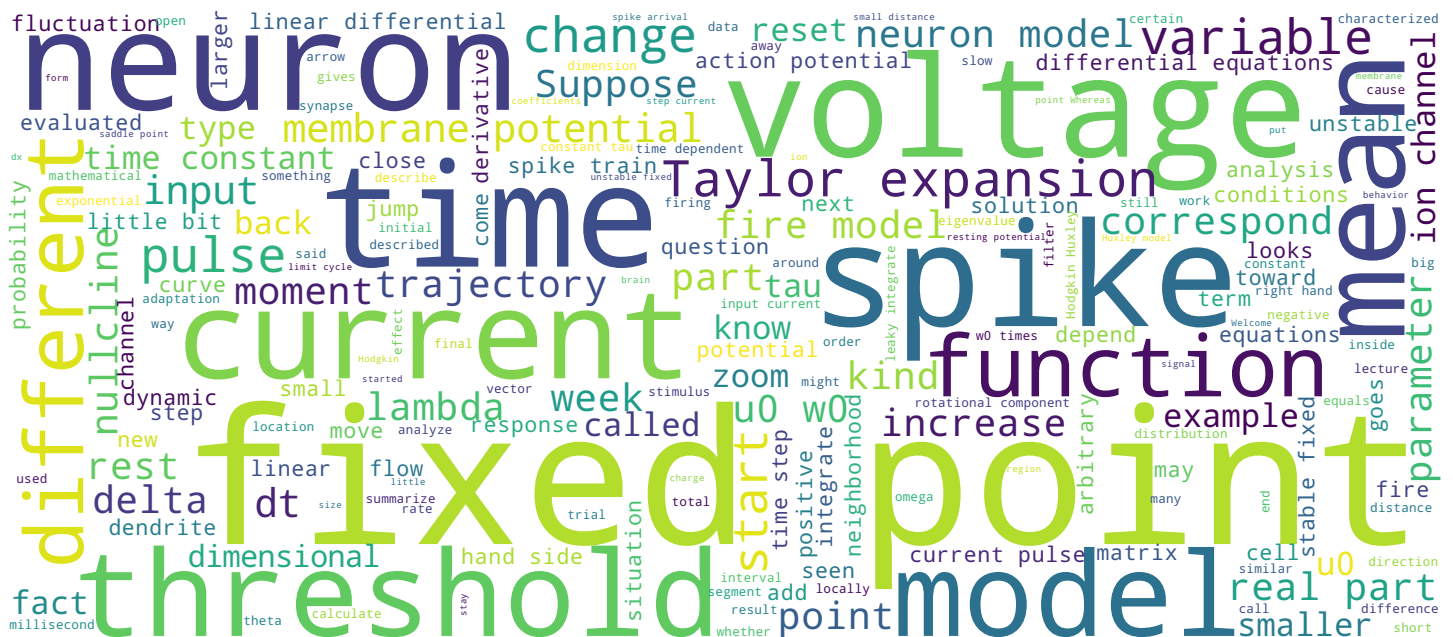


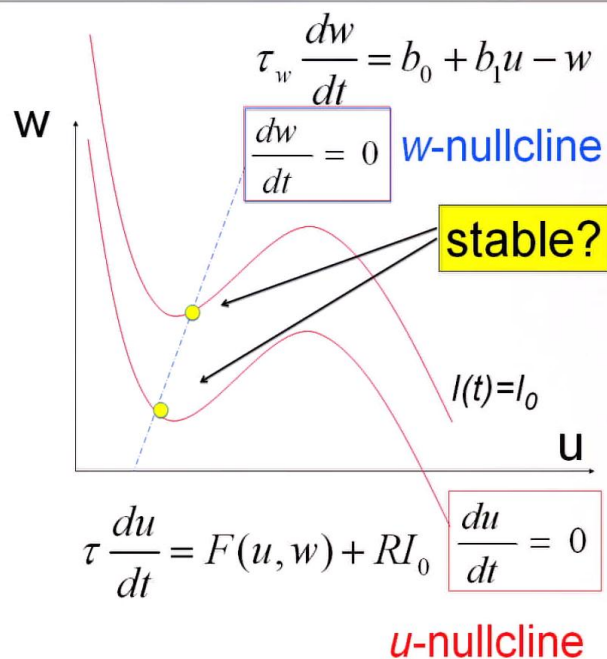
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Neuronal Dynamics – Detour 4.3 : Stability of fixed points.



