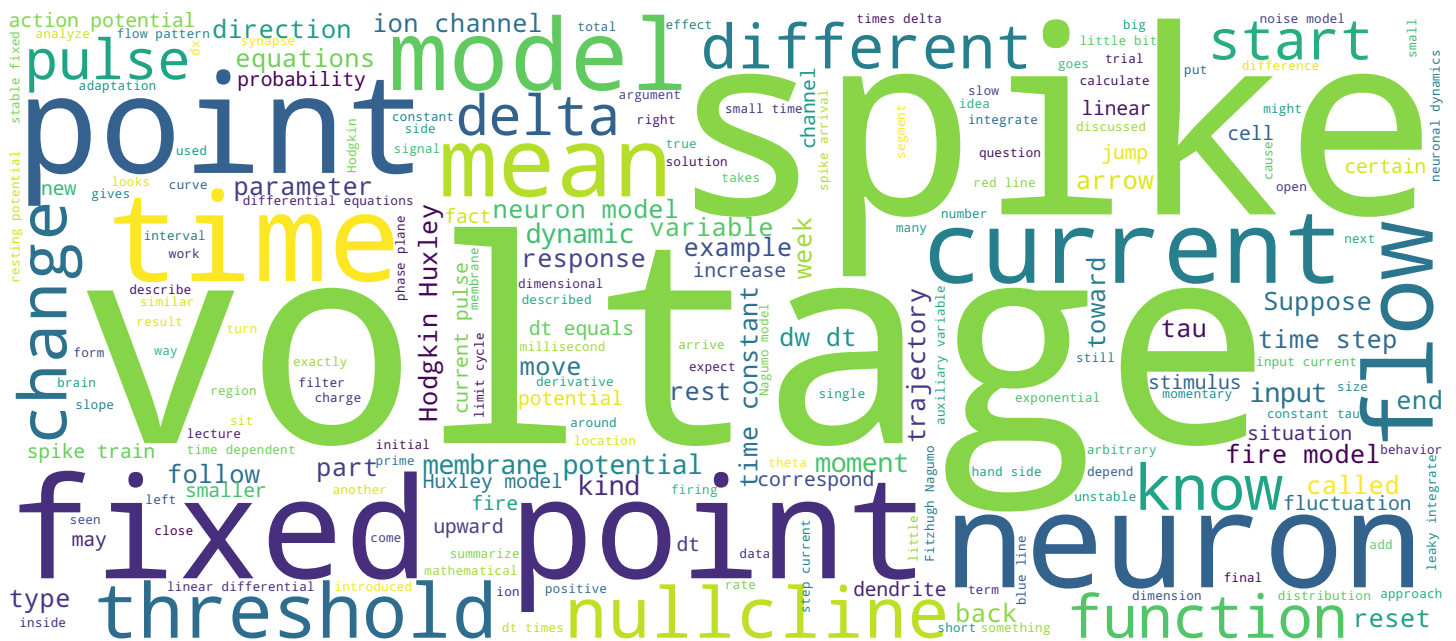


ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

EPFL, Lausanne, Switzerland



EPFL



Week 4 – part 2: Phase Plane Analysis



✓ 4.1 From Hodgkin-Huxley to 2D

- Overview: From 4 to 2 dimensions
- MathDetour 1: Separation of time scales
- MathDetour 2: Exploiting similarities

4.2 Phase Plane Analysis

- Role of nullcline

4.3 Analysis of a 2D Neuron Model

- MathDetour 3: Stability of fixed points

4.4 Type I and II Neuron Models

- where is the firing threshold?
- MathDetour 4: separation of time scales

4.5. Nonlinear Integrate-and-fire

- from two to one dimension

Welcome back to the class on neuronal dynamics. After these mathematical detours, we are now back to the main flow of the argument for this week four.

Notes

Summary



0m 03s

Neuronal Dynamics – 4.2. Phase Plane Analysis



2-dimensional equation

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

Enables graphical analysis!

- Discussion of threshold
- Type I and II

We have reduced the model of Hodgkin-Huxley from four coupled differential equations to two differential equations. We are left with two variables, one for the voltage, u , and the second one, an auxiliary variable called w . We are now going to exploit the mathematical structure of this two-dimensional system and analyze the behavior in the phase plane. Now I'll explain what this means.

