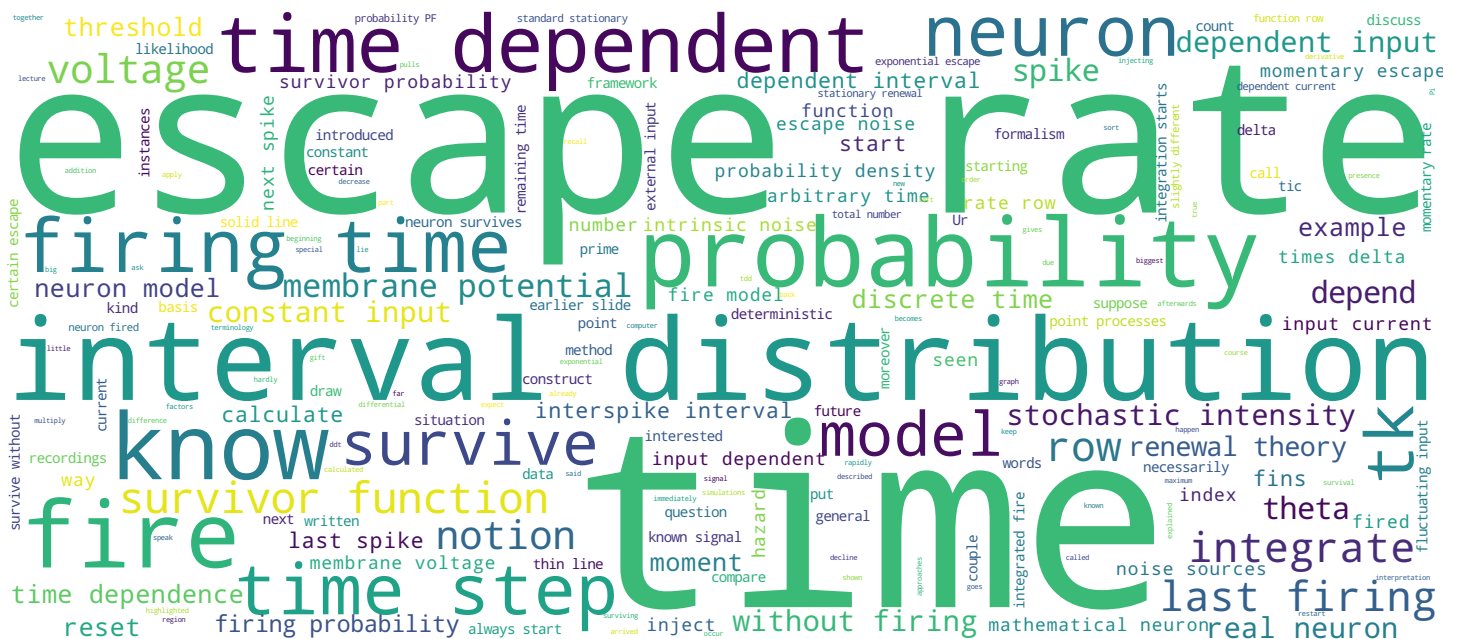


ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

EPFL, Lausanne, Switzerland

- stochastic resonance



EPFL



Week 6 – part 2 : Interspike intervals and renewal processes



✓ 6.1 Escape noise

- stochastic intensity and point process

6.2 Interspike interval distribution

- Time-dependend renewal process
- Firing probability in discrete time

6.3 Likelihood of a spike train

- likelihood function

6.4 Comparison of noise models

- escape noise vs. diffusive noise

6.5. Rate code vs. Temporal Code

- timing codes
- stochastic resonance

Welcome to Part 2 of Week 6 of the class Neuronal Dynamics. In the previous lecture I introduced the notion of stochastic intensity of escape rates and point processes. And now, let's use this mathematical framework to discuss the notion of interspike interval distribution again.

