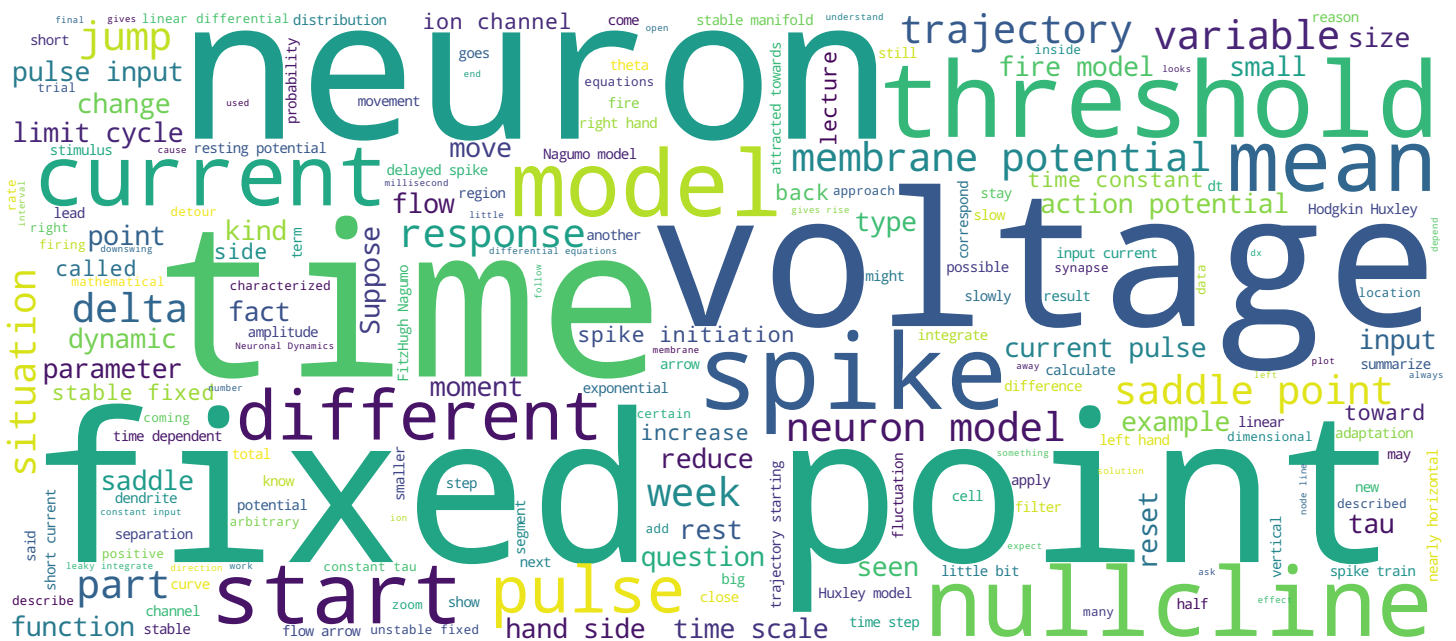




ÉCOLE POLYTECHNIQUE  
FÉDÉRALE DE LAUSANNE

- ✓ 4.1 From Hodgkin-Huxley to 2D
- ✓ 4.2 Phase Plane Analysis
- ✓ 4.3 Analysis of a 2D Neuron Model
- ✓ 4.4 Type I and II Neuron Models
  - where is the firing threshold?
- 4.5. Nonlinear Integrate-and-fire
  - from two to one dimension

Wulfram Gerstner  
EPFL, Lausanne, Switzerland



## Week 4 – part 4b : Firing threshold in 2D models



✓ 4.1 From Hodgkin-Huxley to 2D

✓ 4.2 Phase Plane Analysis

✓ 4.3 Analysis of a 2D Neuron Model

✓ 4.4 Type I and II Neuron Models

- where is the firing threshold?
- separation of time scales

4.5. Nonlinear Integrate-and-fire

- from two to one dimension

Welcome back. This is the second half of the lecture 4.4 week four of the class Neuronal Dynamics. I will ask the question where is the firing threshold? The question that we have already considered in the context of Hodgkin-Huxley model in week two. And also, in the context of integrating file models in week one. So where is the firing threshold? In one of these non-linear, two-dimensional neuron models that we consider this week.

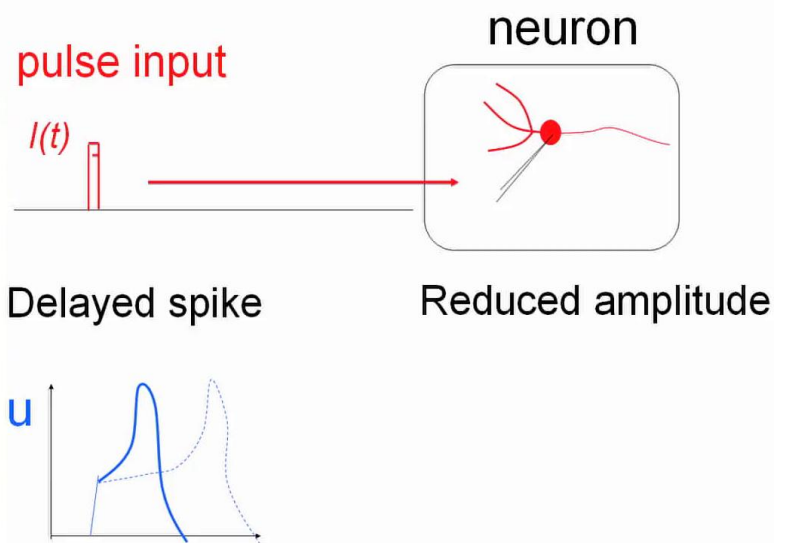
Notes

Summary



0m 04s

# Neuronal Dynamics – 4.4b Threshold in 2dim. Neuron Models



If you give a short current pulse, then you find that if the current pulse is strong enough it evokes an action potential. If you reduce the amplitude slightly you may still find an action potential. If you reduce it more, you may find that there's an action potential. But it's delayed. The question then is, can we understand this delayed spike initiation?

