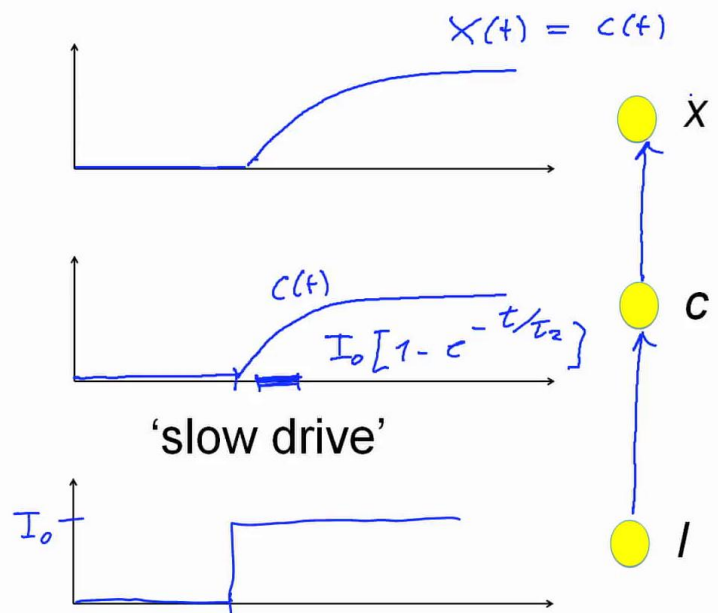
Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Two differential equations

$$\tau_1 \frac{dx}{dt} = -x + c(t)$$

$$\tau_2 \frac{dc}{dt} = -c + I(t)$$

$$\tau_1 \ll \tau_2$$



Now let's look at a system of coupled differential equations. I assume that the time constant of the x variable τ_1 is much shorter than the time constant τ_2 . Now here, I have an external input which, for example, may be switched on at some time to some new value I_0 . Now the variable c is low-pass filtered. It responds slowly to the change in the input. So c itself will approach its new value, I_0 , with a time constant τ_2 . So that's basically the time scale here. That sets τ_2 . Now we have assumed that τ_1 is much smaller than τ_2 . Now from the perspective of the variable x , we know we have exactly the situation that we discussed before. We have a slowly changing target value, c of t . Therefore, the variable x will just follow its target value, c of t . Why is that? The reason is that τ_1 is much smaller than τ_2 . So the fact that the variable c is given by a differential equation means that c is slower than a potential external drive. c of t is guaranteed to react slowly, because its dynamics are characterized by the time constant τ_2 , which is slow. OK, so these are coupling differential equations, but the coupling is one way.

Notes

Summary



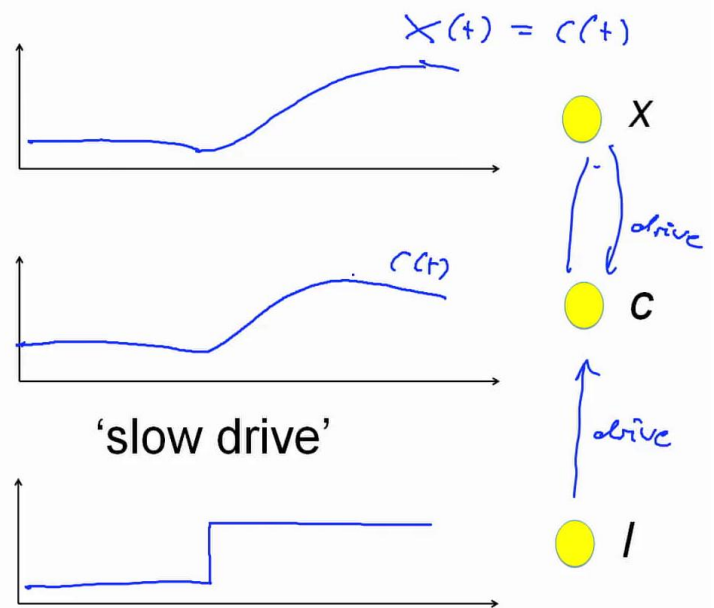
Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + c(t)$$

$$\tau_2 \frac{dc}{dt} = -c + f(x) + I(t)$$

$$\tau_1 \ll \tau_2$$



Let's now generalize this. So I might still have some kind of input which controls my variable c . But then c also gets input from x via this function, f of x . Moreover, as before, c enters the x equation. We make the assumption that τ_1 is much smaller than τ_2 . Now there could be a fast drive here, like as [? occurred?] as before. There can also be a fast drive coming from above. It doesn't really matter where the drive comes from, because whatever comes as a drive is low-pass filtered with this time constant, τ_2 . So this signal here will be slow. I don't know exactly what it is, but it will be slow. Now since it's slow, x of t can follow. x will approach its target of c of t rapidly-- rapidly, because τ_1 is much smaller than τ_2 . Therefore, x will be some copy of c of t . I admit that my hand-drawn copy is not very nice, but I hope you get the idea.