Welcome to this mathematical detour concerning the analysis of stability of fixed points. We have seen, in the previous lectures, that in the phase plane, fixed points are characterized by their crossing points, the intersection points between the two node clients. And the question arises, for given fixed point, is it stable or is it unstable?

Notes

Summary



0m 04s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points

2-dimensional equation

$$\tau \frac{du}{dt} = F(u, w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

How to determine stability of fixed point?

Now, this can be analyzed mathematically in a systematic fashion. All those who know how to analyze the fixed points of differential equations can skip this lecture. For those who are not at ease with stability analysis, please, stay on for the next few slides.

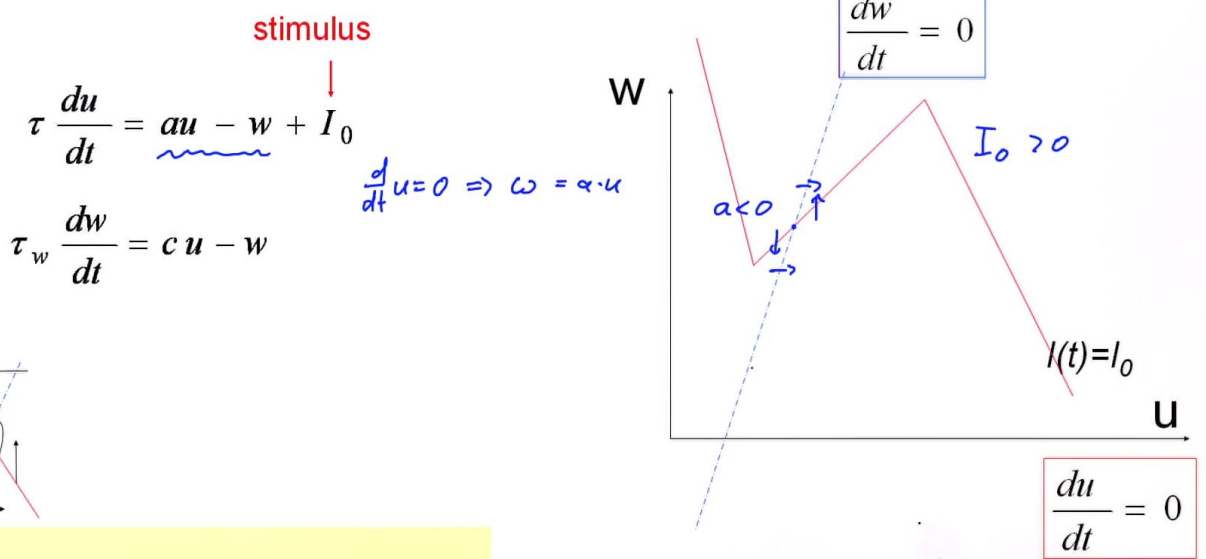
Notes

Summary



0m 32s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points



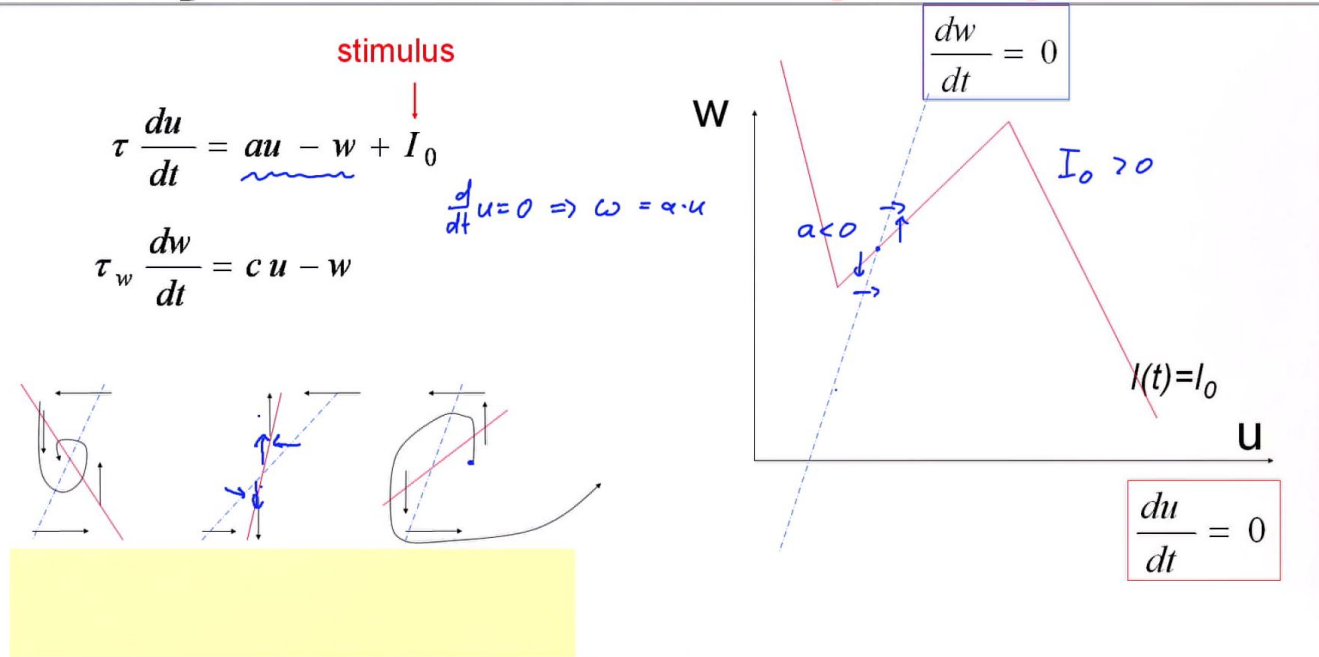
Before we go into the analysis of arbitrary non-linear differential equations, let's look at the analysis of a linear differential equation. In our nullcline picture, this would correspond to a u -nullcline that is linear. If I search for $d/dt u$ equals 0 I find w equal a times u . That means for this fixed point here, I would have a smaller than 0. And the fixed point is chosen such that the voltage is at the fixed point 0. w at the fixed point is 0. Now, with these kind of nullclines I can do the usual picture and say, well, here I expect downward motion, leftward motion, rightward motion. I would expect upward motion here and also upward motion there. Now, if I zoom in just on this little segment, my picture looks like this. And I claim that we can find the conditions that this