

Week 4 – MathDetour 2: Exploiting similarities



Neuronal Dynamics: Computational Neuroscience of Single Neurons

Week 4 – Reducing detail: Two-dimensional neuron models

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4.1 From Hodgkin-Huxley to 2D

- ✓ - Overview: From 4 to 2 dimensions
- ✓ - MathDetour 1: Separation of time scales
- MathDetour 2: Exploiting similarities

4.2 Phase Plane Analysis

- role of nullclines
- MathDetour 3: Stability of fixed points

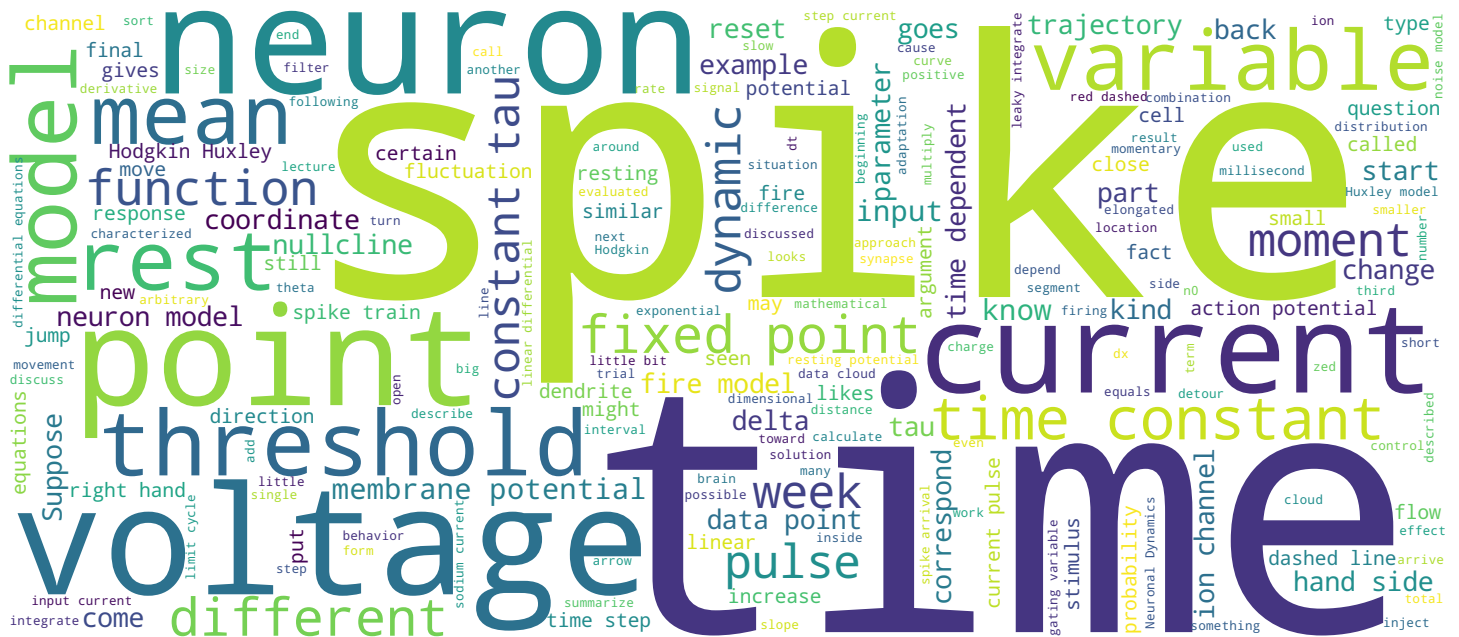
4.3 Analysis of a 2D Neuron Model

4.4 Type I and II Neuron Models

- where is the firing threshold?
- MathDetour 4: separation of time scales

4.5. Nonlinear Integrate-and-fire

- from two to one dimension



EPFL

Video



Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

Reduction of Hodgkin-Huxley Model to 2 Dimension

- step 1:
separation of time scales
(→ 4.1 and 4-Detour1)
- step 2:
exploit similarities/correlations

Now !

Welcome back to Week 4 of the class Neuronal Dynamics. In Lecture 1 of this week, I saw that a reduction of the Hodgkin-Huxley Model, and to do so, I exploited two observations. The first one is the dynamics of m , the gating variable m , is much faster than those of the other variables. The second observation is the dynamics of h and n , the two remaining gating variables, are very similar. And this is what we are going to discuss now in more detail.