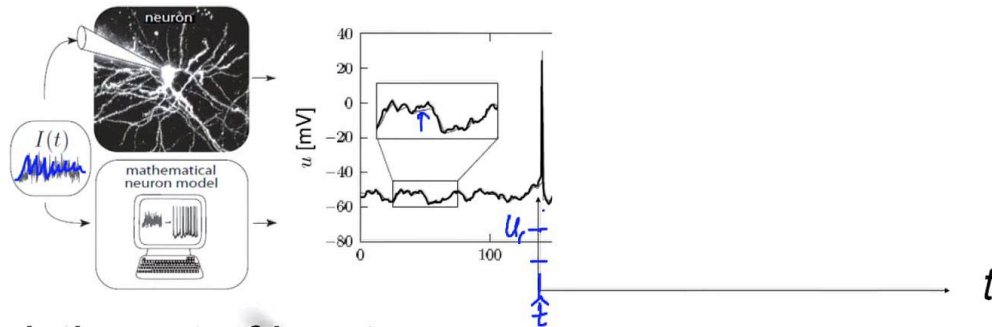
Notes

Summary



0m 04s

Neuronal Dynamics – 6.2. Interspike Intervals



deterministic part of input

$$I(t) \rightarrow u(t) \checkmark$$

noisy part of input/intrinsic noise

→ escape rate

Example:
nonlinear integrate-and-fire model

$$\tau \cdot \frac{d}{dt} u = F(u) + RI(t)$$

if spike at $t^f \Rightarrow u(t^f + \delta) = u_r$ reset

Example:
exponential stochastic intensity

$$\rho(t) = f(u(t)) = \rho_g \exp(u(t) - \vartheta)$$

Let's recall the framework that we use for this escape noise modeling. Let's suppose we inject some fluctuating input current into a neuron. We are interested in injecting the same kind of input current into the mathematical neuron model. Now, if this is a deterministic signal, then this gives rise to a membrane voltage of the model. For example, in the nonlinear integrate-and-fire model, this is where we inject our current. We integrate this up. You generates spikes with our stochastic process. And afterwards we restart at the reset potential, U_r . So if you know the deterministic input, and if you know the firing time, t_f , of the last spike, we know that integration starts at U_r , and we can just integrate this, and we have our deterministic membrane voltage, u of t . Now, this u of t will be sort of the thin line in this graph. This is the model voltage. The real neuron would be a thick solid line. Now, our real neuron, and also our model neuron, now fire at this moment in time. I call this firing time, $t_{\hat{}}$. This is the last firing time-- if I'm at time t , I look back-- this is the last firing time of the neuron. Now, with this method, I now know that for my model, integration starts again at some value-- let's put it here-- some value U_r .

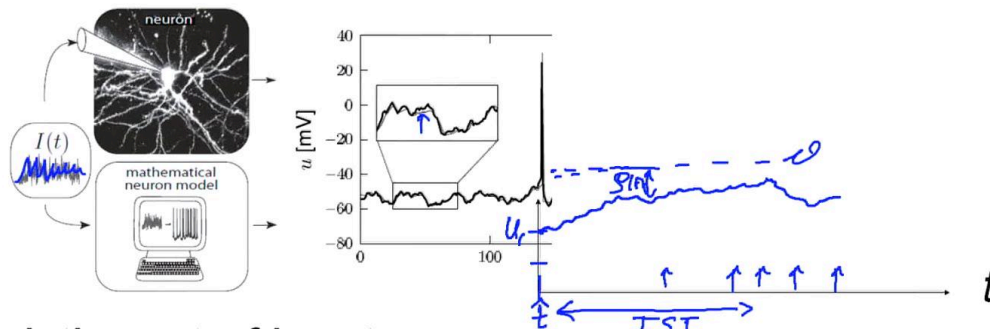
Notes

Summary



0m 30s

Neuronal Dynamics – 6.2. Interspike Intervals (ISIs)



