

Week 4 – part 3B: Analysis of a 2D neuron model – constant input



Neuronal Dynamics: Computational Neuroscience of Single Neurons

Week 4 – Reducing detail:

Two-dimensional neuron models

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4.1 From Hodgkin-Huxley to 2D

4.2 Phase Plane Analysis

- Role of nullcline

4.3 Analysis of a 2D Neuron Model

- MathDetour 3: Stability of fixed points

4.4 Type I and II Neuron Models

- where is the firing threshold?
- separation of time scales

4.5. Nonlinear Integrate-and-fire

- from two to one dimension



Video



Neuronal Dynamics – 4.3. FitzHugh-Nagumo Model

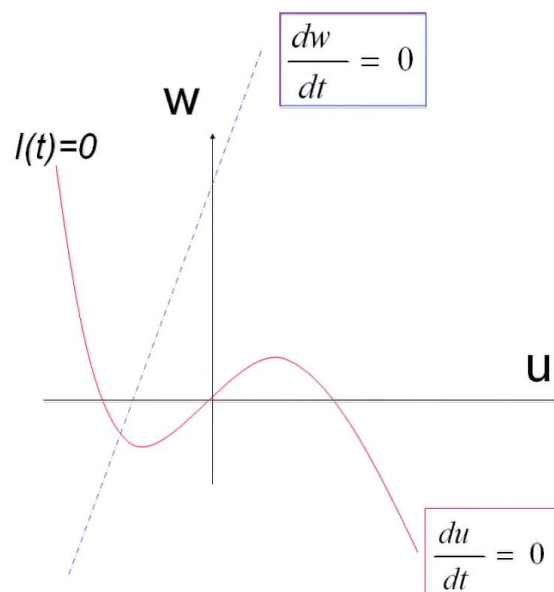
Pulse input:

- jump of voltage
- 'new initial condition'
- spike generation for large input pulses

constant input:

- graphics?
- spikes?
- repetitive firing?

Now!



So far our pulse input-- but what happens now with constant input, another favorite paradigm of experimentalists? What's the kind of graphics, what's the kind of arguments that we should use? What's the kind of firing behavior that we may expect? And this is what we are going to discuss now.

Notes

Summary



0m 04s

Neuronal Dynamics – 4.3. FitzHugh-Nagumo Model: Constant input

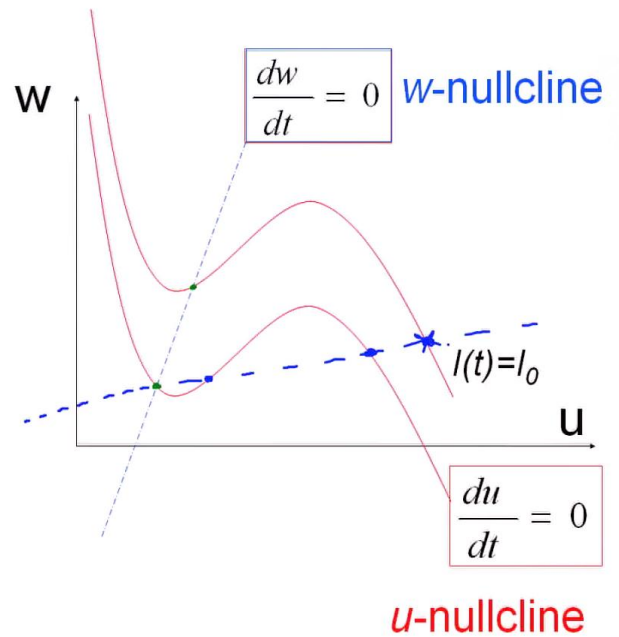
$$\begin{aligned}\tau \frac{du}{dt} &= F(u, w) + RI_0 \\ &= u - \frac{1}{3}u^3 - w + RI_0\end{aligned}$$

$$\tau_w \frac{dw}{dt} = G(u, w) = b_0 + b_1 u - w$$

Intersection point (fixed point)