Notes

Summary



0m 41s

# Neuronal Dynamics – 4.4 Bifurcations, simplifications

Bifurcations in neural modeling,  
Type I/II neuron models,  
Canonical simplified models

*Nancy Koppell,  
Bart Ermentrout,  
John Rinzel,  
Eugene Izhikevich  
and many others*



Now, we have already seen, in the first half of lecture 4.4 that bifurcations can help us to understand neuronal behavior. The difference between type one and type two neural models, for example. In fact, the field of nonlinear dynamics, in particular, bifurcations, applied to neural modeling has been a very lively field during the last 20 years. And there are many people who have made substantial contributions to that part of theoretical and mathematical neuroscience. I may name Nancy Koppell, Bart Ermentrout, John Rinzel, Eugene Izhikevich and some others like Boris Gutkin, or David Hansel, and many others more. It's an active field that has been beautifully developed over the last years. And I would like to show, in this second half of that lecture 4.4, how the bifurcation analysis in two dimensions can help to understand thresholds in neuron models.

Notes

Summary



1m 07s

# Neuronal Dynamics – 4.4b Pulse input

stimulus

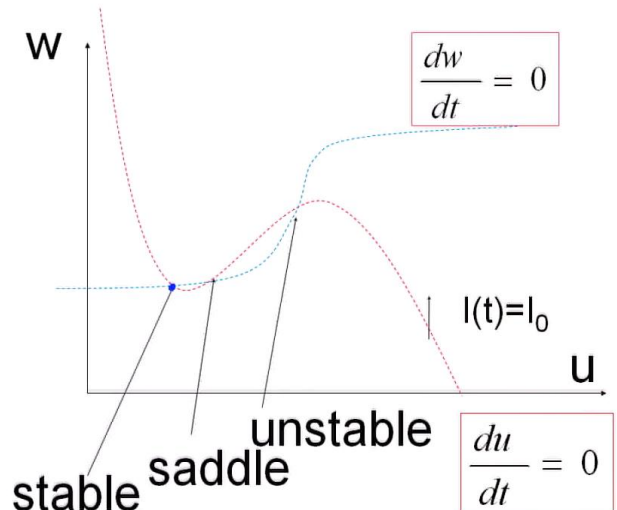
$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

pulse input

$I(t)$

$u$   
 $u_{rest}$



So in the first half off this lecture I have introduced a saddle-node onto limit cycle bifurcation. This is the situation here where I have a w-nullcline and a u-nullcline, And they have three fixed points, one, a stable one and a saddle, and followed by an unstable one. The stable one-- what's it called? A node-- will merge with a saddle when I apply a constant input current. And the two fixed points disappear. You have seen that, in this case, if the two fixed points have disappeared, a limit cycle emerges, a limit cycle around the remaining unstable fixed point. And this limit cycle has a part where the trajectory moves incredibly slowly. And the reason is that it moves through the region where, just before the bifurcation, there have been fixed points. The flow at the fixed point is, by definition, zero. Now, in this second half of the lecture 4.4, I would like to stay in the region where they are still the three fixed points, a stable one, a settled one, and an unstable one. And now I consider the situation that we give a pulse input. So suppose the neuron at the beginning is addressed. The voltage,  $U$ , is at the resting potential,  $U_{rest}$ . And now I give my pulse current.

Notes

Summary



2m 21s

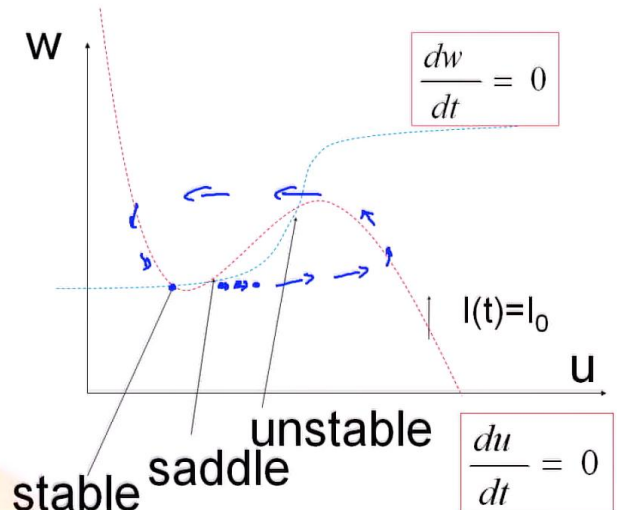
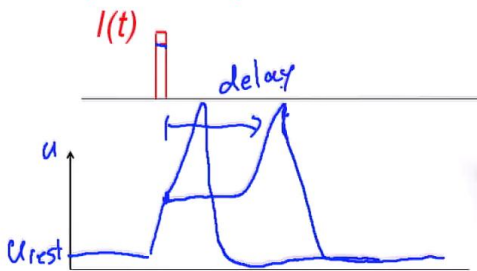
# Neuronal Dynamics – 4.4b Pulse input

stimulus

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

pulse input



As we have seen, the pulse current will lead to a horizontal jump of the trajectory to a new initial condition, say, this initial condition here. Now, if you look at the arrows in this phase diagram you could find that the movement is towards the u-nullcline. There it turns around. And then it goes back and it turns around. And it goes back to the stable fixed point. If I plot this over here the pulse leads to a jump. And afterwards I have this big voltage trajectory followed by downswing. And then I come back to the resting potential from below. OK, now, if I reduce the size of the pulse then the jump will become smaller. If I reduce the size of the pulse even further, the jump may end up very close to the saddle point. Now, at the saddle point, the flow is slow. The trajectory will move very, very slowly. The size of the arrows is very small. And this leads to a situation that initially, from this new starting point, just a little bit below, the trajectory increases very, very slowly before it finally takes over. And you have, basically, the same kind of pulse as before. However, with a delay, delayed pulse initiation.