deterministic part of input

$$I(t) \rightarrow u(t) \checkmark$$

noisy part of input/intrinsic noise

→ escape rate

Example:
nonlinear integrate-and-fire model

$$\tau \cdot \frac{d}{dt} u = F(u) + RI(t)$$

if spike at $t^f \Rightarrow u(t^f + \delta) = u_r$ reset

Example:
exponential stochastic intensity

$$\rho(t) = f(u(t)) = \rho_g \exp(u(t) - \vartheta)$$

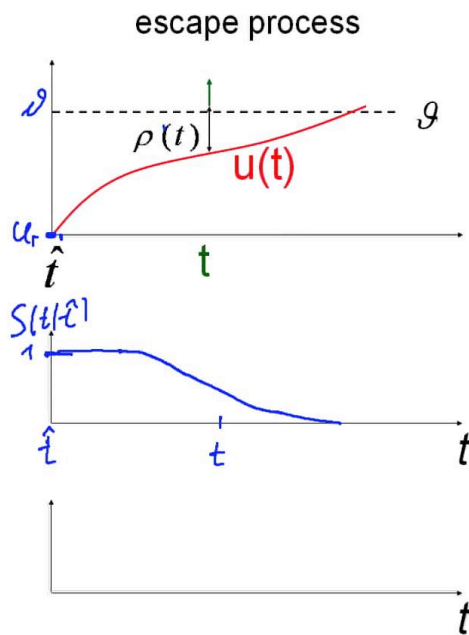
I know my current, it's known, for all the future. So I can construct this deterministic trajectory. And then the question arises, will this neuron fire again here, or will it fire again here, or there, or there? Now, we know that the stochastic intensity or escape rate depends on the momentary difference between a threshold value and my membrane voltage. So there's a likelihood that it fires here. There's a certain stochastic intensity that it would fire here, or there, or there, or there. And now the question is, what's-- this was the last spike-- what's the distribution of these interspike intervals, I-S-I-- Interspike intervals.

Notes

Summary



Neuronal Dynamics – 6.2. Interspike Interval distribution



escape rate

$$\rho(t) = f(u(t) - g)$$

Survivor function

$$\frac{d}{dt} S_I(t|\hat{t}) = -\rho(t) S_I(t|\hat{t})$$

$$S(t|\hat{t}) = e$$

Let's construct this together. So at any moment in time, there's a certain momentary escape rate. I'm at time t . There is this momentary escape rate row of t . Now I ask-- this was my last firing time, t -hat. What's the probability that the neuron survives-- starting at t -hat, last firing time-- that it survives up to this moment, t , without firing? So immediately after the spike at t -hat, we start at U_r . U_r is a long way below the threshold. Therefore, the probability that it survives for a short moment is 1. So this survivor function, S of t , given t -hat, will always start at 1. We know that at the beginning it survives. And now, as the membrane potential approaches the threshold, there's a certain probability that it fires, instantaneous radius row. So we expect that this survival probability will decrease with this momentary rate row of t . This is the momentary rate. So as u of t approaches the threshold, there's an escape rate, and therefore, the probability to survive will go down. So this is the probability of surviving. Now we can integrate this equation. So S of t , given t -hat, always starts at 1. Now, this is just like radioactive decay, but there's a time-dependent decay constant.

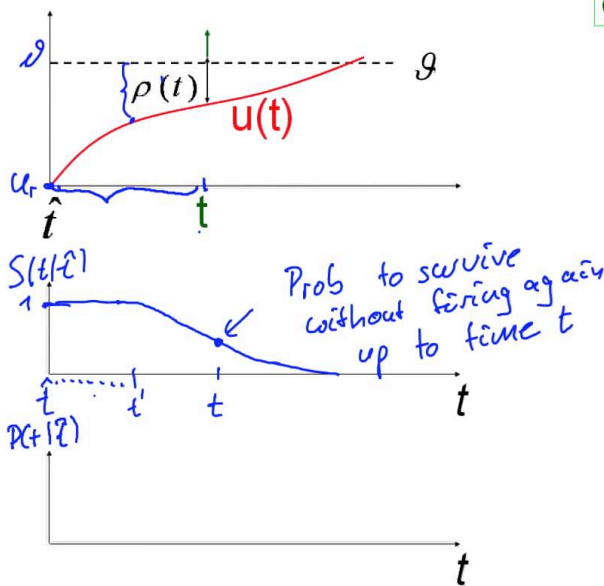
Notes

Summary



Neuronal Dynamics – 6.2. Interspike Interval distribution

escape process



escape rate

$$\rho(t) = f(u(t) - \vartheta)$$

Survivor function

$$\frac{d}{dt} S_I(t|\hat{t}) = -\rho(t) S_I(t|\hat{t})$$

rate
↓
last firing $\hat{t}$

$$S(t|\hat{t}) = e^{-\int_{\hat{t}}^t \rho(t') dt'}$$

Interval distr.

$$P(t|\hat{t}) = \rho(t) \cdot S(t|\hat{t})$$

↑ ↑
next sp. last spike

So we integrate this row of t prime starting at t -hat. And then we integrate this-- this is some arbitrary prime time t prime, between t -hat and t . And in this moment, t prime, I have already a certain escape rate. And this lowers my survivor function, and I integrate up to time t . And this is the value, this is the probability to survive without firing again, up to time t . Now, I said we are interested in the interspike interval distribution. So far, we just have the survivor probability. So how do we get from this survivor probability to the interval distribution? Well, what's the probability that a neuron that has fired its last spike t -hat will fire again at time t ? P of t , given t -hat, gives this as a probability density. Now, for the neuron, to fire its next spike at time t , it has first to survive from t -hat to t without firing. So this interval distribution, P of t , t -hat, will have two factors. The neuron has to survive from t -hat to t without firing again. But then it actually has to fire at time t . Well, what's the rate that it fires at time t ? Well, it's ρ of t . So this is the probability density that, given the last spike at t -hat, the next spike would occur at time t .