Notes

Summary



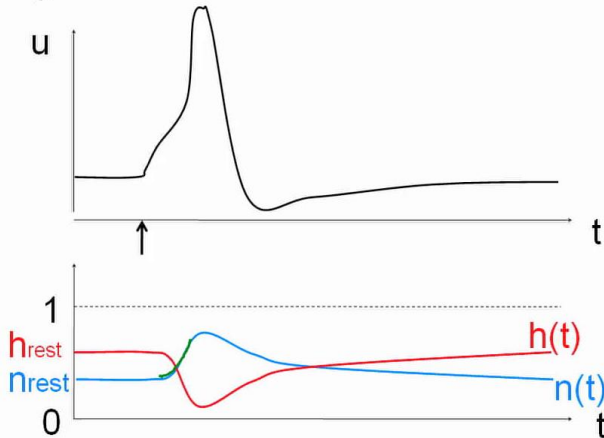
0m 04s

Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

$$C \frac{du}{dt} = -g_{Na} m^3 h (u - E_{Na}) - g_K n^4 (u - E_K) - g_l (u - E_l) + \overset{\text{stimulus}}{\downarrow} I(t)$$

dynamics of h and n is similar

$$\longrightarrow 1 - h(t) = \underline{a} n(t)$$



So here is the Hodgkin-Huxley equation, and as I've discussed in Lecture 1 of this week, the dynamics of h and n are similar in the sense that if n goes up, h goes down. If h goes up, n goes down. Therefore, we have a mirror symmetry, which led us to say, well, $1 - h$ is very similar to n of t with some constant a . And I would like to make this argument more precise now.

Notes

Summary



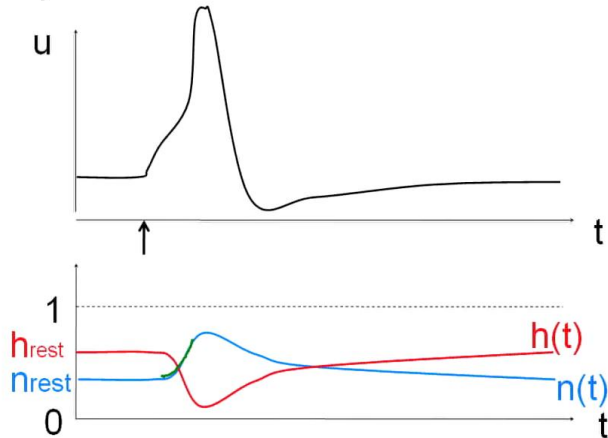
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Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

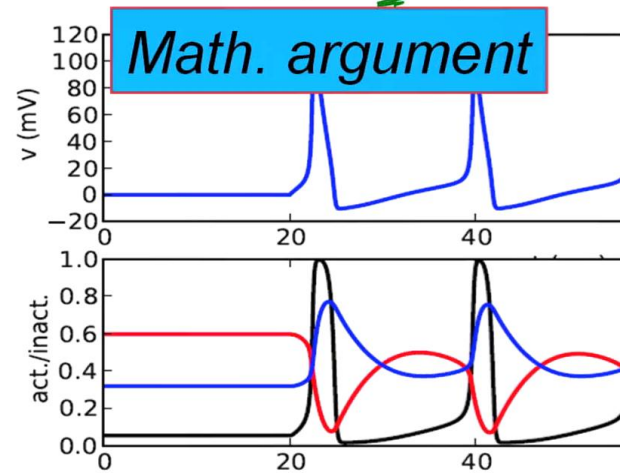
$$C \frac{du}{dt} = -g_{Na} m^3 h (u - E_{Na}) - g_K n^4 (u - E_K) - g_l (u - E_l) + I(t)$$

stimulus

dynamics of h and n is similar



$$1 - h(t) = a n(t)$$



So let's look at the graph here on the right hand side.