is stable. Stable means I start at some initial condition. And then it will zoom in to the fixed point. Now suppose now, that I have a stimulus, I_0 , which is positive. Then the nullcline would be shifted upwards. And it would have a new situation. Well, this is downward slope. This is left. This is Right And the old arrows are no longer valid. Well, actually, the picture looks very similar.

Notes

Summary



Neuronal Dynamics – 4.3 Detour. Stability of fixed points



I also have this kind of rotational component. However, we will find the conditions under which such a picture implies instability of the fixed point. If I start at this point here, [? the trajectory ?] will circle outwards. It will go away from the fixed point. You can imagine yet another scenario. In this case, it doesn't have this rotational component. Rather, in this direction, it's attracted towards the fixed point. Whereas in the other direction it's going away from the fixed point.

Notes

Summary

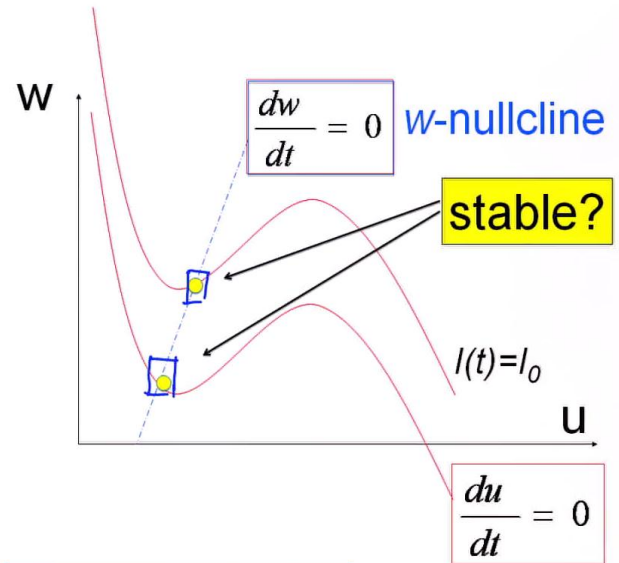
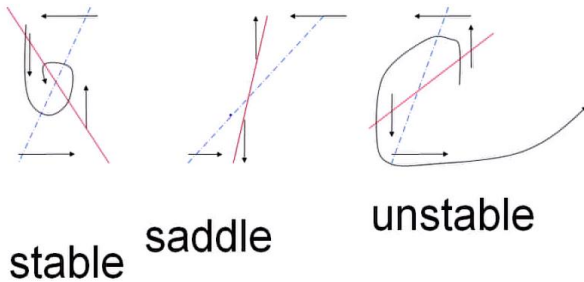


Neuronal Dynamics – 4.3 Detour. Stability of fixed points

$$\tau \frac{du}{dt} = F(u, w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

zoom in:



Math derivation
now

That would be called a saddle. Now, the picture with these linear red and blue lines looks rather artificial. However, it's not. Because if you just zoom in locally in a region very close to the fixed point we'll find that locally it looks like straight lines. And then this would be a stable fixed point. Whereas up here, locally, it looks like the case we had here, which is an unstable fixed point. And yet, there could also be these saddle points. That's the overview. Now, this zooming in corresponds to a mathematical operation that we will discuss now.