Notes

Summary



Neuronal Dynamics – 6.2. Interspike Interval distribution

escape process

escape rate

$$\rho(t) = f(u(t) - \vartheta)$$

Survivor function

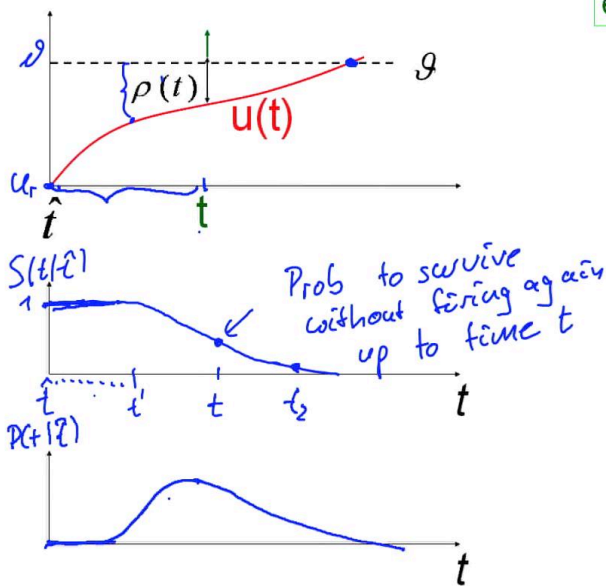
$$\frac{d}{dt} S_I(t|\hat{t}) = -\rho(t) S_I(t|\hat{t})$$

$$S(t|\hat{t}) = e^{-\int_{\hat{t}}^t \rho(t') dt'}$$

Interval distr.

$$P(t|\hat{t}) = \rho(t) \cdot S(t|\hat{t})$$

↑
next sp.
↑
last spike



Well, it has first to survive from $t\text{-hat}$ to t , and then it has to fire. So while the survivor probability stays at 1, we know that no spikes can happen. And then as this starts to decline, the interval distribution increases. Now, the instantaneous rate of firing is very big here. But hardly any probability is left that the new one would survive to this time, t_2 . Therefore, the interval distribution goes already down again. So it's not true that the maximum of the interval distribution would lie at the crossing point of the deterministic potential. Now, it's absolutely possible that the maximum lies well before.

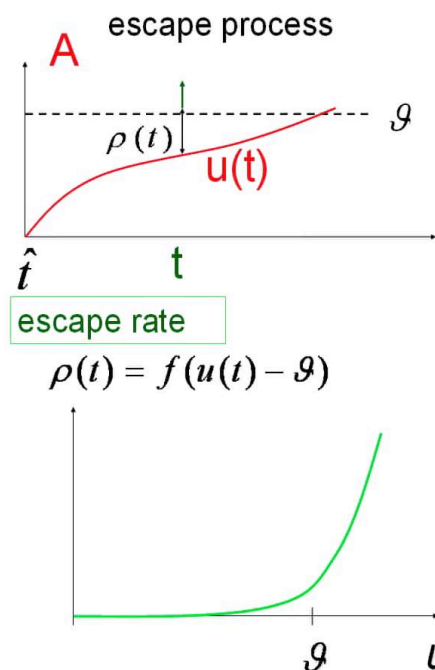
Notes

Summary



8m 50s

Neuronal Dynamics – 6.2. Interspike Intervals



Survivor function

Examples now

$$\frac{d}{dt} S_I(t|\hat{t}) = -\rho(t) S_I(t|\hat{t})$$

$$S_I(t|\hat{t}) = \exp\left(-\int_{\hat{t}}^t \rho(t') dt'\right)$$

Interval distribution

$$P_I(t|\hat{t}) = \underbrace{\rho(t)}_{\text{escape rate}} \cdot \underbrace{\exp\left(-\int_{\hat{t}}^t \rho(t') dt'\right)}_{\text{Survivor function}}$$