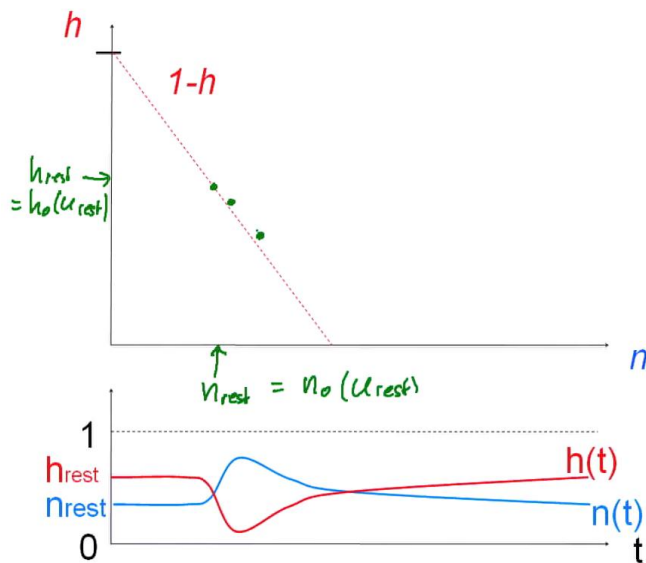
Notes

Summary

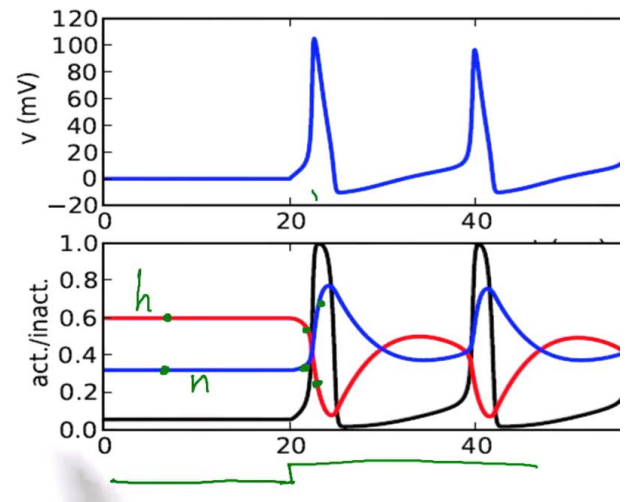


Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

dynamics of h and n are similar



$$1 - h(t) = a n(t)$$



At the beginning, the voltage here is at rest. I've chosen the voltage scaled such that the resting potential is at 0. While the voltage is at the rest, the activation variable, n , is at a certain value and the inactivation variable, h , of the sodium current also takes a certain value. I can take the combination of those two, n and h , and plug them in the two-dimensional graph. Say here is my resting value of the two, so this is n_{rest} . More precisely, it's n_0 evaluated at U_{rest} . Here is h_{rest} . More precisely, it's h_0 evaluated at the resting potential. Now this is one point, one combination of n and h . Now at some point, here at 20 milliseconds, a step current is switched on and the neuron starts to emit spikes. During the spikes, the h variable does some detours. The n variable also does some detours, and the two of them are sort of mirror images. For example, immediately after the onset of the stimulus, I see that h is going down. This might be this point here. Then a little bit later, h has gone down even further. n has gone up even further. So this might be this point. A little bit later, h goes through a minimum at the same moment h goes through a maximum, and then they move down again together.