Notes

Summary



0m 19s

Neuronal Dynamics – 4.2. Nullclines of reduced HH model

$$\tau \frac{du}{dt} = F(u, w) + I(t)$$

stimulus

$$\tau_w \frac{dw}{dt} = G(u, w)$$

there is a factor R
missing here
(assume $R=1$)

$$\frac{dw}{dt} = 0$$

$$\frac{du}{dt} = 0$$

u-nullcline

w-nullcline

So here is, again, our system of equations. If you follow the reduction step study we've explained starting from the Hodgkin-Huxley model, we arrive at this kind of situation, which I need to explain.

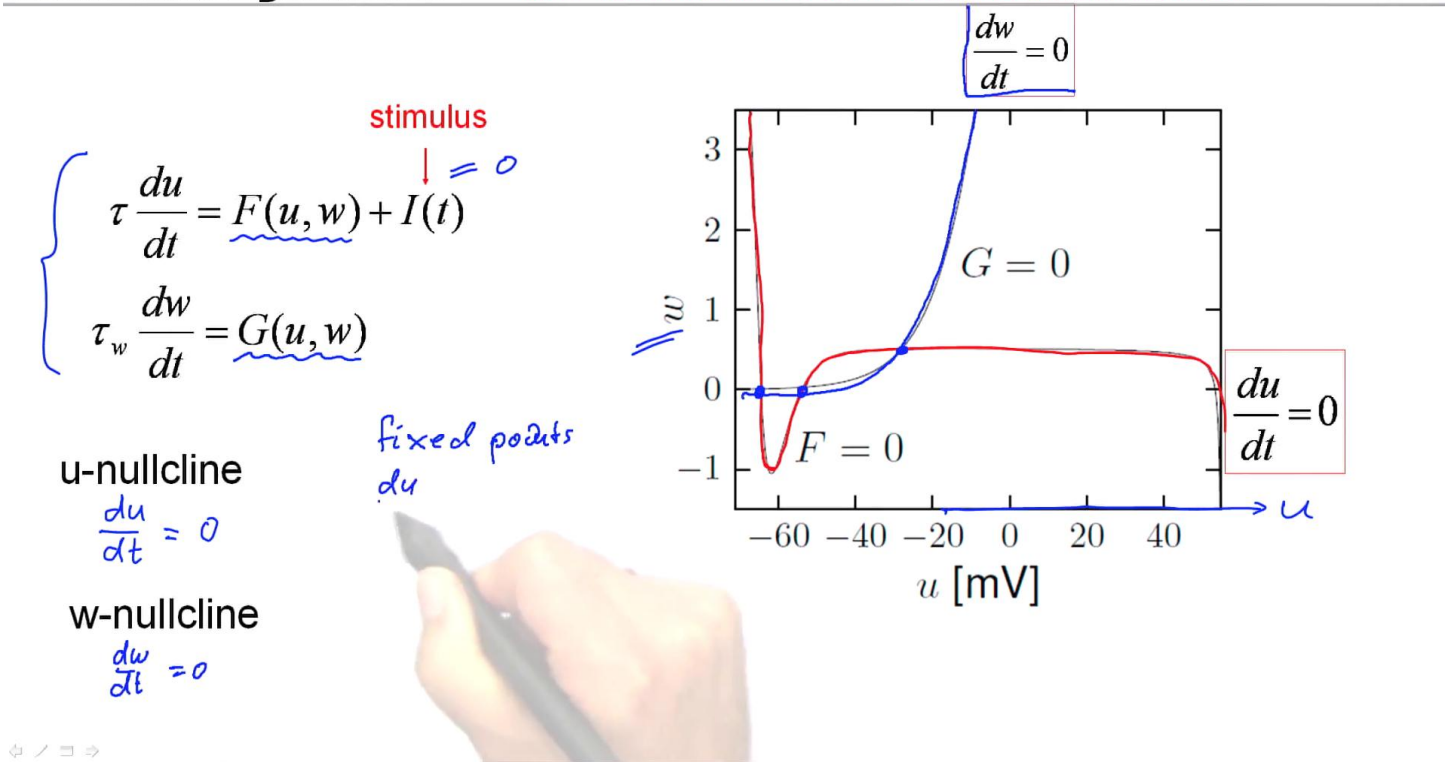
Notes