Notes

Summary



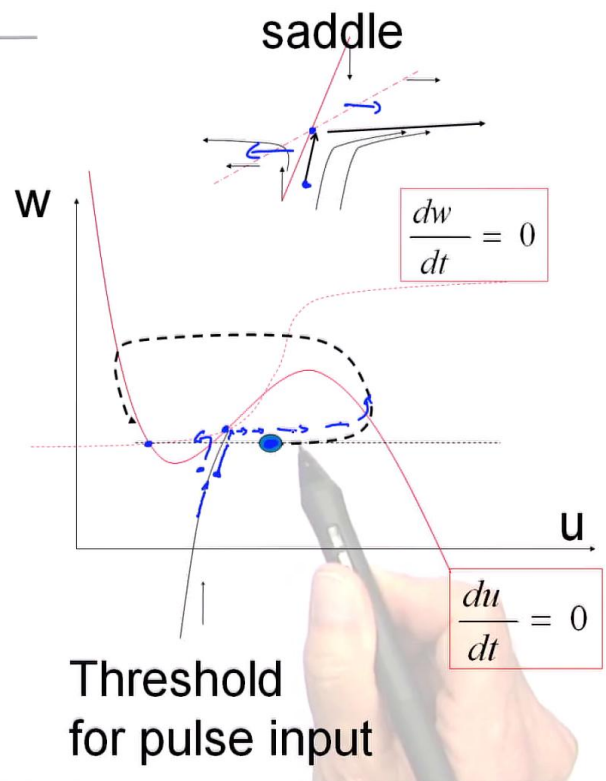
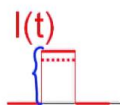
3m 57s

## 4.4b Type I model: Pulse input

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

pulse input



So let's discuss this again. This is the saddle point. The saddle point is characterized by a flow pattern where in one direction there's a movement towards the threshold, towards the saddle point. There is, in another directions, there's a movement away from the saddle point. So if the trajectory starts close to the saddle point and it's first attracted towards the saddle before it moves out, let's apply this to our situation at hand. The strong current pulse that I apply first leads to a jump to this new initial condition here. And we expect a movement back to the stable fixed point. Now, if I start at a situation that's closer to the saddle point, I might be attracted towards the saddle point first very slowly-- here, the movement is slow-- before it moves out again. If I start on the left-hand side, then it will turnover as indicated over here. Thus, there's one unique trajectory. A saddle point is characterized by one unique trajectory. There's only one trajectory from below that will really approach the saddle and stay at the saddle. All other trajectories will either bend to the left or to the right. Therefore, this attractive manifold of the saddle, the stable manifold, will serve as a threshold for pulse input.

Notes

Summary



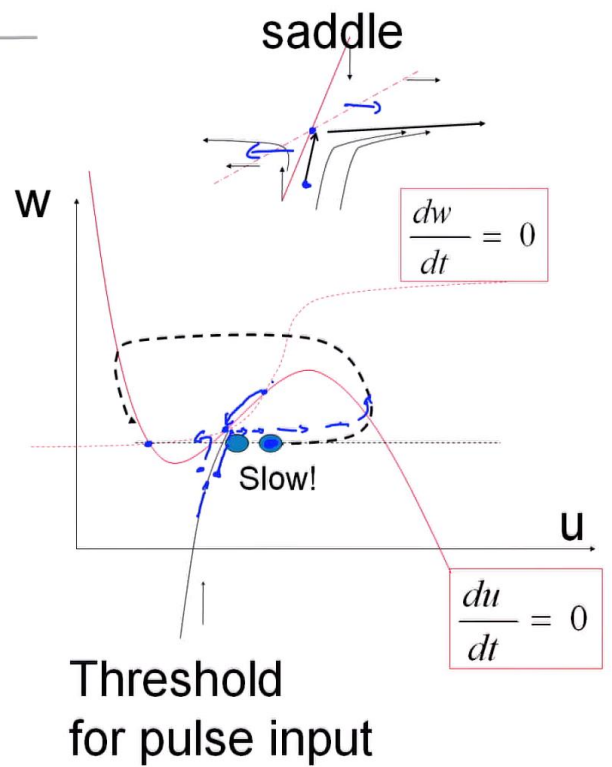
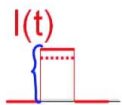
5m 40s

## 4.4b Type I model: Pulse input

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

pulse input



I said, there's only one trajectory coming from below. In fact, there is another one coming from above. So if you start close to this stable manifold that's attracted towards the saddle, then we can expect a slow movement and therefore, delayed spike initiation.