Notes

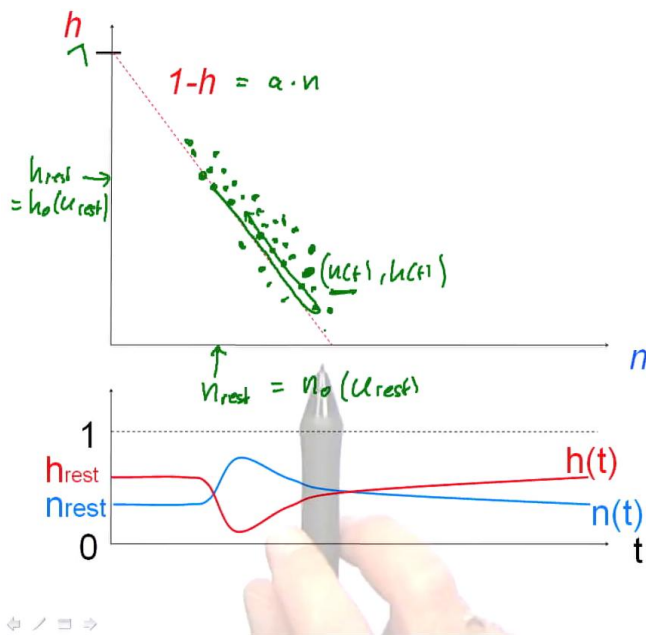
Summary



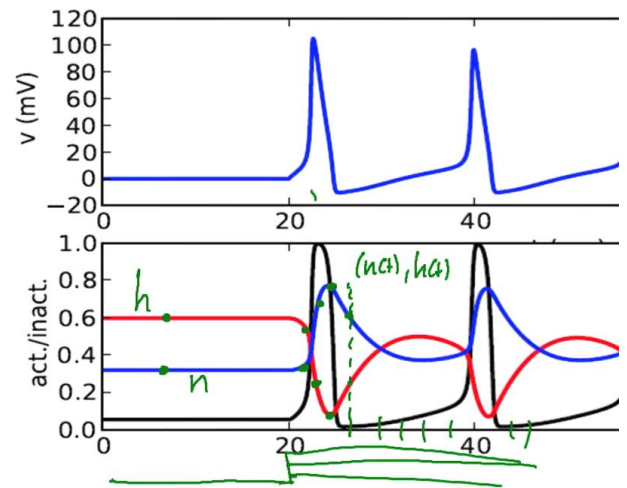
1m 38s

Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

dynamics of h and n are similar



$$1 - h(t) = a n(t)$$



So at each moment in time, I can read out a combination n of t , h of t . For example, this might be the point n of t , h of t . And there are many points. I can evaluate this at each moment in time. I can take one of these samples, and it'll find many points in this two-dimensional plane, and each point is characterized by two coordinates, its value n and its value h , which gives the vertical axis. So during one of these spikes, n and h do some kind of movement in this two-dimensional coordinate space. If I gave a different stimulus, the trajectories might be different. However, what you find is that all data points lie in close proximity to this red dashed line. There's a cloud of data points, and this cloud of data points does not fill the two-dimensional space equally. But this cloud of data points is elongated and it has a preferred direction, and this preferred direction is given by the red dashed line. So the red dashed line represents the function $1 - h$ equals a times n . So if n is 0, I'm here. Then the red curve is at 1. More generally, this could be some arbitrary constant, some constant C , but for the sake of simplicity, I've put this constant to 1 so that it's easy to draw this line.