Summary

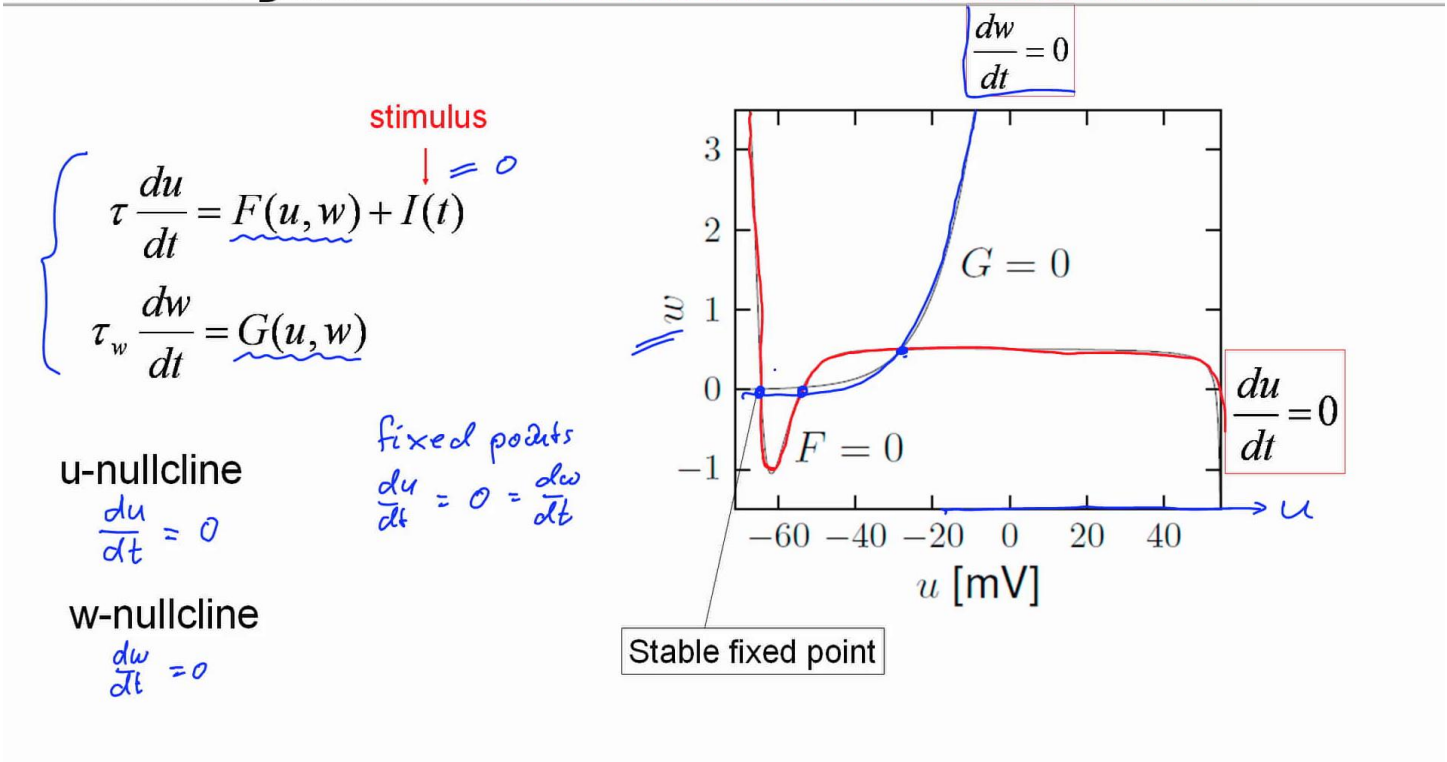


0m 53s

Neuronal Dynamics – 4.2. Nullclines of reduced HH model



Neuronal Dynamics – 4.2. Nullclines of reduced HH model



Once the voltage, u , and the auxiliary variable, w , sit here, they no longer change. This is the first fixed point, second fixed point, third fixed point. It turns out that this fixed point is stable. So the nullclines allow us to find graphically fixed points of the dynamics. However, with nullclines, we can do much more than just finding fixed points.