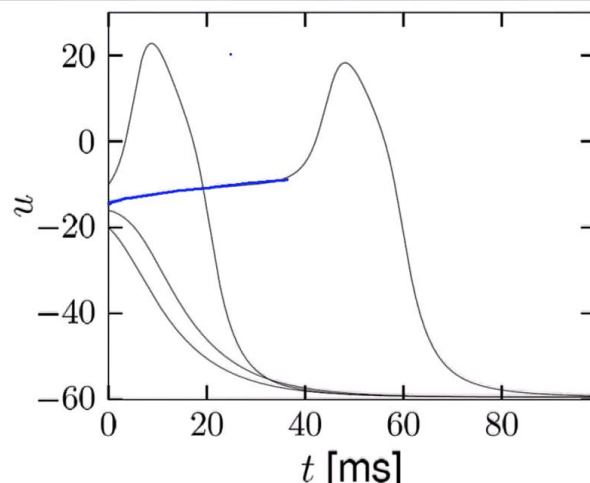
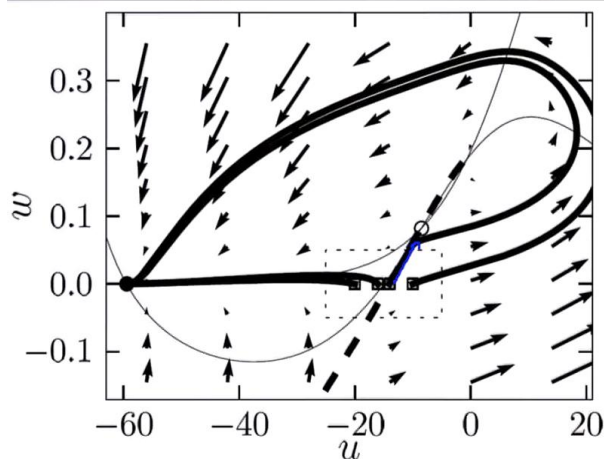
Notes

Summary



7m 38s

## 4.4b Type I model: Delayed spike initiation for Pulse input



Delayed spike initiation close to  
'Threshold' (for pulse input)

Image: *Neuronal Dynamics*,  
Gerstner et al.,  
Cambridge Univ. Press (2014)

⏮ ⏪ ⏩ ⏭

Now, here's a situation coming from a real simulated model. Here is my  $u$ -nullcline. This one here is the dashed line. The blue dashed line is the  $w$ -nullcline. Here is my  $u$ -nullcline. And I have the three fixed points, a stable fixed point here, a saddle point over there, and an unstable fixed point over here. Now, if a trajectory starts to the right-hand side of this stable manifold it makes the detour. If it starts to the left-hand side it goes straight back. If I zoom in onto this region I see the different arrows. And I can predict that if you start here you still move towards the stable fixed point towards the rest. Well, if you start here, just a little bit on the other side, we will move up here. And this is shown in these iterations here. We are now very close to this threshold, to the separatrix. And if you start just above, just on the right-hand side of the separatrix, the trajectory is first attracted towards the saddle. Remember, this is the saddle. And then makes to a large detour. This movement here is very slow. And this slow movement gives rise to the delayed spike initiation. Now, notice that there's no shortcut possible.

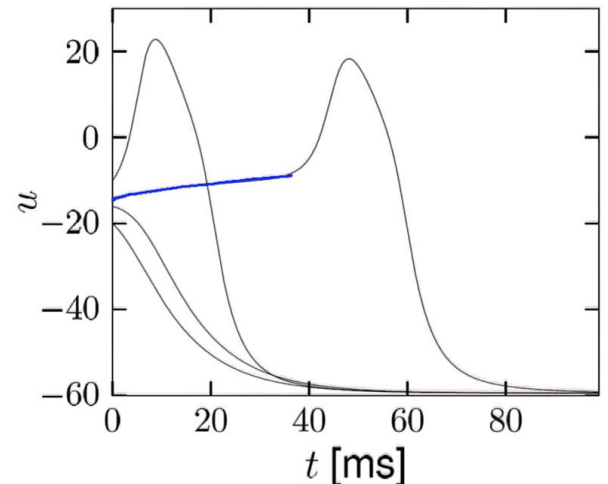
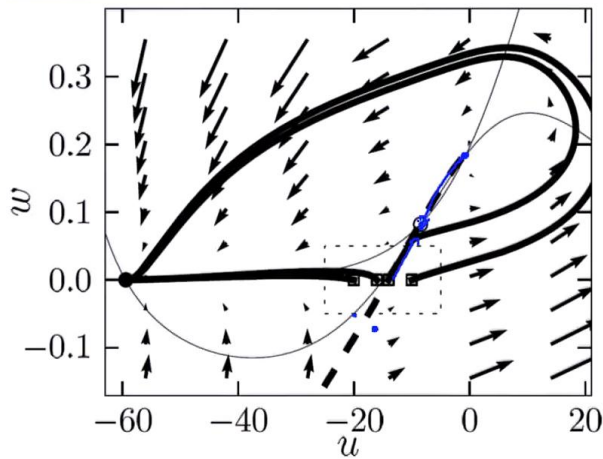
Notes

Summary



8m 04s

## 4.4b Type I model: Delayed spike initiation for Pulse input



Delayed spike initiation close to  
'Threshold' (for pulse input)

Image: *Neuronal Dynamics*,  
Gerstner et al.,  
Cambridge Univ. Press (2014)

Because there's another attractive trajectory starting from this unstable fixed point towards the saddle. So any trajectory starting on that side will have to make the detour to go back. Any trajectory starting on this side can go back directly. So this stable manifold of the saddle point plays the role of a threshold. This trajectory cannot be crossed.