$I(t)$

$$P(t|\hat{t}) = -\frac{d}{dt} S(t|\hat{t})$$

So let's write this up again. My survivor function can be calculated by this differential equation, which I can integrate. In order to get to the interval distribution, I take the survivor function part, and then I multiply with the momentary escape rate. Now, there is something interesting, and that's this P_I , this interval distribution-- t of $[\hat{t}, t]$ t -hat-- if I compare this and that, then I see I can also write it as minus dS of t , given t -hat. If I take the derivative of this exponential term, it just pulls down this row of t . Now, this is a time-dependent interval distribution. As I explained on an earlier slide, we give some time-dependent input, some fluctuating input, a deterministic, a known signal, the $I(t)$ is in here. And in the presence of this time-dependent input-- and that's highlighted by this index I here-- I can calculate the probability density that the next spike occurs at t , given I know the last firing time was at t -hat. So this survivor function can be written in the general case with this index I . And of course, this row also is input dependent.

Notes

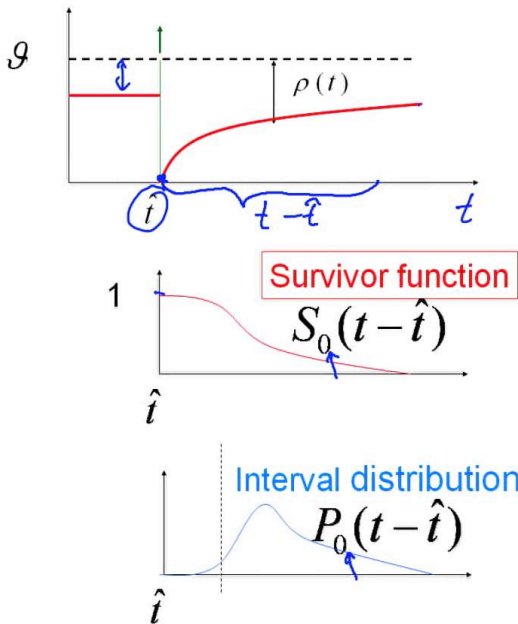
Summary



9m 51s

Neuronal Dynamics – 6.2. Renewal theory

Example: I&F with reset, constant input



escape rate

$$\rho(t|\hat{t}) = f(u(t|\hat{t})) = \rho_g \exp(u(t|\hat{t}) - g) = S_0(t-\hat{t})$$

$$S(t|\hat{t}) = \exp\left(-\int_{\hat{t}}^t \rho(t'|\hat{t}) dt'\right) = S_0(t-\hat{t})$$

$$P(t|\hat{t}) = \rho(t|\hat{t}) \exp\left(-\int_{\hat{t}}^t \rho(t'|\hat{t}) dt'\right) = -\frac{d}{dt} S(t|\hat{t})$$

Now let's discuss the two examples. The first example is that of constant input. As a constant input, which I apply all the time, now with this constant input, stochastically, because of this stochastic intensity, a spike happened here. Now I know the last firing time for all the future times, t , has been here. Now, after the last firing time, the membrane potential was reset. The survivor function starts at a value of 1, and then it goes down again. The interval distribution is biggest in the region here where their survivor function goes down rapidly. Now, if I have constant input, then the input is constant. That's this little 0 here. It has no time-dependent input, so I don't keep the index I . But moreover, the remaining time dependence, you see that the membrane potential still has a time dependence. Now, this remaining time dependence is due to the reset at time t -hat. And so, the time dependence is completely described by the time difference, t minus t -hat. So this becomes a function which just depends on t minus t -hat. And so this would be, since this only depends on t minus t -hat, this will also only depend on t minus t -hat. And so does the interval distribution.

Notes

Summary



Neuronal Dynamics – 6.2. Renewal theory

Example: I&F with reset, constant input

Renewal theory

"hazard"

escape rate

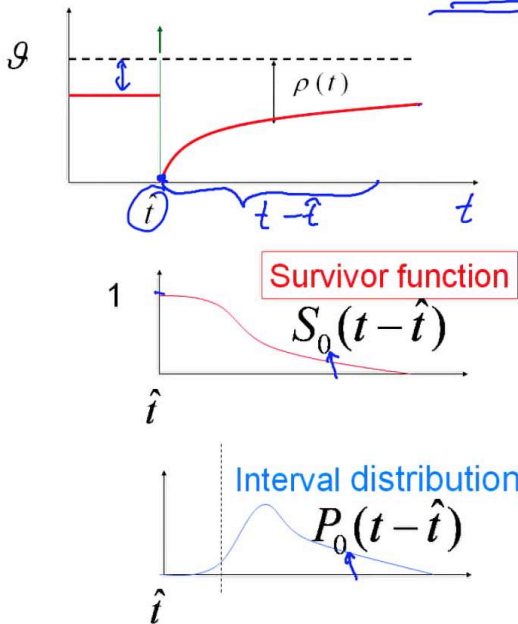
$$\rho(t|\hat{t}) = f(u(t|\hat{t})) = \rho_g \exp(u(t|\hat{t}) - g) = S_0(t - \hat{t})$$