Notes

Summary



Neuronal Dynamics – 4.2. FitzHugh-Nagumo Model

$$\tau \frac{du}{dt} = F(u, w) + \underbrace{RI(t)}_{=0}$$

$$= u - \frac{1}{3}u^3 + \cancel{w} + RI(t)$$

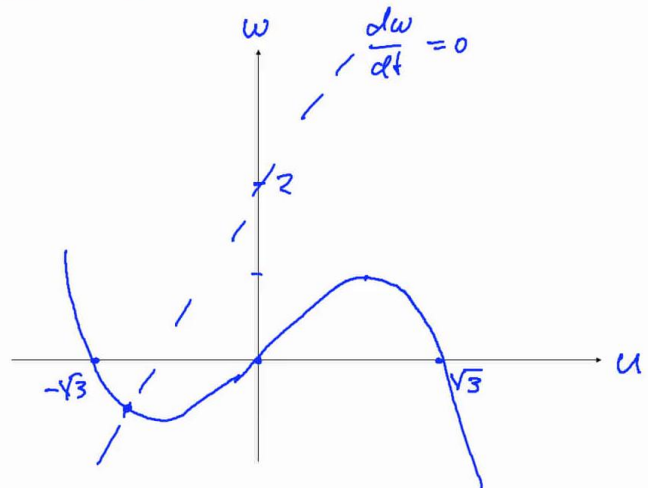
$$\tau_w \frac{dw}{dt} = G(u, w) = b_0 + b_1 u - w$$

u-nullcline

$$\frac{du}{dt} = 0 \Rightarrow w = u - \frac{1}{3}u^3$$

w-nullcline

$$\frac{dw}{dt} = 0 \Rightarrow w = \overset{=2}{b_0} + \overset{=2}{b_1} u$$



And to see that, let's look at a somewhat simplified model. This is called the FitzHugh-Nagumo model. Again, I have a stimulus, which, for the moment, I set equal to zero. And this function, f of u and w , has a u dependence, u minus $1/3u$ cubed minus w . It's missing here. Minus w plus r times i , which for the moment is zero. Now if you look at the zero point, the u -nullcline, du/dt equals 0, so the left hand side is zero. The current is zero. So I can put w to the other side, and I find w is u minus $1/3$ third u cubed. And this is a function-- w takes the value of 0 if u is 0. It also takes a value of zero if u is square root of 3, and it starts with a slope of 1. So the function will more or less look like this. And then it's symmetric. So roughly it's something like this. And then I have my w -nullcline, but the w -nullcline is linear. dw/dt equals 0. That means w is b_0 plus b_1 times u . Now let's put both constants equal to 2. So if u is 0, then w takes a value of 2, and, moreover, it has a slope of 2. So roughly my w -nullcline, the set of points with dw/dt equals 0 is a straight line which passes through 2. The two lines cross here, so this would be the fixed point. These are the first steps. But now what can we do with this kind of graph? I promised that they can do more than just finding the fixed points. In fact, we can look at the flow in this kind of graph.

Notes

Summary



4m 15s

Neuronal Dynamics – 4.2. flow arrows

$$\tau \frac{du}{dt} = F(u, w) + RI(t) \quad \text{Stimulus } I=0$$

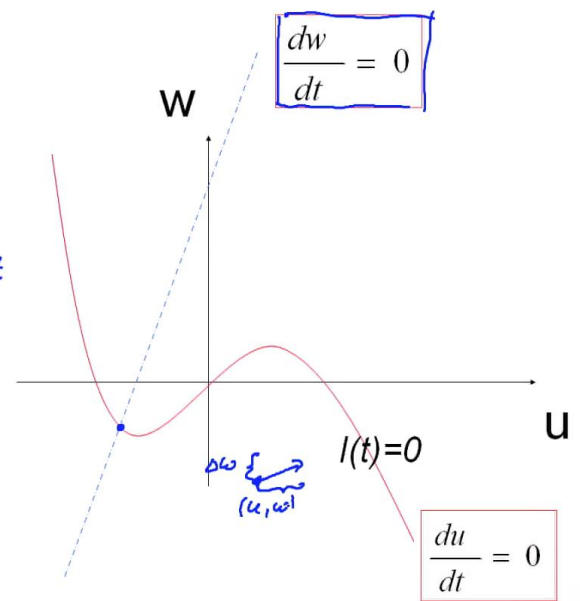
$$\tau_w \frac{dw}{dt} = G(u, w)$$

Consider change in small time step Δt

$$\begin{pmatrix} \Delta u \\ \Delta w \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \cdot \Delta t \\ \frac{dw}{dt} \cdot \Delta t \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \\ \frac{dw}{dt} \end{pmatrix} \cdot \Delta t$$