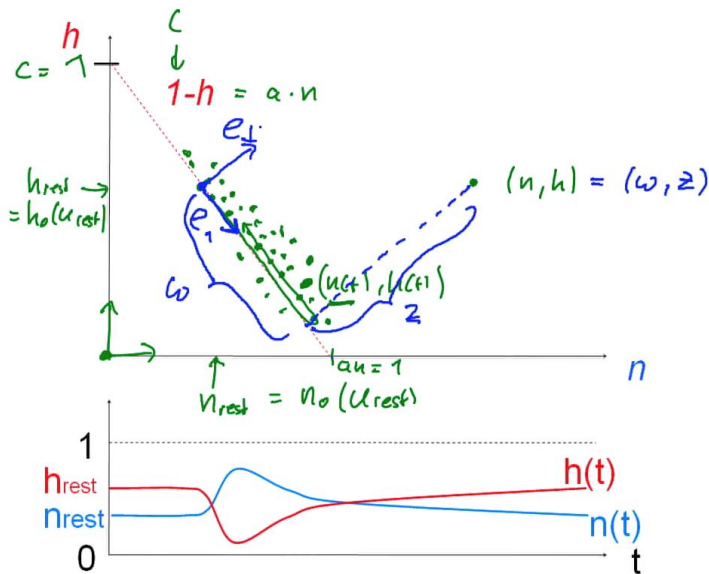
Notes

Summary

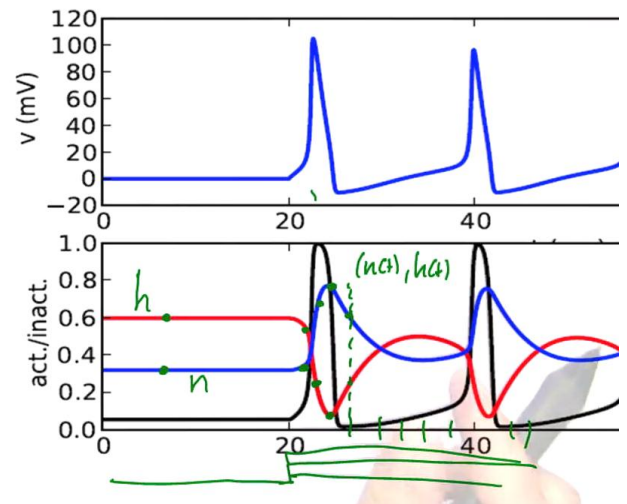


Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

dynamics of h and n are similar



$$1 - h(t) = a n(t)$$



So if h has a value of 0, then I'm at this point here, a times n equal to 1. Now any arbitrary data point, like this point down here, can be described by the two coordinates n and h . This is the coordinate system that is sitting here at the origin, and this is the direction of the n variable. This is the direction of the h variable. However, I could choose a completely different coordinate system. For example, I could say, well given that the data cloud is elongated, I put my coordinate system with a origin here, and then I define my first axis in this direction and my second axis in this direction. And the same point that I have here can now be also expressed in this new coordinate system by its distance from this new origin along the red dashed line. That would be this value here-- let's call it w . w would be the location along the red dashed line, and z would be the second coordinate. So this is my first coordinate vector and the second one is orthogonal to the first one. Now any data point in the two-dimensional space can be expressed by two coordinates, and after change of the coordinate system, the values are different, but it's still the same point.

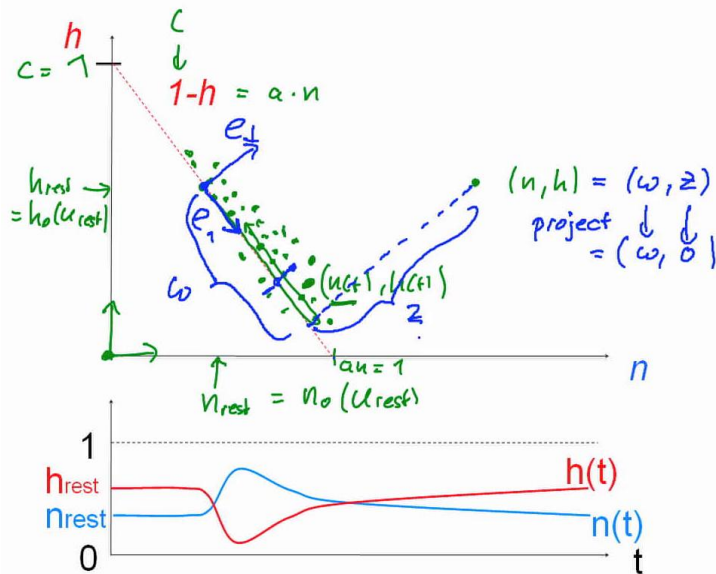
- Notes

Summary

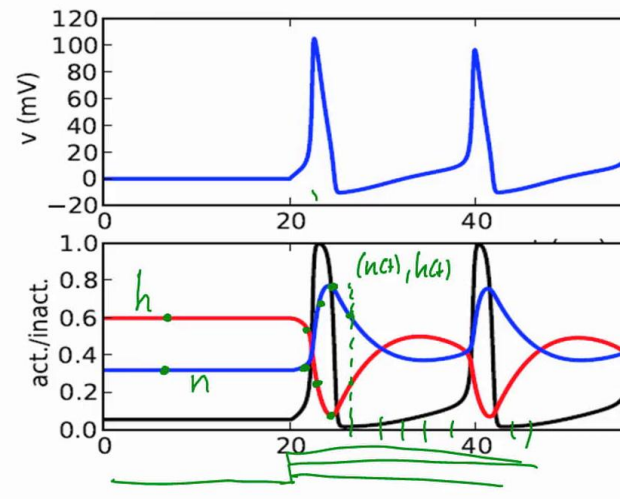


Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

dynamics of h and n are similar



$$1 - h(t) = a n(t)$$



Now as we have seen all the data points line a cloud of points that's elongated, and in fact, it's very close to this red line. So this point here, for example, it will have a w coordinate given by the distance here, but it's z value will be very small. So I can project a data point by keeping w but putting the z value down to 0. I suppress my second coordinate because it's always small, and this is what gives the reduction of dimensionality.