Notes

Summary



3m 59s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points

$$\begin{cases} \tau \frac{du}{dt} = F(u, w) + RI_0 \\ \tau_w \frac{dw}{dt} = G(u, w) \end{cases}$$

$\uparrow$ *
 $\uparrow$ *

Fixed point at (u_0, w_0)

At fixed point

$$\frac{d}{dt}u = 0 \Rightarrow 0 = F(u_0, w_0) + RI_0$$

$$\frac{d}{dt}w = 0 \Rightarrow 0 = G(u_0, w_0)$$

zoom in:

$$x = u - u_0$$

$$y = w - w_0$$

$$u = u_0 + x$$

$$w = w_0 + y$$

$$\begin{aligned} \frac{d}{dt}x &= \frac{d}{dt}(u - u_0) = \frac{d}{dt}u = F(u, w) + R \cdot I_0 \\ &\quad \uparrow \text{constant} \\ &= F(u_0 + x, w_0 + y) + R \cdot I_0 \end{aligned}$$

So this is my arbitrary two-dimensional system of differential equations. The change of voltage, w , u is given by some function, F , which depends on the two variables u and w . And for the w variable, it also depends on the two variables, u and w . Now, at the fixed point I will have a certain relation. At the fixed point, they note it at u_0, w_0 . There is no change of $d/dt u$. That means this right-hand side must be equal to 0. At the fixed point there is no change of w . That means the G function at the fixed point is evaluated to 0. Now, let's look in the close neighborhood of the fixed point. The fixed point is at u_0 . x is a small distance from u_0 . y is a small distance from w_0 . Now, let's look at the change of x . I can write down a $d/dt x$ is $d/dt u$ minus u_0 . But u_0 is a constant. So it's the same as $d/dt u$. But $d/dt u$ is F of u and w plus R times I_0 . Now, we said that we zoom in into the neighborhood of u_0 . So I can write u as u_0 plus x . So I have here F of u_0 plus x . This is u . Now, what about w ? w is w_0 plus y . So let's plug that in. w_0 plus y . And I copy the rest. x is small. We stay in the neighborhood of u_0 . F is assumed to be a very smooth function. So we can do a Taylor expansion.

Notes

Summary



4m 49s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points

$$\begin{cases} \tau \frac{du}{dt} = F(u, w) + RI_0 \\ \tau_w \frac{dw}{dt} = G(u, w) \end{cases}$$

Fixed point at (u_0, w_0)

At fixed point

$$\frac{d}{dt}u = 0 \Rightarrow 0 = F(u_0, w_0) + RI_0$$

$$\frac{d}{dt}w = 0 \Rightarrow 0 = G(u_0, w_0)$$

zoom in:

$$x = u - u_0$$

$$y = w - w_0$$

$$u = u_0 + x$$

$$w = w_0 + y$$

$$\begin{aligned} \frac{d}{dt}x &= \frac{d}{dt}(u - u_0) = \frac{d}{dt}u = F(u, w) + RI_0 \\ &\quad \text{constant} \\ &\quad \text{Taylor exp.} \downarrow \\ &= F(u_0 + x, w_0 + y) + RI_0 \\ &= F(u_0, w_0) + \frac{\partial F}{\partial u} \bigg|_{u_0, w_0} x + \frac{\partial F}{\partial w} \bigg|_{u_0, w_0} y + RI_0 \\ &\quad \frac{d}{dt}u = \frac{\partial F}{\partial u} \bigg|_{u_0, w_0} x + \frac{\partial F}{\partial w} \bigg|_{u_0, w_0} y \\ &= F_u \cdot x + F_w \cdot y \end{aligned}$$

The Taylor expansion gives the evaluation at u_0, w_0 F of u_0, w_0 plus dF/du , the first derivative. Evaluate it at u_0, w_0 times x plus dF/dw evaluated at u_0, w_0 times y . And then you need to add the plus RI_0 over here. Now, let's look up here. We said that at the fixed point F of u_0, w_0 plus RI_0 adds up to 0. So this term together with the RI_0 term here adds up to 0. So I have $d/dt u$ equals dF/du evaluated at u_0, w_0 times x plus dF/dw evaluated at u_0, w_0 times y . Now, this looks like an awkward notation. Right? I've had dF/du evaluated at u_0, w_0 . Let's write this shorthand notation as F_u . That's the first derivative of this function, F , with respect to u evaluated at the fixed point, shorter, F_u . Same thing here, F_w times y . So that was the Taylor expansion of this first equation. I can do a similar Taylor expansion of the second equation. And it will look very much the same.