Flow on nullcline

Flow in regions between nullclines



✎ ✎ ✎ ✎

And this is now redrawn the same kind of figure. The u -nullcline is the red line. The w -nullcline is the blue line. And my fixed point is here. We know that if the system is here, then there's no change anymore. But what can we say if the momentary value of u and w is here? Well, we know that the voltage will change, because all the points where the voltage does not change are on the red line, and this point is not on the red line. We know that w will change because all the points where w is constant would lie on the blue line, and this is not the case here. So let's consider small time step Δt . In a small time step, Δt , I expect the change, Δu , which is du/dt times Δt . I expect a change of w , which is dw/dt times Δt . Or I can write it as $du/dt, dw/dt$ times Δt . So I expect that in our small time step, there's a change of u . This is the speed at which the system moves in the voltage direction. And this is the speed at which the w variable moves. So the flow might be like this. This would be Δu , and this would be Δw , starting from this point, u and w . Now there's something interesting now happening on the nullclines.

Notes

Summary



7m 01s

Neuronal Dynamics – 4.2. flow arrows

$$\tau \frac{du}{dt} = F(u, w) + RI(t) \quad \text{Stimulus } I=0$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

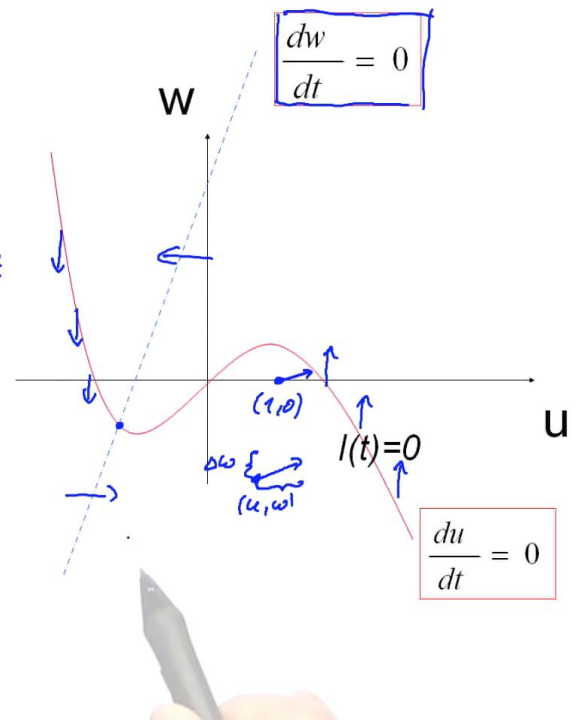
Consider change in small time step Δt

$$\begin{pmatrix} \Delta u \\ \Delta w \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \cdot \Delta t \\ \frac{dw}{dt} \cdot \Delta t \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \\ \frac{dw}{dt} \end{pmatrix} \cdot \Delta t$$

Flow on nullcline

$$u\text{-nullcline} \quad \frac{du}{dt} = 0 ; \quad w\text{-nullcline} \quad \frac{dw}{dt} = 0$$

Flow in regions between nullclines



✎ ✏ ✂ ➡

On the nullcline I know-- on the u -nullcline, I know that du/dt is 0, but that means that the change in the u direction, du/dt is 0. Delta u is zero. If delta u is zero, then the flow arrows must be vertical. At this point, I don't know yet whether they would be vertically upward or vertically downward. Similarly on the w -nullcline, I know that dw/dt is zero. If dw/dt is zero, and then there's no change in the direction of w possible. Delta w is zero. And that means the direction of flow must be horizontal, either leftward or rightward pointing. Now, how do I know whether the arrows point towards the left or towards the right? Well, one way to find out is to evaluate our equations at some point. For example, I take this point, u equal 1, w equal 0. I insert this into this equation, and for the model that I discussed before, with a set of parameters I discussed before, I find that, in this case, at this location, the flow would be slightly upward and slightly to the right. Now, as I get closer to the red line, the upward component remains, whereas the horizontal component disappears, and therefore the flow here must be upward, as I indicated.