Notes

Summary



Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

dynamics of h and n are similar

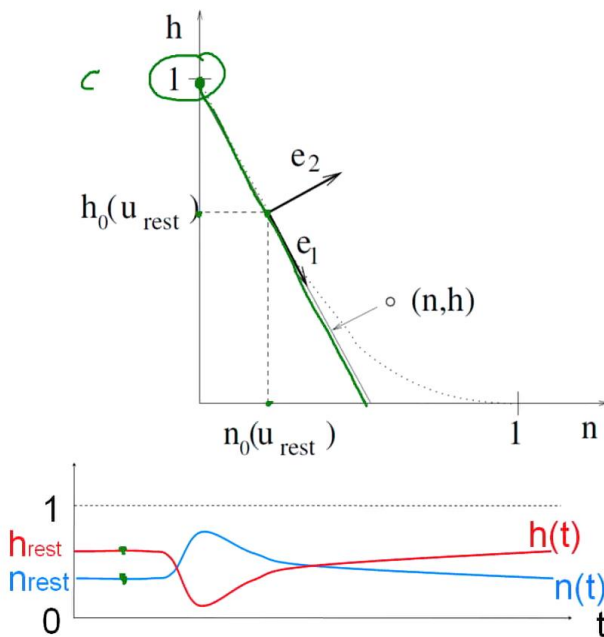
$$1 - h(t) = a n(t)$$

at rest

$$\frac{1 - h_0(u_{\text{rest}})}{h_0(u_{\text{rest}})} = a$$

$$\frac{dh}{dt} = - \frac{h - h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = - \frac{n - n_0(u)}{\tau_n(u)}$$



So here, again the idea-- an arbitrary point is projected onto, in this case, the solid line. It's projected, and its position along the solid line is represented by the vector e_1 that corresponds to the biggest elongation of this data cloud. In general, this point here does not need to be 1. It could be some arbitrary constant C . And the question is, how do we get this slope? Well it comes out of this elongation of the data cloud. But for the case of simplicity, let's assume we put this here at 1. Now a simple way to determine the value a is to say, well, at least when the system is at rest, I have my n_{rest} -- that's this value-- I have my h_{rest} -- that's this value. At least when the system is at rest, I don't want to make a mistake. So the resting point should lie on my straight line. At the same time, I know this point here. The line starts at 1, which means now I know two points. I know the upper point and I know the resting point, and therefore, the slope is fixed. So at rest, I have the condition h_0 of u_{rest} equals a times n_0 of u_{rest} . I put this to the other side. I have n_0 of u_{rest} , and that would give me this slope parameter a once I fix my constant, C , to 1. OK.

Notes

Summary



Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

dynamics of h and n are similar

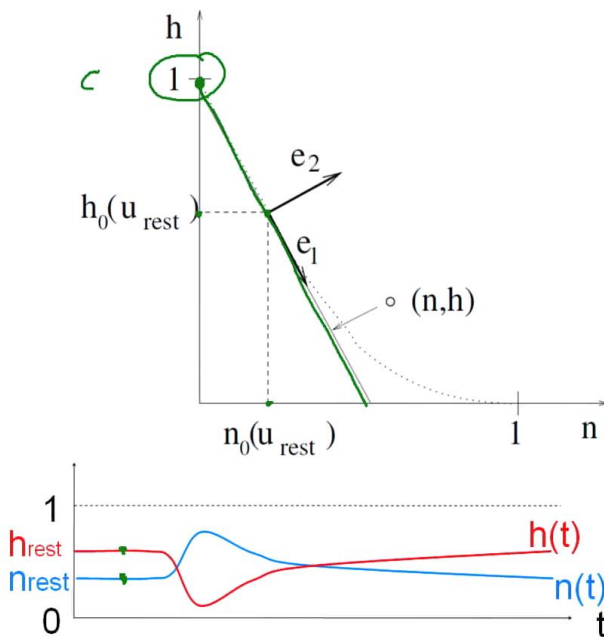
$$1 - h(t) = \underline{a} n(t)$$

at rest

$$\boxed{\frac{1 - h_0(u_{\text{rest}})}{h_0(u_{\text{rest}})} = a}$$

$$\frac{dh}{dt} = - \frac{h - h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = - \frac{n - n_0(u)}{\tau_n(u)}$$



So this is how we can reduce a two-dimensional description in this two-dimensional plane to a one-dimensional coordinate that corresponds to the projection onto my green line. But what about the dynamics? We know that h likes to go towards h_0 with some time constant, τ_h . n likes to go to n_0 with some time constant, τ_n .