Notes

Summary



7m 37s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points

$$\tau \frac{du}{dt} = F(u, w) + RI_0 \quad (1)$$

Fixed point at (u_0, w_0)

At fixed point

$$\tau_w \frac{dw}{dt} = G(u, w) \quad (2)$$

$$0 = F(u_0, w_0) + RI_0$$

$$0 = G(u_0, w_0)$$

zoom in:

$$x = u - u_0$$

$$y = w - w_0$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left. \begin{aligned} \tau \frac{dx}{dt} &= F_u x + F_w y \quad (1') \\ \tau_w \frac{dy}{dt} &= G_u x + G_w y \quad (2') \end{aligned} \right\}$$

$$\frac{d}{dt} \vec{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \vec{x}$$

✎ ✎ ✎ ✎

So let's do this. That's my Taylor expansion of the first equation. And from the second equation I get this Taylor expansion where this means the derivative of the function G with respect to u evaluated at the fixed point u_0, w_0 , analogously for G sub w. Now, I can introduce a vector, which is just x and y. And then this set of two equations becomes one single vector equation. This is the vector x. And this is a matrix of coefficients. And the coefficients arise from the Taylor expansion of the functions F and G.

Notes

Summary



10m 04s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points

Linear matrix equation

$$\frac{d}{dt} \mathbf{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \mathbf{x}$$

Search for solution

$$\mathbf{x}(t) = e \exp(\lambda t)$$



Two solution with Eigenvalues λ_+, λ_-

$$\lambda_+ + \lambda_- = F_u + G_w$$

$$\lambda_+ \lambda_- = F_u G_w - F_w G_u$$

So we have, now, a linear matrix equation. And the question is whether x -- that's the to small distance away from the fixed point. The question is whether this x is going to increase. So if I go down to one dimension, x as a function of time, this distance, x from 0. 0 means I'm at the fixed point. I'm a little bit away. Now it could be that this distance, x , increases. That's the case if the real part of λ is positive. Or it could be that I start at this value x . But in fact, if I start a little bit away from the fixed point-- the fixed point is at 0-- and I go back to the fixed point. That corresponds to the case that this λ , exponential approach and the real part of λ is smaller than 0. So this is the matrix equation. Since we started with two differential equations we are in two dimensions. This is a two-dimensional matrix. And I have two solutions. And they are characterized by eigenvalues λ_+ and λ_- . And if you search for this solution and you plug it into this equation you find that the eigenvalues are related to these derivatives arising from the Taylor expansion. And we can sum up this in two conditions, condition A and B.

Notes

Summary



11m 07s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points

Linear matrix equation

$$\frac{d}{dt} \mathbf{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \mathbf{x}$$

Search for solution

$$\mathbf{x}(t) = e \exp(\lambda t)$$

Two solution with Eigenvalues λ_+, λ_-

$$\lambda_+ + \lambda_- = F_u + G_w$$

$$\lambda_+ \lambda_- = F_u G_w - F_w G_u$$

$$\lambda_+ = \lambda_o + i\omega$$

Stability requires:

$$\Re\{\lambda_+\} < 0 \quad \text{and} \quad \Re\{\lambda_-\} < 0$$



$$\boxed{\begin{array}{l} F_u + G_w < 0 \quad (1) \\ \text{and} \\ F_u G_w - F_w G_u > 0 \quad (2) \end{array}}$$

Now, stability requires that the real part of both eigenvalues be negative. So it could be that the eigenvalues are real. In which case, I don't need to write down the real part of. But it can also be that these eigenvalues are complex numbers. Complex numbers means that lambda plus, for example, could be some real part. Let's call it lambda 0 plus some imaginary part, I omega. OK, now, stability requires that both of these have negative real parts. And I can combine this with my rules here. And what I find is, I find conditions on my derivatives arising from the Taylor expansions at the entries of this matrix here. And these are the conditions. Let's call this condition one and this here, condition two. So if you want to check whether your system of two equations has a stable fixed point you have to do a Taylor expansion around that fixed point. And you have to check the coefficients of these Taylor expansion and see that F_u plus G_w is smaller than 0 while this combination should be larger than 0.