Notes

Summary



9m 25s

Neuronal Dynamics – 4.2. flow arrows

$$\tau \frac{du}{dt} = F(u, w) + RI(t) \quad \text{Stimulus } I=0$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

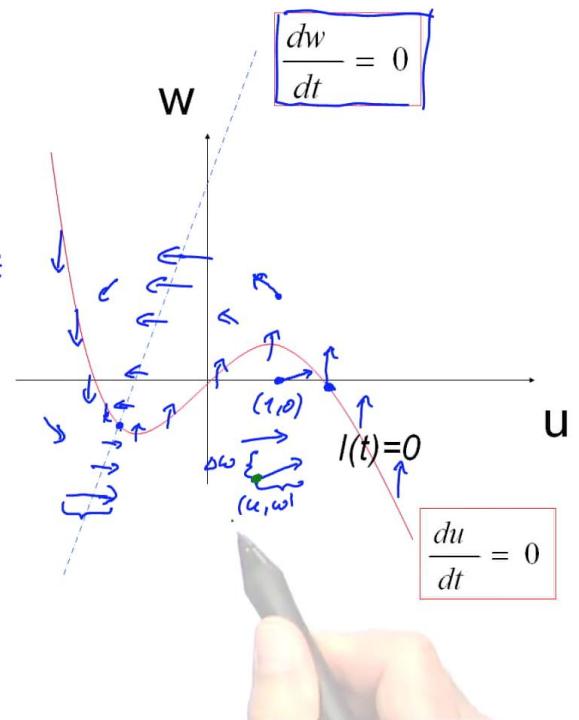
Consider change in small time step Δt

$$\begin{pmatrix} \Delta u \\ \Delta w \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \cdot \Delta t \\ \frac{dw}{dt} \cdot \Delta t \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \\ \frac{dw}{dt} \end{pmatrix} \cdot \Delta t$$

Flow on nullcline

$$u\text{-nullcline} \quad \frac{du}{dt} = 0 ; \quad w\text{-nullcline} \quad \frac{dw}{dt} = 0$$

Flow in regions between nullclines



✎ ✎ ✎ ✎

Similarly, as I move closer to the blue line, to the w-nullcline, the vertical component disappears, but the horizontal component remains. Therefore, the flow is towards the right as indicated. Now dw/dt has a certain 0 here, but du/dt has a certain value. As we get closer to the fixed point, this value of du/dt gets smaller and smaller, because at the fixed point, the length of the arrows is exactly zero. Now, on the other side the fixed point, the direction of arrows has changed. The same is true for the u-nullcline. Arrows get smaller. They turn at the fixed point, and then this is my total diagram. Now what does this mean? This means that if my momentary value of u and w sits here, then the system-- the dynamics will move the momentary value upwards. If I sit here, the dynamics will move towards the left. If I sit in between-- well, I cannot say exactly what the angle is, but I know that it's some combination of upward and leftward. And so I can write down the flow in the regions between the nullclines approximately. I know roughly the direction of the arrows. So what does that mean? Suppose I start at some arbitrary point, like the point here. I know at this point, the flow is in this direction.

Notes

Summary



11m 40s

Neuronal Dynamics – 4.2. flow arrows

$$\tau \frac{du}{dt} = F(u, w) + RI(t) \quad \text{Stimulus } I=0$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

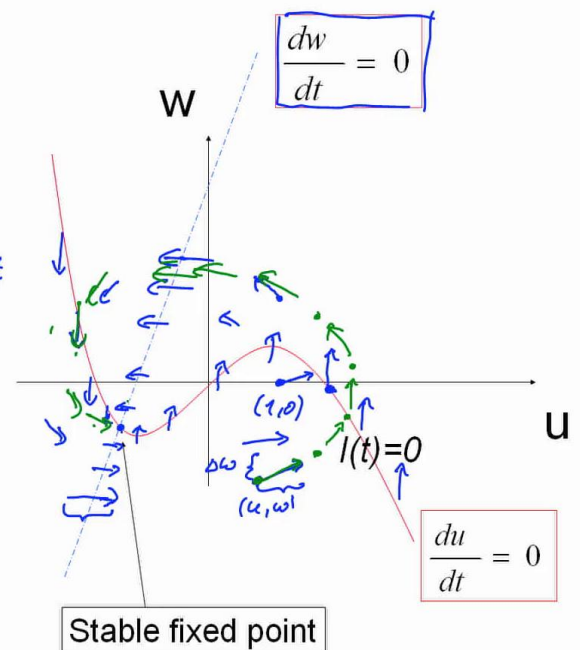
Consider change in small time step Δt

$$\begin{pmatrix} \Delta u \\ \Delta w \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \cdot \Delta t \\ \frac{dw}{dt} \cdot \Delta t \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \\ \frac{dw}{dt} \end{pmatrix} \cdot \Delta t$$