Notes

Summary



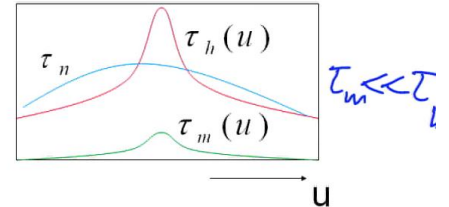
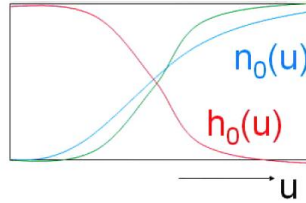
Neuronal Dynamics – Reduction of Hodgkin-Huxley model

$$C \frac{du}{dt} = - \overbrace{g_{Na} m^3 h (u - E_{Na})}^{I_{Na}} - \overbrace{g_K n^4 (u - E_K)}^{I_K} - \overbrace{g_l (u - E_l)}^{I_{leak}} + \overbrace{I(t)}^{\text{stimulus}}$$

$$\frac{dm}{dt} = - \frac{m - m_0(u)}{\tau_m(u)} \quad \text{fast}$$

$$\frac{dh}{dt} = - \frac{h - h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = - \frac{n - n_0(u)}{\tau_n(u)}$$



dynamics of m is fast

$$m(t) = m_0(u(t))$$

Fast compared to what?

Now this is exactly the kind of argument we used for Hodgkin-Huxley model. We have an m , and this m variable is characterized by a time constant-- a voltage-dependent time constant. And τ_m is much smaller than the other τ , τ_h or τ_n . So we have a fast variable. m is the fast variable. And it's called fast compared to what? Well, it's fast compared to the external. It's fast compared to the dynamics of the other two [INAUDIBLE] variables. τ_h is slower, τ_n is slower. And it must also be fast compared to the stimulus. The stimulus should not be too fast in this case, so that we can say m follows immediately the voltage. m approaches m_0 of u . If the voltage is slow-- which is the case if the stimulus is not too fast, and the other variables cannot be fast because they are controlled by slower time constants. So m will approach 0 of u rapidly-- fast compared to the voltage. And that's why we can replace m by its instantaneous value, m_0 of u .

Notes

Summary



Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + h(y)$$

$$\tau_2 \frac{dy}{dt} = f(y) + g(x)$$

Separation of time scales

$$\tau_1 \ll \tau_2 \rightarrow x = h(y)$$

Reduced 1-dimensional system

$$\tau_2 \frac{dy}{dt} = f(y) + g(h(y))$$



Now this works for all sorts of coupled differential equations. Whenever you observe a difference in time scale, you can exploit this difference, and you can eliminate the fast variable.

Notes

Summary



10m 44s

Neuronal Dynamics – Quiz 4.2.

A- Separation of time scales:

We start with two equations

$$\tau_1 \frac{dx}{dt} = -x + y + I(t)$$

$$\tau_2 \frac{dy}{dt} = -y + x^2 + A$$

[] If $\tau_1 \ll \tau_2$ then the system can be reduced to

$$\tau_2 \frac{dy}{dt} = -y + [y + I(t)]^2 + A$$

[] If $\tau_2 \ll \tau_1$ then the system can be reduced to

$$\tau_1 \frac{dx}{dt} = -x + x^2 + A + I(t)$$

[] None of the above is correct.

B- Separation of time scales:

A channel with gating variable r , given by

$$\tau_1 \frac{dr}{dt} = -r + r_0(u)$$

influences the voltage

$$\tau_2 \frac{du}{dt} = -(u - u_0) + r^2 A$$

We assume that $\tau_1 \ll \tau_2$

In this case a reduction of dimensionality

[] is not possible

[] is possible and the result is

$$\tau_2 \frac{du}{dt} = -u + u_0 + [r_0(u)]^2 A$$

[] is possible and the result is

$$\tau_1 \frac{dr}{dt} = -r + r_0(u_0 + r^2 A)$$

Now before you go on, please check that you understood what I said by looking at the quiz questions now.

Notes

Summary



11m 01s