survivor function

$$S(t|\hat{t}) = \exp\left(-\int_{\hat{t}}^t \rho(t'|\hat{t}) dt'\right) = S_0(t - \hat{t})$$

$$P(t|\hat{t}) = \rho(t|\hat{t}) \exp\left(-\int_{\hat{t}}^t \rho(t'|\hat{t}) dt'\right) = P_0(t - \hat{t})$$

$$= -\frac{d}{dt} S(t|\hat{t})$$



It's just a function of t minus t -hat. Now, this situation of constant input where the escape rate just depends on t minus t -hat is called renewal theory. For the case of renewal theory, the stochastic intensity is also called the hazard. This is, again, the survivor function. And if we use this terminology of survivals, then the hazard is, so to speak, the hazard of time. Now what I've shown on the earlier slides is that this renewal theory for constant input is a special case of our more general scenario.

Notes

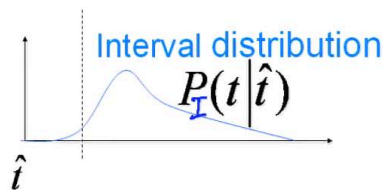
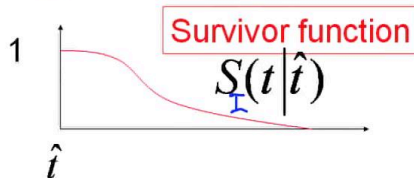
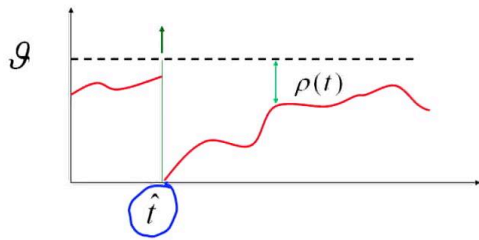
Summary



13m 49s

Neuronal Dynamics – 6.2. Time-dependent Renewal theory

Example: I&F with reset, time-dependent input,



escape rate

$$\rho(t|\hat{t}) = f(u(t|\hat{t})) = \rho_g \exp(u(t|\hat{t}) - g)$$

$$S_{\hat{t}}(t|\hat{t}) = \exp\left(-\int_{\hat{t}}^t \rho(t'|\hat{t}) dt'\right)$$

$$P_{\hat{t}}(t|\hat{t}) = \rho(t|\hat{t}) \exp\left(-\int_{\hat{t}}^t \rho(t'|\hat{t}) dt'\right) = -\frac{d}{dt} S_{\hat{t}}(t|\hat{t})$$

If I have a time-dependent input, and the membrane potential is truly time-dependent, the time dependence arises from both the external input. And, in addition, for integrated fire model with reset, I need to know the last firing time, $t\text{-hat}$. And that's why I've written here the voltage as u of t , given $t\text{-hat}$. I have to know $t\text{-hat}$. In this case, I have an input dependent survivor function, which depends on this row here. And I can calculate this input dependent interval distribution just the same way as in renewal theory. In other words, you have arrived at a time-dependent version of standard stationary renewal theory.