-moves

-changes Stability



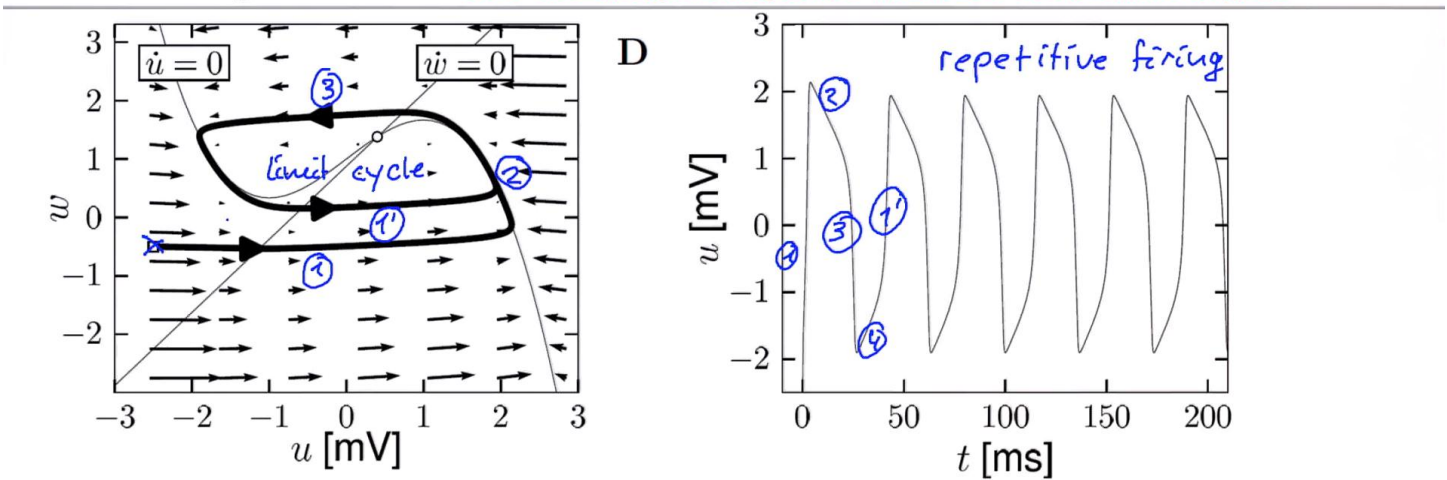
Again, our FitzHugh-Nagumo model, with our choice of parameters-- the u -nullcline is a straight line, the w -nullcline is a cubic function. And this value has been calculated for the case that the current is 0. So the u -nullcline for I equals 0, we have w equal u minus one third u cubed. Set this equation to 0, and we have it. Now for a positive I , I have $w = u$ minus $1/3 u$ cubed plus RI_0 . Now what does this mean? This means that the original curve of the u -nullcline is shifted vertically upward if I apply a constant current. So it's shifted upward, and as it's shifted upward, you see that the intersection point-- the fixed point-- changes. The new fixed point is here, the old fixed point was there. And as the location of the fixed point changes, its stability may change, as well. While this fixed point is stable it could be that this fixed point here is unstable. Note that if we have a different slope of the blue curve-- for example, if the w -nullcline looks like this, then it's not just the stability of the fixed point which may change, but it's also the number of fixed points which may change. I have three fixed points here for 0 current, I have only fixed point here for positive current.

Notes

Summary



Neuronal Dynamics – 4.3. FitzHugh-Nagumo Model : Constant input



FN model with $b_0 = 0.9; b_1 = 1.0; \underline{RI_0 = 2}$
 constant input: u -nullcline moves
 limit cycle

Image: Neuronal Dynamics,
 Gerstner et al.,
 Cambridge Univ. Press (2014)

⌂ / ☰ ➡

So with constant input, the u -nullcline shifts upwards-- vertically upwards-- for positive input. While the previous argument was with hand-drawn figures, here's a real figure. We have our FitzHugh-Nagumo model with a certain set of parameters, and a constant current equivalent of 2 units. The fixed point is up here, suppose the initial condition is here, then we follow the flow. It's a rapid rise upwards, then it takes some time to move over here, and it's a rapid downswing. And then the whole pattern repeats. This repeating pattern is called a limit cycle. If you project this out on the voltage trajectory, I have initial phase of upswing-- that's this part here. And there's a first time when it's close to the maximum-- that's this part here-- and there's a downswing. A return, but it's not a return to rest, because the fixed point would be up here, more where this fixed point is unstable. No, once it returns, it starts all over again, and with a new upswing-- 1 prime. And afterwards, the pattern repeats. So if we identify each of the pulses as an action potential, then we have here the situation of repetitive firing-- periodic firing. And this periodic firing is mathematically described as a limit cycle of my two-dimensional system of equations.