Flow on nullcline

$$u\text{-nullcline} \quad \frac{du}{dt} = 0 ; \quad w\text{-nullcline} \quad \frac{dw}{dt} = 0$$

Flow in regions between nullclines



But then I'm here. I know at this point the flow is, again, in this direction, and I'm here. I know at this point the flow is vertically upward, and I'm here. At this point, I'm in the region where the flow is upward and toward the left. I can draw it like this. I'm here. I will follow the directional flows. I will pass here horizontally. I will move down. I will move vertically down. I will follow the flow here. And in the end, I will end up at the fixed point. Once you have constructed the flow pattern, as I do with the blue arrows here, you can basically read off the dynamics by just following the flow. The flow carries you along. It's like throwing a little wooden stick into a lively river. It's taken along by the flow until, at some point, it may stop. It might be a fixed point where it gets stuck. Now this is a simplified scenario here and drawn.

Notes

Summary

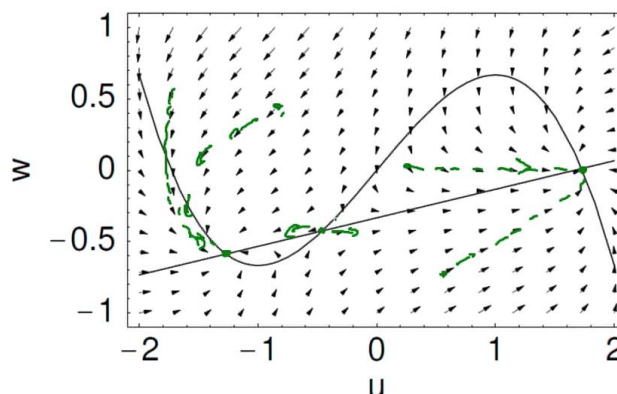


Neuronal Dynamics – 4.2. FitzHugh-Nagumo Model

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$= u - \frac{1}{3}u^3 + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w) = b_0 + b_1 u - w$$



✎ ✎ ✎ ✎

You can think of other situations like this one here-- again, a Fitzhugh-Nagumo model. There's a slightly different set of parameters so that we have fixed points. Let's start at some arbitrary initial condition, say here. I follow the flow, and I would end up here. Let's start with the initial condition here. I follow the flow. I follow the flow. I go here, vertically downward. I follow the flow. I end up here. So from any initial condition I can construct flow patterns which will end at one of the fixed points. And in this case, we would have two stable fixed points, here and there. This fixed point is unstable. The flow is away from the fixed point.

Notes

Summary



15m 17s

Neuronal Dynamics – 4.2. Phase Plane Analysis

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

Enables graphical analysis!

Important role of

- nullclines
- flow arrows



Let me summarize this. We made all this effort for reducing the Hodgkin-Huxley model from four equations to two equations. And, yes, it was worth the effort, because once we are done in two dimensions, we can analyze the dynamics graphically. In this lecture, I've introduced the idea of flow patterns. I've introduced the idea of nullclines.. And in the following lectures, we will exploit these very ideas to really get a deeper understanding of why neurons spike, how neurons spike. We will discuss thresholds. We will discuss type one, type two behavior. Everything will become transparent in the phase plane during the discussion of the dynamics in two dimensions.

Notes

Summary



16m 07s

Neuronal Dynamics – Quiz 4.4.

A. u-Nullclines

- ☐ On the u-nullcline, arrows are always vertical
- ☐ On the u-nullcline, arrows point always vertically upward
- ☐ On the u-nullcline, arrows are always horizontal
- ☐ On the u-nullcline, arrows point always to the left
- ☐ On the u-nullcline, arrows point always to the right

B. w-Nullclines

- ☐ On the w-nullcline, arrows are always vertical
- ☐ On the w-nullcline, arrows point always vertically upward
- ☐ On the w-nullcline, arrows are always horizontal
- ☐ On the w-nullcline, arrows point always to the left
- ☐ On the w-nullcline, arrows point always to the right
- ☐ On the w-nullcline, arrows can point in an arbitrary direction

Please take a moment and answer the quiz questions before you go on to the next lecture.

Notes

Summary



17m 05s