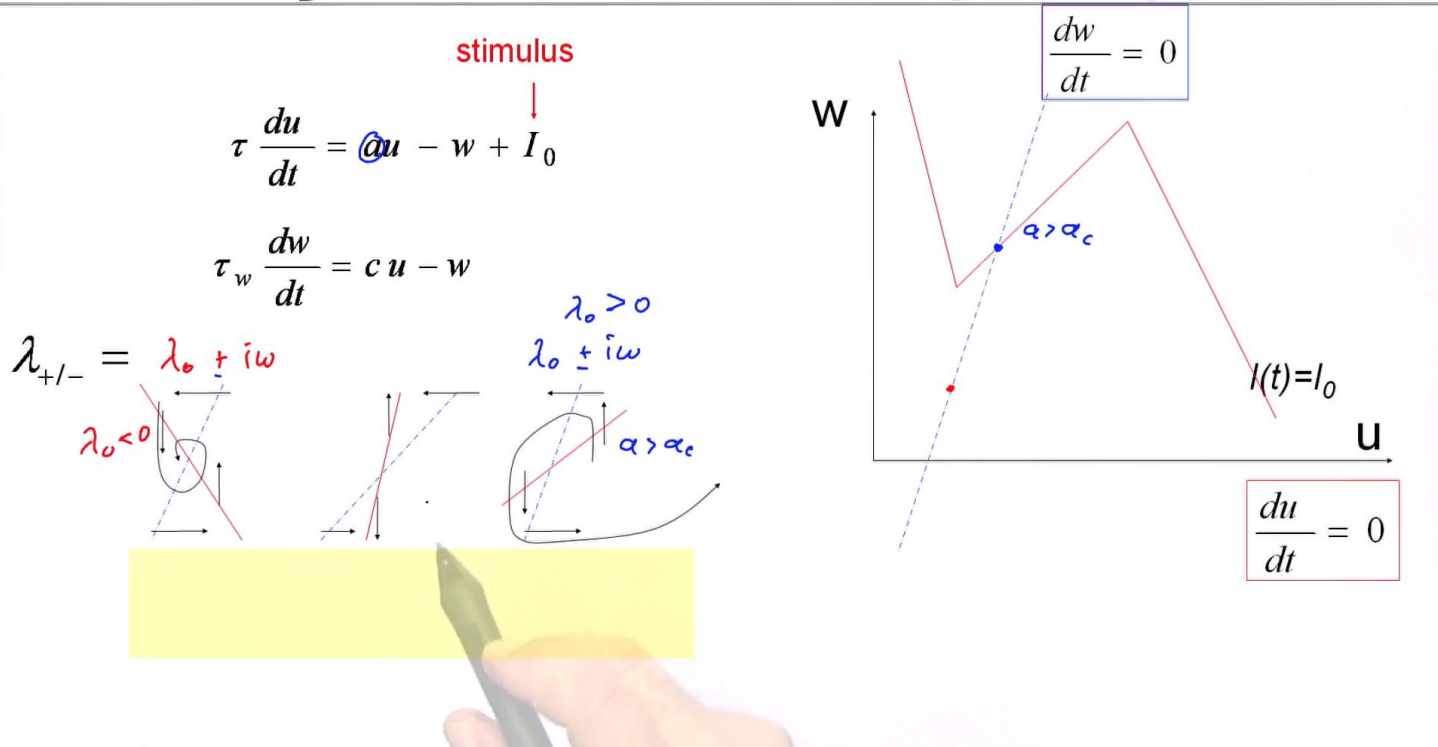
Notes

Summary



13m 09s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points



OK, now, we zoom in. I have started some arbitrary two-dimensional system. I zoom in on one fixed point, save this fixed point here. And what I find is that this fixed point here, where u is now the distance from the fixed point, this fixed point here is stable. In fact, I have a real part λ_0 . And the rotation corresponds to an imaginary part, $i\omega$. And λ_0 is negative. Which means we zoom into the fixed point. Any trajectory starting in the neighborhood of the fixed point will converge with some quotation or component towards the fixed point. Now, let's look at the next situation, at the fixed point up here. Again, this value, u , would be the distance from the fixed point. If you do the analysis we find in this situation that it's unstable. It's unstable. And again, the structure is plus minus $i\omega$. And the λ_0 here is larger than 0. Under the condition that this slope parameter a -- which corresponds to the slope a -- is larger than some critical value α_c . That depends on the other parameters. In particular, on the time constants. In the same way, we can also analyze the third pattern which is not here, visible on the right-hand side. And in this case, I have two real parts.