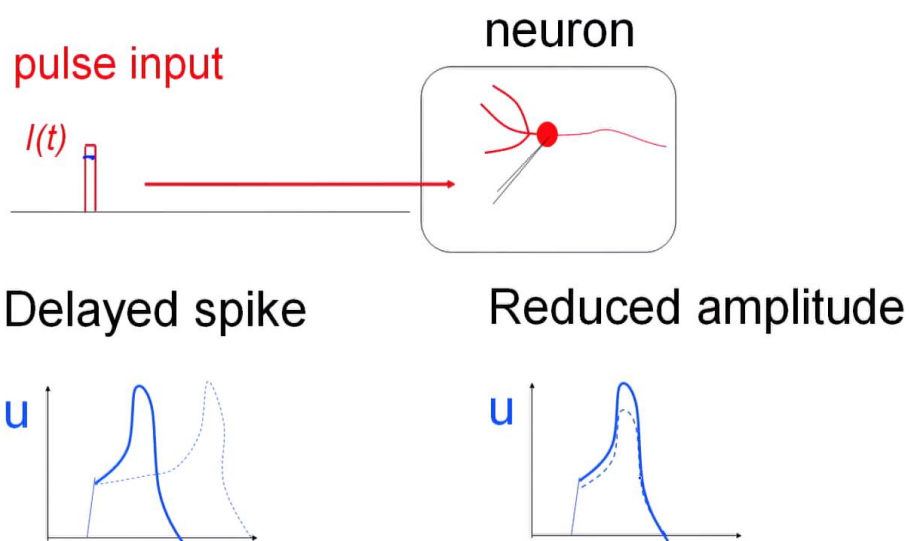
Notes

Summary



9m 57s

# Neuronal Dynamics – 4.4b Threshold in 2dim. Neuron Models



So what we have seen so far is that a system that has a saddle node onto the limit cycle bifurcation-- and that's not yet at the bifurcation-- will generate delayed spike output in response to short input pulses. However, some neuron models don't show this. If I reduce the amplitude of the pulse the amplitude of the response is simply reduced.

Notes

Summary



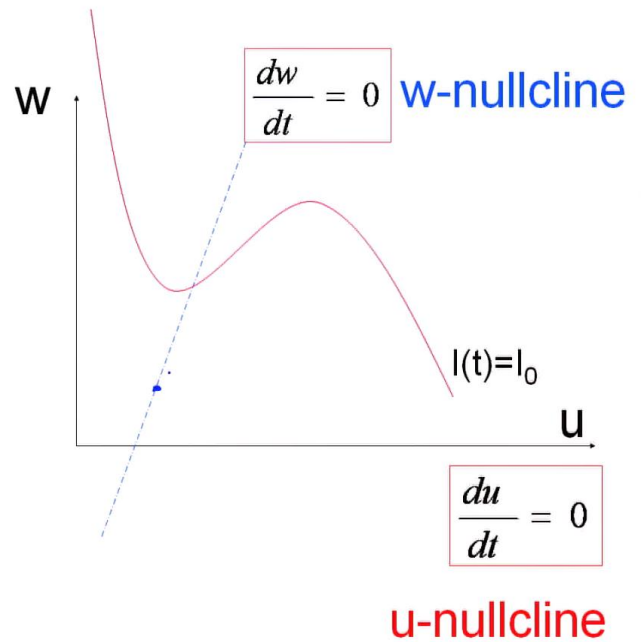
10m 34s

# FitzHugh-Nagumo Model: Hopf bifurcation

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

apply constant stimulus  $I_0$



So let's go back to our FitzHugh-Nagumo Model. The FitzHugh-Nagumo model has a  $w$ -nullcline, which is a straight line. And we have a stable fixed point down here. Now, the FitzHugh-Nagumo model, upon application of a constant current, will show a Hopf bifurcation. The point moves. The fixed point moves.

Notes

Summary

11m 07s



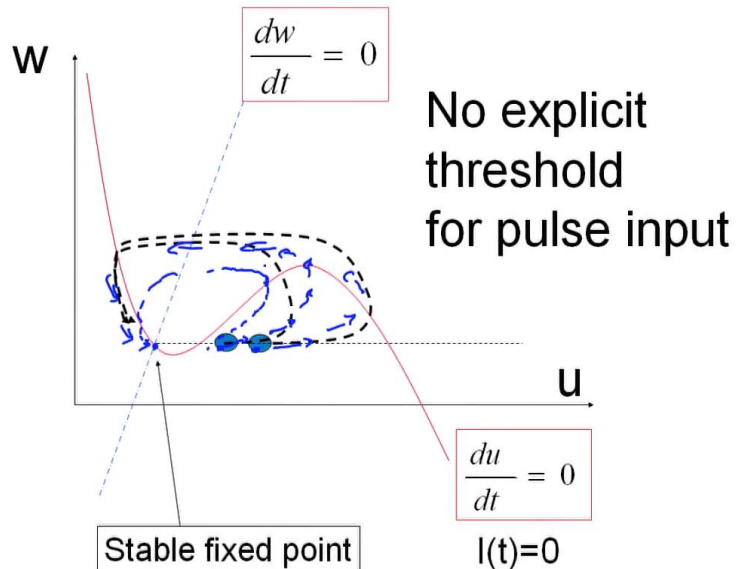
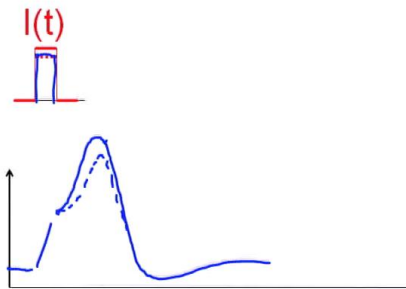
# FitzHugh-Nagumo Model - pulse input