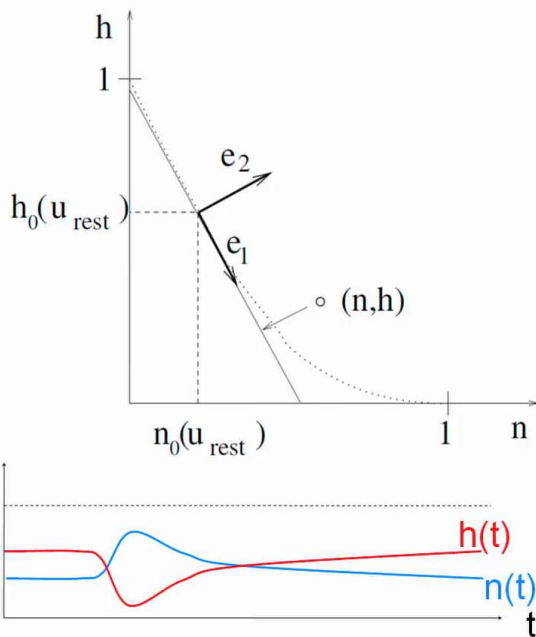
Notes

Summary



11m 27s

Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations



dynamics of h and n are similar

- (i) Rotate coordinate system
- (ii) Suppress one coordinate
- (iii) Express dynamics in new variable

$$1 - h(t) = a n(t) = w(t)$$

$$\left. \begin{aligned} \frac{dh}{dt} &= -\frac{h - h_0(u)}{\tau_h(u)} \\ \frac{dn}{dt} &= -\frac{n - n_0(u)}{\tau_n(u)} \end{aligned} \right\} \frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_{\text{eff}}(u)}$$

Therefore, it's not surprising that we can reformulate a dynamics for the new variable, w , such that we say, w likes to go to w_0 with some effective time constant, τ_{eff} .

Notes

Summary

11m 57s



Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

$$C \frac{du}{dt} = \underbrace{-g_{Na} [m(t)]^3 h(t) (u(t) - E_{Na})}_{I_{Na}} - \underbrace{g_K [n(t)]^4 (u(t) - E_K)}_{I_K} - \underbrace{g_l (u(t) - E_l)}_{I_{leak}} + I(t)$$

$$C \frac{du}{dt} = -g_{Na} m_0(u)^3 (1-w)(u - E_{Na}) - g_K \left[\frac{w}{a}\right]^4 (u - E_K) - g_l (u - E_l) + I(t)$$

1) dynamics of m are fast $\longrightarrow m(t) = m_0(u(t))$

2) dynamics of h and n are similar $\longrightarrow \underbrace{1-h(t)}_{w(t)} = \underbrace{a n(t)}_{w(t)}$

$$\left. \begin{aligned} \frac{dh}{dt} &= -\frac{h - h_0(u)}{\tau_h(u)} \\ \frac{dn}{dt} &= -\frac{n - n_0(u)}{\tau_n(u)} \end{aligned} \right\} \longrightarrow \frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_{eff}(u)}$$

And with this, we have arrived at the end of the argument. In the Hodgkin-Huxley equation, we have inserted the momentary value m_0 of u . We have replaced h by 1 minus w , we have replaced n by w over a , and we have found an additional equation for this new equation, w . w likes to go towards w_0 with some time constant, τ_{eff} . Constructing this w_0 -- constructing the time constant, τ_{eff} , out of those two functions is not trivial. It's some additional calculation, but it is possible.

Notes

Summary

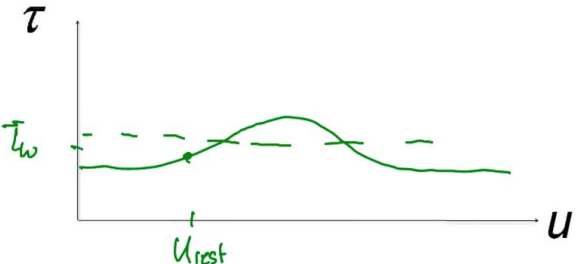


12m 18s

Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

$$\begin{aligned}
 \underbrace{RC}_{\tau} \frac{du}{dt} &= - \overbrace{g_{Na} m_0(u)^3 (1-w)(u-E_{Na})}^{I_{Na}} - \overbrace{g_K \left(\frac{w}{a}\right)^4 (u-E_K)}^{I_K} - \overbrace{g_l (u-E_l)}^{I_{leak}} + \underbrace{RI(t)}_{\mathcal{R}} \\
 \tau_w \frac{dw}{dt} &= - \frac{\underbrace{w - w_0(u)}_{\frac{1}{\tau_w} \tau_{eff}(u)}}{\tau_w}
 \end{aligned}$$