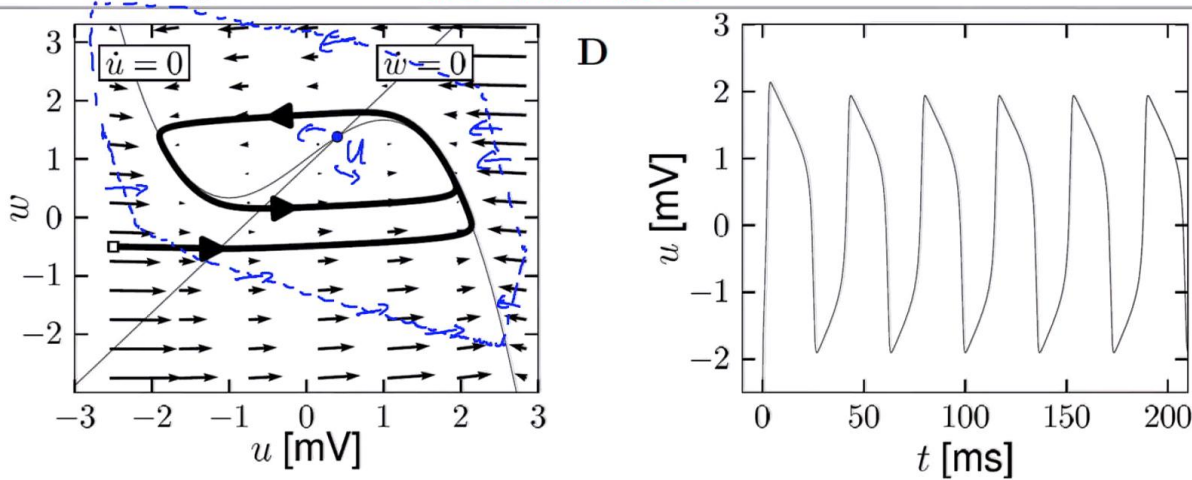
Notes

Summary



2m 54s

Neuronal Dynamics – 4.3. Limit Cycle



- unstable fixed point in 2D
- bounding box with inward flow
- limit cycle (*Poincaré Bendixson*)

Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

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Now how can I know that there's a limit cycle? Well, this is one of the beauties of phase plane analysis in a two-dimensional system of equations. Suppose for some reason I can calculate that this fixed point is unstable-- unstable fixed point. Now suppose that I find some surface-- and I try to draw one here-- and I construct the surface such that all the arrows point inward. So let's continue like this. I would have to go up here. So this is the kind of forces that would do the trick. So I know that the fixed point is unstable, so any trajectory starting in neighborhood of fixed point will leave the fixed point, will go away from the fixed point. So one could argue, well, the trajectory could maybe leave towards infinity. Well, it's not possible, because along this bounding surface, I know that the flow is inward. A trajectory cannot leave. Rather, trajectories enter. Now, the only way a trajectory could stop would be at a stable fixed point. But if inside of my bounding surface there's only one fixed point, and this is unstable, then the only possibility for the system to keep moving is to move in a circle, to move on a limit cycle. And this is the famous theorem of Poincare-Bendixson. It works in 2D.

Notes

Summary



5m 00s

Neuronal Dynamics – 4.3. Limit Cycle

In 2-dimensional equations,
a limit cycle must exist, if we can
find a surface

- containing one unstable fixed point
 - bounding box with inward flow
- limit cycle (*Poincare Bendixson*)

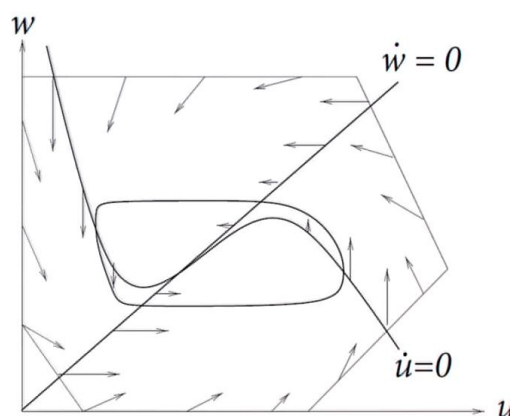


Image: Neuronal Dynamics,
Gerstner et al.,
Cambridge Univ. Press (2014)

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If you find a surrounding surface, such that it contains one unstable fixed point, no outer fixed points, and a bounding box with inward flow, then you're bound to have a limit cycle.

Notes

Summary



7m 08s

Neuronal Dynamics – 4.3. Analysis of a 2D neuron model

2-dimensional equation
stimulus

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

Enables graphical analysis!

- Pulse input
→ AP firing (or not)
- Constant input
→ repetitive firing (or not)



Let me sum up. With our two-dimensional system of equations, we can analyze firing behaviors that are relevant for biology. Famous biological paradigms used by many experimentalists are-- A, injection of short current pulses; B, injection of constant currents. For both paradigms, we can discuss graphically and predict the type of solution that must appear. For short current pulses, we have found that if their pulse amplitude is small, and there's a return-- potentially an oscillatory return towards the rest-- if the pulse amplitude is big, then the neuron model will generate an action potential-- a huge excursion of the voltage. Similarly, for constant current, we can understand why above some critical value a neuron emits spikes in a periodic fashion. The periodic firing regime corresponds to a limit cycle-- a limit cycle that we can easily visualize in the phase plane.

Notes

Summary



7m 23s

Neuronal Dynamics – Quiz 4.5.

A. Short current pulses. In a 2-dimensional neuron model, the effect of a delta current pulse can be analyzed

- ☐ By moving the u-nullcline vertically upward
- ☐ By moving the w-nullcline vertically upward
- ☐ As a potential change in the stability or number of the fixed point(s)
- ☐ As a new initial condition
- ☐ By following the flow of arrows in the appropriate phase plane diagram

B. Constant current. In a 2-dimensional neuron model, the effect of a constant current can be analyzed

- ☐ By moving the u-nullcline vertically upward
- ☐ By moving the w-nullcline vertically upward
- ☐ As a potential change in the stability or number of the fixed point(s)
- ☐ As a new initial condition
- ☐ By following the flow of arrows in the appropriate phase plane diagram

Before we continue, please have a look at the quiz which I've prepared for you.

Notes

Summary



8m 49s