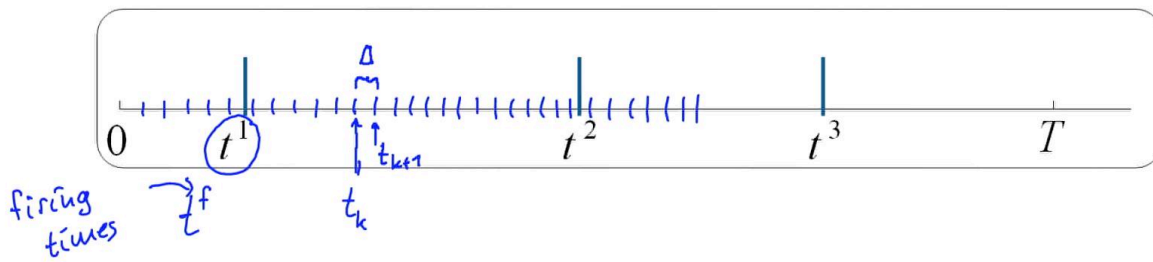
Notes

Summary



15m 01s

Neuronal Dynamics – 6.2. Firing probability in discrete time



Probability to survive 1 time step

$$S(t_{k+1} | t_k) = \exp\left[-\int_{t_k}^{t_{k+1}} \rho(t') dt'\right]$$



Now let's see what we can do with these notions. One thing is for an interpretation of this escape rate, but also for simulations on a computer, we have to work in discrete time. So suppose I have some discrete time steps, and from time to time there is a spike. So let's draw a couple of these discrete time steps, and I have a couple of spikes. So these are the firing times. The firing times have an upper index. Now, an arbitrary time step, for example, this guy here, is t_k . It's the lower index. And I should be more precise. So let me redraw this arrow. t_k is the tic. And then this here, this tic here, would be $t_k + 1$, the next moment in time. So a time step has a certain duration. So this duration here would be Δ . And now the question is, I have my newer model, the one with escape noise, with stochastic intensity. And I would like to know what's the probability that the neuron survives from t_k to $t_k + 1$, through the whole time step Δ , without firing. So the probability to survive for one time step is-- now, starting at t_k , this is not necessarily to reset, this is not necessarily the way of firing time. This is my time bin-- this is the start of my time bin-- up to the next time bin.

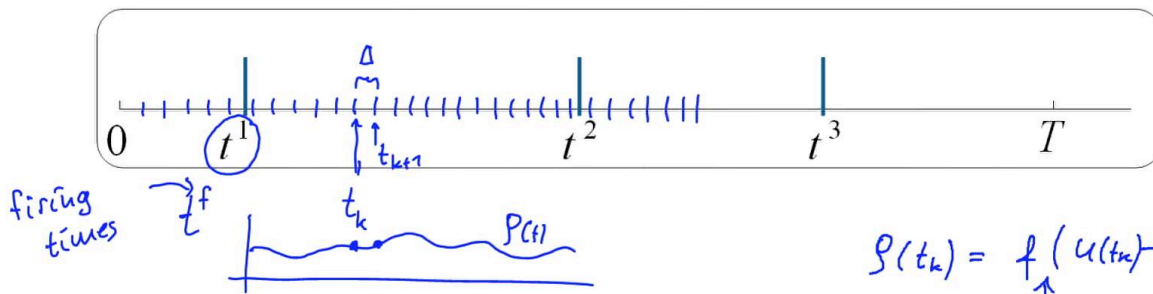
Notes

Summary

16m 08s

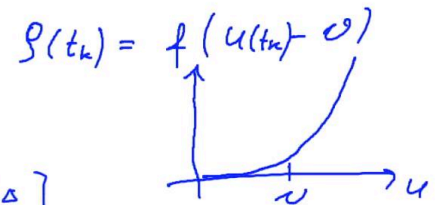


Neuronal Dynamics – 6.2. Firing probability in discrete time



Probability to survive 1 time step

$$S(t_{k+1} | t_k) = \exp\left[-\int_{t_k}^{t_{k+1}} \rho(t') dt'\right] = \exp[-\rho(t_k) \Delta]$$



Probability to fire in 1 time step

$$P_k^F = 1 - S(t_{k+1} | t_k) = 1 - e^{-\rho(t_k) \Delta}$$

What's the probability to survive for this time, delta, for one time step? Well, just as before, it's just exponentially we have to integrate this. Now, if the time steps are small, then this function row of t, which I would like to draw now here, roughly, this function row of t will not change a lot within t_k and t_k plus 1. So for this integral here, I can approximate this by row of t_k times delta. Now, the next question is, from time to time we have these spikes. So this would be a time bin where the neuron has fired. So what's the probability to fire in one time step? Well, this probability, let's call it P_F, F of firing, firing in the time step k. Well, the probability to fire in one time step, that's the probability that you do not survive without firing. So this is 1 minus S of t_k plus 1, starting at t_k. So this would be using this expression here, 1 minus e to the-- and there's a minus sign missing-- e to the minus row of t_k times delta. Now, row of t_k is my escape function, f of u minus theta evaluated at this moment, t_k. We have seen that this function f may explode for u [INAUDIBLE] and theta. It just goes up to infinity. It does not saturate.