$\mathcal{R} = \frac{1}{g_l}$



$$\begin{aligned}
 \tau \frac{du}{dt} &= F(u(t), w(t)) + RI(t) \\
 \tau_w \frac{dw}{dt} &= G(u(t), w(t))
 \end{aligned}$$

So this is our final set of two equations. What I can do now is I can try to pull out a time constant. I multiply with r where r is 1 over g_l . I multiply with a leak resistance, so I have RC , and then I have here g_{Na} times R , R times g_K . And then R times g_l will go away, R times I . And this is the membrane time constant, τ , that controls the dynamics of the voltage variable, u . Let me do something similar for the w equation. I have here some effective time constant-- say it looks like this. Just a guess. Say here's my resting value, u_{rest} . This turns out to be a typical value, so this would be the resting value, and I call this τ_w . So let's multiply with τ_w . Then I have here τ_f divided by τ_w . So this is a normalization factor, and the net result is that I can write my w equation with a time constant, τ_w , which is representative. It could be the resting value. It could also be the mean value of these different values that I see here. Some representative value of τ that controls the w dynamics on the right hand side of the variables u and w . u comes in here and there, w comes in here. And I have my voltage equation again with u and w .

Notes

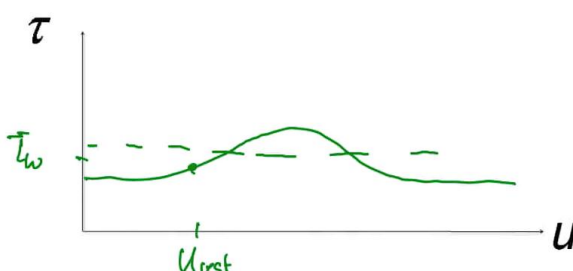
Summary



Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

$$\begin{aligned}
 \underbrace{RC}_{\tau} \frac{du}{dt} &= - \underbrace{g_{Na} m_0(u)^3 (1-w)}_{I_{Na}} (u - E_{Na}) - \underbrace{g_K \left(\frac{w}{a}\right)^4}_{I_K} (u - E_K) - \underbrace{g_l}_{I_{leak}} (u - E_l) + \underbrace{RI(t)}_{\mathcal{R}} \\
 \tau_w \frac{dw}{dt} &= - \frac{w - w_0(u)}{\tau_w \tau_{eff}(u)}
 \end{aligned}$$

$\mathcal{R} = \frac{1}{g_e}$



$$\begin{cases} \tau \frac{du}{dt} = F(u(t), w(t)) + RI(t) \\ \tau_w \frac{dw}{dt} = G(u(t), w(t)) \end{cases}$$

u is here and here and here and there, and w appears here and here. Now this is an explicit form. u is a variable, which is time dependent. w is a variable, which is time dependent. u is the variable, time dependent. w is a variable, which is time dependent. However, it's convenient and normal to suppress the arguments in these functions, f and g. u is the variable-- you see it, because on the left-hand side you have the time derivative.

Notes

Summary



Neuronal Dynamics – 4.1. Reduction to 2 dimensions

2-dimensional equation

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

Enables graphical analysis!

- Discussion of threshold
- Repetitive firing
- Type I and II



And therefore, we arrive at a final set of two equations. The voltage variable is characterized by the membrane time constant, τ , and some function, and on the right-hand side, we have some function, f of u and w . The w value variable is characterized by a time constant, τ_w , and on the right-hand side, we have some arbitrary function, g of u and w . With such a two-dimensional system of equations, we can do marvelous things. We can analyze the behavior of the neuron models. And this is the topic of the following few lectures.

Notes

Summary



15m 53s

Neuronal Dynamics – Quiz 4.3.

Exploiting similarities:

A sufficient condition to replace two gating variables r, s by a single gating variable w is

- ☐ Both r and s have the same time constant (as a function of u)
- ☐ Both r and s have the same activation function
- ☐ Both r and s have the same time constant (as a function of u) AND the same activation function
- ☐ Both r and s have the same time constant (as a function of u) AND activation functions that are identical after some additive rescaling
- ☐ Both r and s have the same time constant (as a function of u) AND activation functions that are identical after some

multiplicative

Please have a moment to look at the quiz.

Notes

Summary