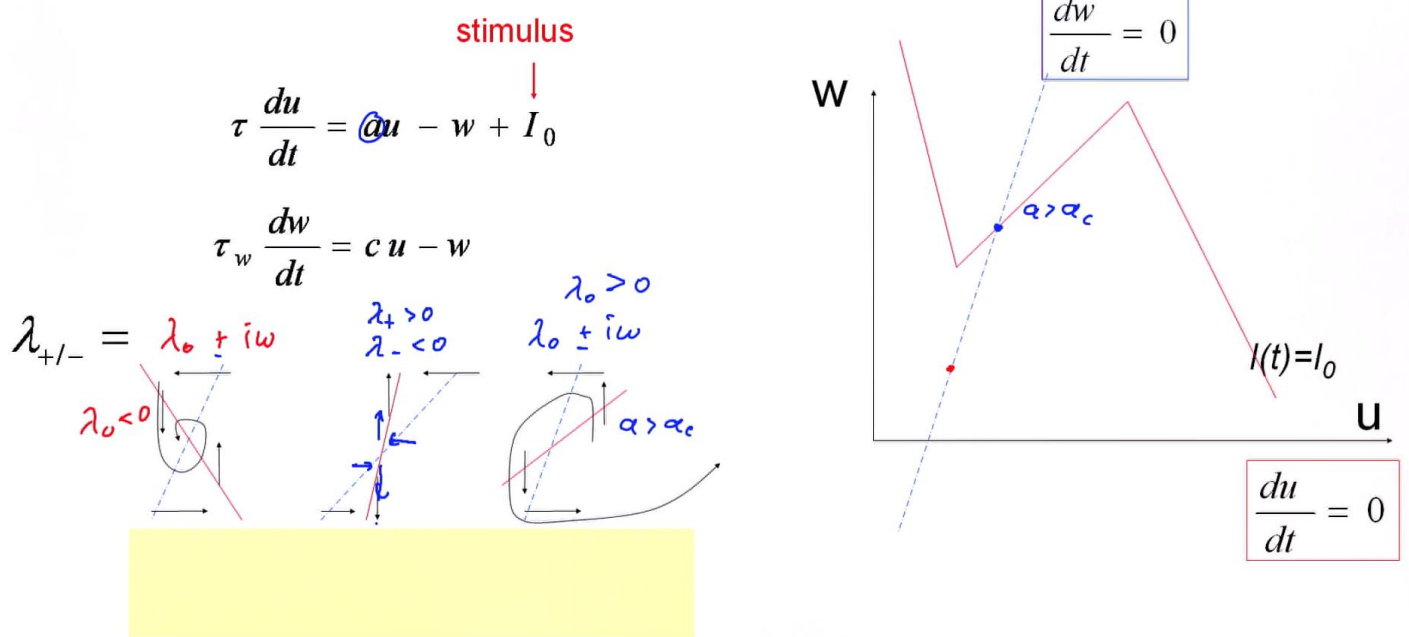
Notes

Summary



14m 54s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points



I have two real eigenvalues, lambda plus is positive. And lambda minus is negative. The negative eigenvalue corresponds to the attraction towards the fixed point, whereas the positive eigenvalue corresponds to the divergence of the fixed point here in the vertical direction.