$$\tau \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w)$$

stimulus

pulse input



And at some point it becomes unstable. But let's go back to the situation where the fixed point is stable. And let's apply the pulse input. A pulse input will lead to a jump, a horizontal jump of the membrane potential. Then you follow the flow pattern. And what we see is that it will go back like this. So in response to the input pulse I have an increase of the voltage. And then it goes down. It comes back from below. Now, if I reduce the size of the pulse and I start here, well, all on this side, the flow is to the right and slightly to the left. And now, a new trajectory can develop, which looks different. And this new trajectory will, basically, look like this. If I move over here, there's no reason that there shouldn't be a shortcut back to the fixed point. In this kind of model, with a Hopf bifurcation in the regime where we have a stable fixed point there's no such thing as a strict threshold. There's no stable manifold that could serve as a threshold. Short cuts are possible. So here, again, this situation, normally I make the detour. But there's no explicit threshold. If I start somewhere else a shortcut might be possible.

Notes

Summary



# FitzHugh-Nagumo Model - pulse input

threshold?

$$\tau_u \frac{du}{dt} = F(u, w) + RI(t)$$

stimulus

$$\tau_w \frac{dw}{dt} = G(u, w) \quad \text{slow}$$

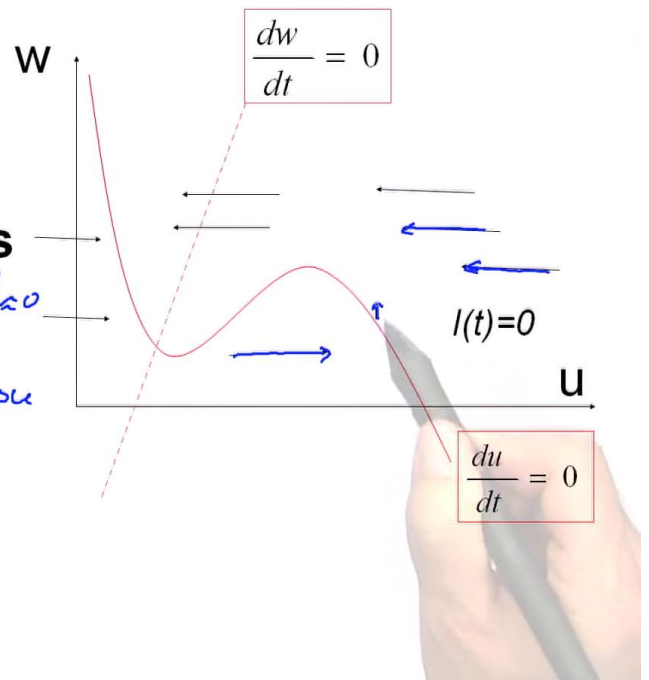
**Separation of time scales**

pulse input



$$\tau_w \gg \tau_u \quad \frac{dw}{dt} = \frac{G(u, w)}{\tau_w} \approx 0$$

$$\Delta w \ll \Delta u$$



However, the dynamics of the variable,  $w$ , arises-- if I come from the Hodgkin-Huxley model-- from  $n$  negating variable  $n$ ,  $n$  negating variable  $h$ . Both of these variables are fairly slow, slow compared, for example, to the  $n$  variable. Therefore, the dynamics of  $w$  is slow. Now, let's take the limit where  $\tau_w$ , this time scale here, is much larger than the voltage time scale,  $\tau_u$ . Now, this means that the change,  $dw/dt$ -- I bring the  $\tau_u$  to the other side-- is  $G$  divided by  $\tau_w$ . But  $\tau_w$  is very, very big. Therefore, this is nearly 0. That means if I construct a typical flow arrow, which would have two components-- you would have a component  $\Delta w$ . And you would have a component  $\Delta u$ . Then normally the  $\Delta w$  is much smaller than the  $\Delta u$ . And that means that flow arrows are nearly horizontal. Wherever I am, flow arrows are nearly horizontal. Now, there's one exception. And that is on the  $u$ -nullcline. On the  $u$ -nullcline,  $\Delta u$  is exactly 0. Therefore, the flow arrows must be vertical. But the amount  $\Delta w$  is very small. Therefore, I have a very slow, slow, vertically [ $\uparrow$  upward  $\uparrow$ ]. Now, the trajectory cannot get very far away from the  $u$ -nullcline.

Notes

Summary



13m 17s

# FitzHugh-Nagumo Model - pulse input

threshold?

stimulus

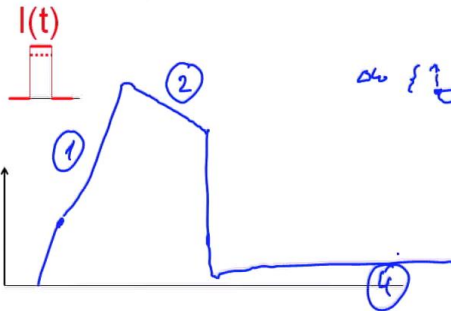
$$\tau_u \frac{du}{dt} = F(u, w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u, w) \quad \text{slow}$$