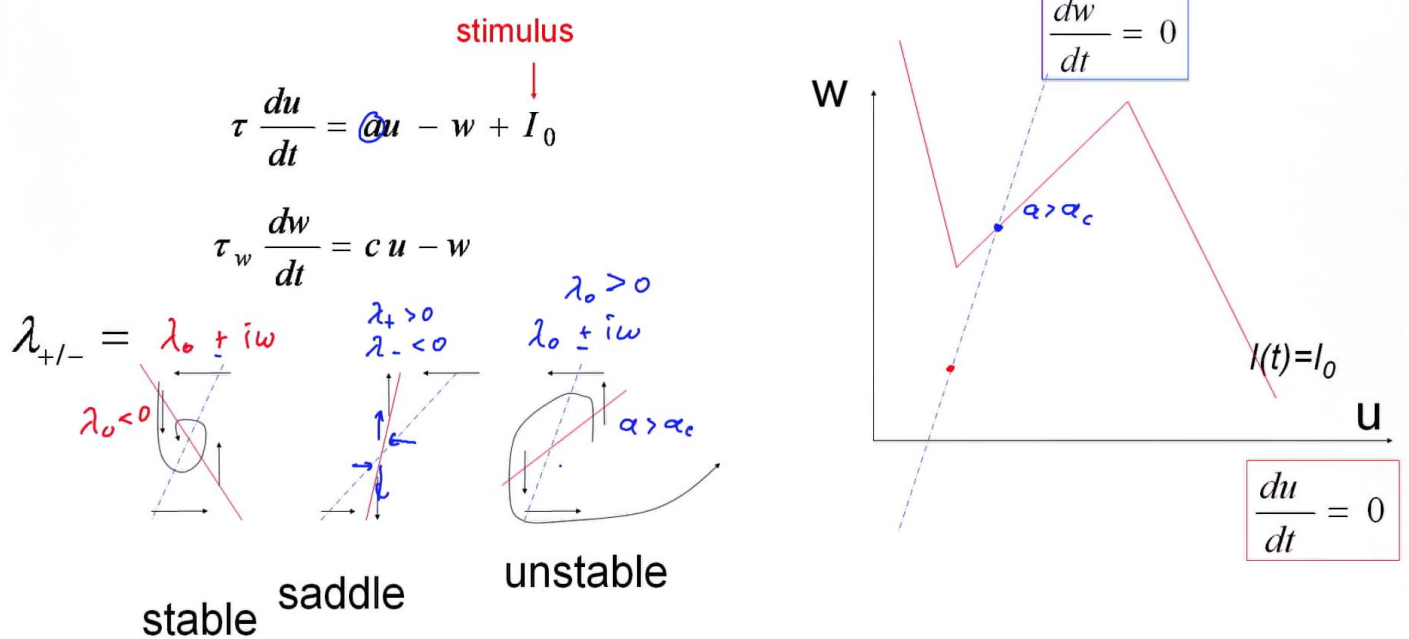
Notes

Summary



17m 07s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points



So we have the scenario of a stable fixed point with a rotational component. Of a saddle point and of an unstable fixed point with a rotational component.

Notes

Summary



17m 31s

Neuronal Dynamics – 4.3 Detour. Stability of fixed points

2-dimensional equation

$$\tau \frac{du}{dt} = F(u, w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

Stability characterized
by Eigenvalues of
linearized equations

$$\frac{d}{dt} \mathbf{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \mathbf{x}$$



To summarize, if you have a two-dimensional system of equations you have to go towards the fixed point, do a Taylor expansion in the neighborhood of the fixed point, look at the coefficients of the Taylor expansion. Write them into a matrix. Do an analysis of that matrix in terms of eigenvalues. Stability of the fixed point means that both eigenvalues have negative real parts. If you are not in two dimensions but in higher dimensions, the same type of analysis applies. Again, you have to look at the eigenvalues of the corresponding matrix. With this, I would like to close my review of stability analysis of fixed points. I hope this little review is useful for you in order to analyze the stability of the fixed points that we have for neuron models, which we are interested in.

Notes

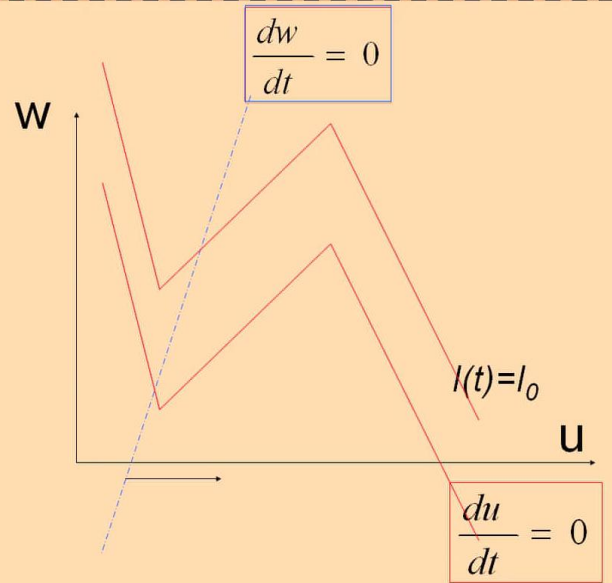
Summary



17m 43s

Neuronal Dynamics – Assignment.

Stability analysis of 2-dimensional equations is important for the homework assignment of week 4.



Notes

Summary