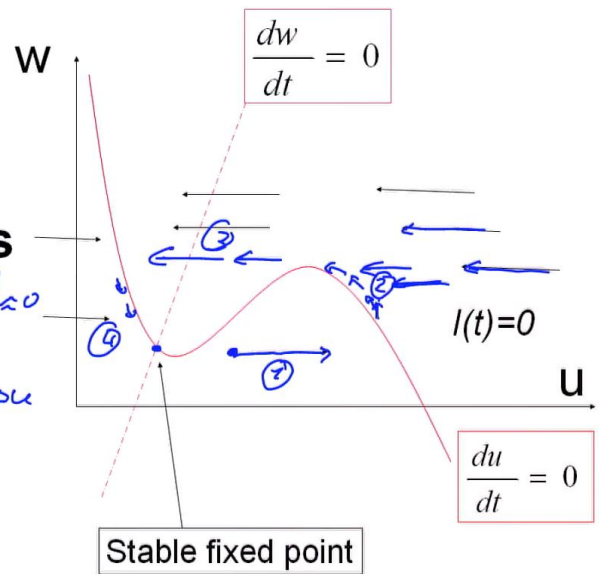
**Separation of time scales**

pulse input

$$\tau_w \gg \tau_u \quad \frac{dw}{dt} = \frac{G(u, w)}{\tau_w} \approx 0$$



$$\Delta w \ll \Delta u$$



Because if you're away from the u-nullcline the nearly horizontal arrows will push you back onto the u-nullcline. So if I start here I apply a short current pulse. The short current pulse leads to a horizontal jump. Now I have a rapid rise. And then I have a very slow movement along the u-nullcline. If you are on the u-nullcline you're pushed upward. But as soon as you're away you're pushed leftward back towards the u-nullcline. So take some time. And you go along the u-nullcline. If the upswing is phase one then this would be phase two. And now, suddenly, there's no u-nullcline anymore that hinders your movement. So you move rapidly down. And then you come back slowly, very slowly from below, towards the resting potential.

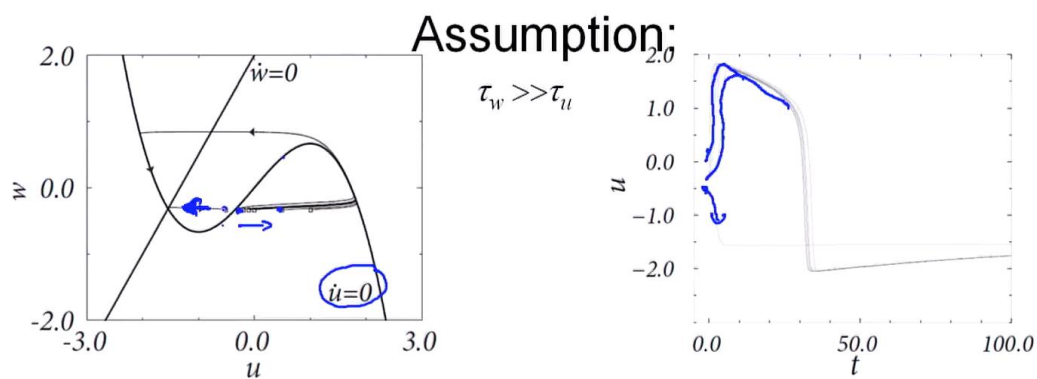
Notes

Summary



15m 32s

## 4.4b FitzHugh-Nagumo model: Threshold for Pulse input



Middle branch of  $u$ -nullcline  
plays role of  
'Threshold' (for pulse input)

Image: *Neuronal Dynamics*,  
Gerstner et al.,  
Cambridge Univ. Press (2014)

✎ ✎ ✎ ✎

Now, in this case, if the movement is always nearly horizontal, if you start here you're going to move rightward. If you start here you will move leftward. So this is a leftward arrow. If you start here you have a trajectory starting here coming down. If you start here you have a trajectory starting here and then coming down. If you start on the left-hand side of this branch you will go down immediately. In other words, the middle branch of the  $u$ -nullcline-- this is the  $u$ -nullcline-- acts effectively as a threshold for spike initiation on the pulse input.

Notes

Summary



16m 43s

## 4.4b Detour: Separation fo time scales in 2dim models

fast  $\tau_u \frac{du}{dt} = F(u, w) + \overset{\text{stimulus}}{RI(t)}$

slow  $\tau_w \frac{dw}{dt} = G(u, w)$

Assumption:

$$\tau_w \gg \tau_u$$

$$\frac{\tau_u}{\tau_w} = \epsilon \ll 1$$

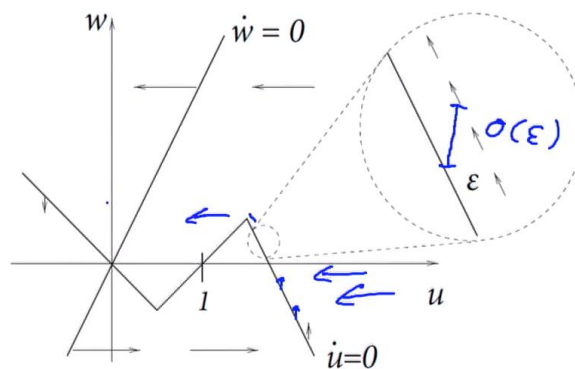


Image: *Neuronal Dynamics*,  
Gerstner et al.,  
Cambridge Univ. Press (2014)

Let's just make a little detour about this notion of separation of time scales. So we said that the time scale,  $\tau_w$ , is slow compared to that of the voltage variable  $u$ . So let's look at the ratio  $\tau_u$  over  $\tau_w$ . Now, this is, then, a small number. It's unit free. Let's set it to epsilon, much smaller than 1. So in this region down here, arrows are nearly horizontal. There is a little component of size, roughly epsilon, [? upward ?] but effectively, they are horizontal. And as epsilon gets close to 0, they become more and more horizontal. Now, on the  $u$  node line, arrows have to be vertical. But they are very small. And they are argued that the trajectory is always pushed back towards the  $u$  node line. In fact, you can show that if you zoom in, the distance within the trajectory and the  $u$  node line is always of order epsilon. So as the separation of times scales becomes strong and stronger, as epsilon goes to 0, the trajectory becomes slower. And it gets closer to the  $u$ -nullcline. Once it's up here, you can jump. And it jumps with a fast time scale.

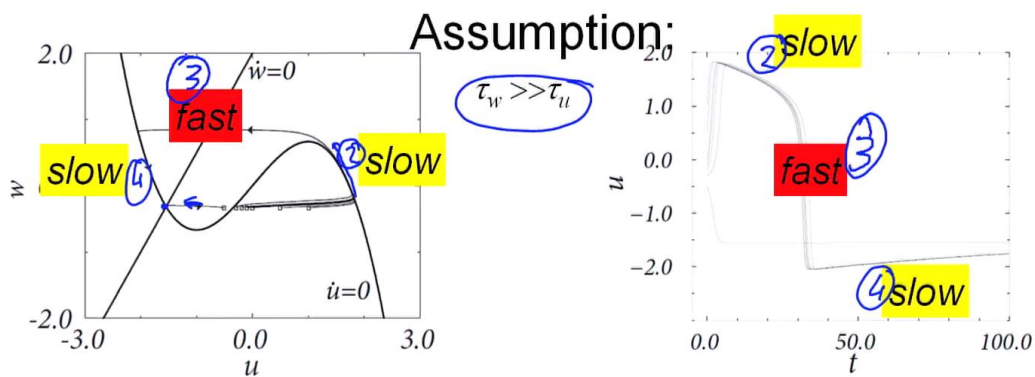
Notes

Summary



17m 45s

## 4.4b FitzHugh-Nagumo model: Threshold for Pulse input



trajectory

- follows  $u$ -nullcline: **slow**
- jumps between branches: **fast**

Image: *Neuronal Dynamics*,  
Gerstner et al.,  
Cambridge Univ. Press (2014)

So here's a simulation of the FitzHugh-Nagumo model and the pulse input under the assumption of a separation of time scales. You see that while it moves along the  $u$ -nullcline the movement is very slow. While it moves back to the resting potential the movement is very slow. On the other hand, the downswing of the action potential is very, very fast.

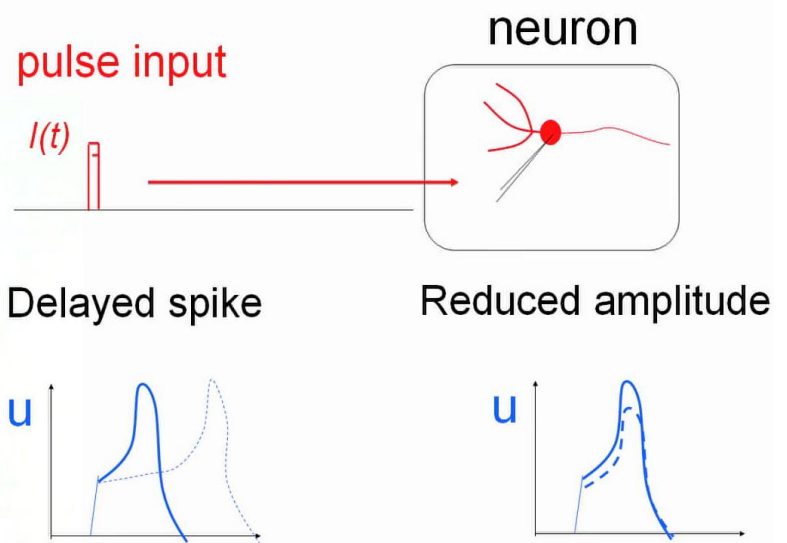
Notes

Summary



19m 31s

# Neuronal Dynamics – 4.4b Threshold in 2dim. Neuron Models



To summarize, the methods of phase-plane analysis allow us to discuss delayed spike initiation on the pulse input or other changes of spike initiation if you reduce the amplitude of the pulse. In a type-one neuron model that's canonically characterized by a saddle node onto limit cycle bifurcation. Delayed spike initiation is the norm. Now, for neurons-- for example, for the half bifurcation the situation is different. They don't really have a threshold. Formally, amplitudes of the response can have any size. However, in the limit of a separation of time scale, the middle branch of the  $u$ -nullcline plays, effectively the role of a threshold. Now, these are, as I said, classical topics of vertical and mathematical neuroscience, of computational neuroscience.

Notes

Summary



20m 17s

# Neuronal Dynamics – 4.4 Literature

**Reading:** W. Gerstner, W.M. Kistler, R. Naud and L. Paninski, *Neuronal Dynamics: from single neurons to networks and models of cognition*. Chapter 4: *Introduction*. Cambridge Univ. Press, 2014  
OR W. Gerstner and W.M. Kistler, *Spiking Neuron Models*, Ch.3. Cambridge 2002  
OR J. Rinzel and G.B. Ermentrout, (1989). Analysis of neuronal excitability and oscillations.  
In Koch, C. Segev, I., editors, *Methods in neuronal modeling*. MIT Press, Cambridge, MA.

## Selected references.

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- Fourcaud-Trocme, N., Hansel, D., van Vreeswijk, C., and Brunel, N. (2003). *How spike generation mechanisms determine the neuronal response to fluctuating input*. J. Neuroscience, 23:11628-11640.
- Badel, L., Lefort, S., Berger, T., Petersen, C., Gerstner, W., and Richardson, M. (2008). Biological Cybernetics, 99(4-5):361-370.
- E.M. Izhikevich, Dynamical Systems in Neuroscience, MIT Press (2007)



And I invite you to look at the literature. And please, also have a look at the quiz.

Notes

Summary



21m 30s