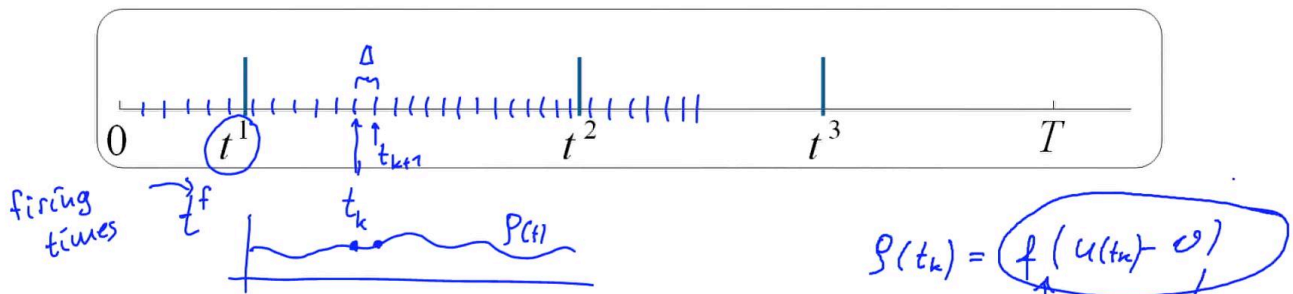
Notes

Summary



18m 30s

Neuronal Dynamics – 6.2. Firing probability in discrete time



Probability to survive 1 time step

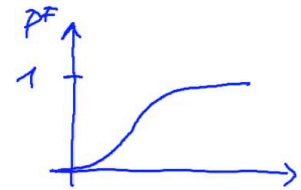
$$S(t_{k+1} | t_k) = \exp\left[-\int_{t_k}^{t_{k+1}} \rho(t') dt'\right] = \exp[-\rho(t_k) \Delta]$$

$$\rho(t_k) = f(u(t_k) - \theta)$$

Graph of the function $f(u)$ showing a sigmoidal shape.

Probability to fire in 1 time step

$$P_k^F = 1 - S(t_{k+1} | t_k) = 1 - e^{-\rho(t_k) \Delta}$$



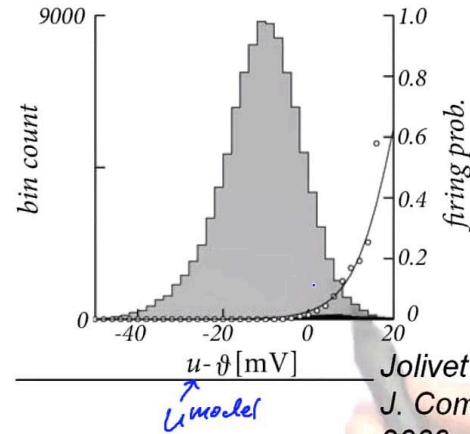
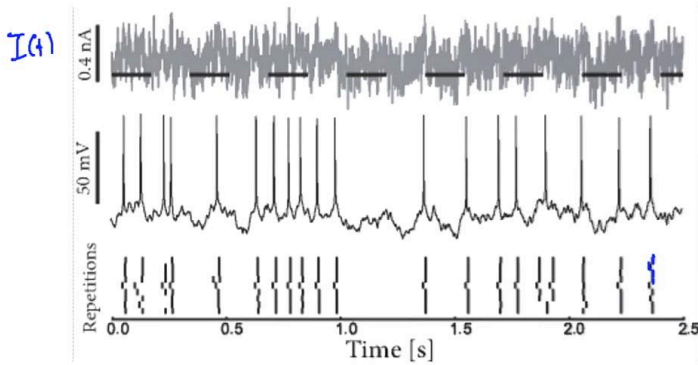
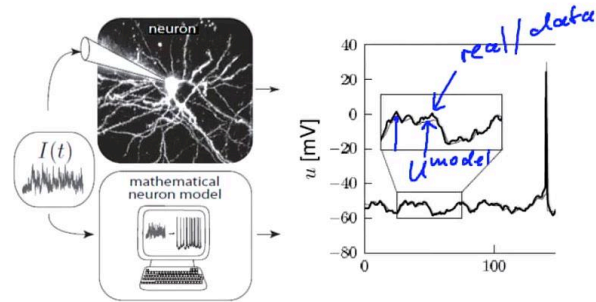
Now, if you evaluate this finite time step probability, you take this unbounded f , you plug it into this formula here. Then this probability of firing is bounded to 1, as it should be. Now this can be used, in fact, to determine, experimentally, the shape of the function f .

Notes

Summary



Neuronal Dynamics – 6.2. Escape noise - experiments



Jolivet et al.,
J. Comput. Neurosc.
2006

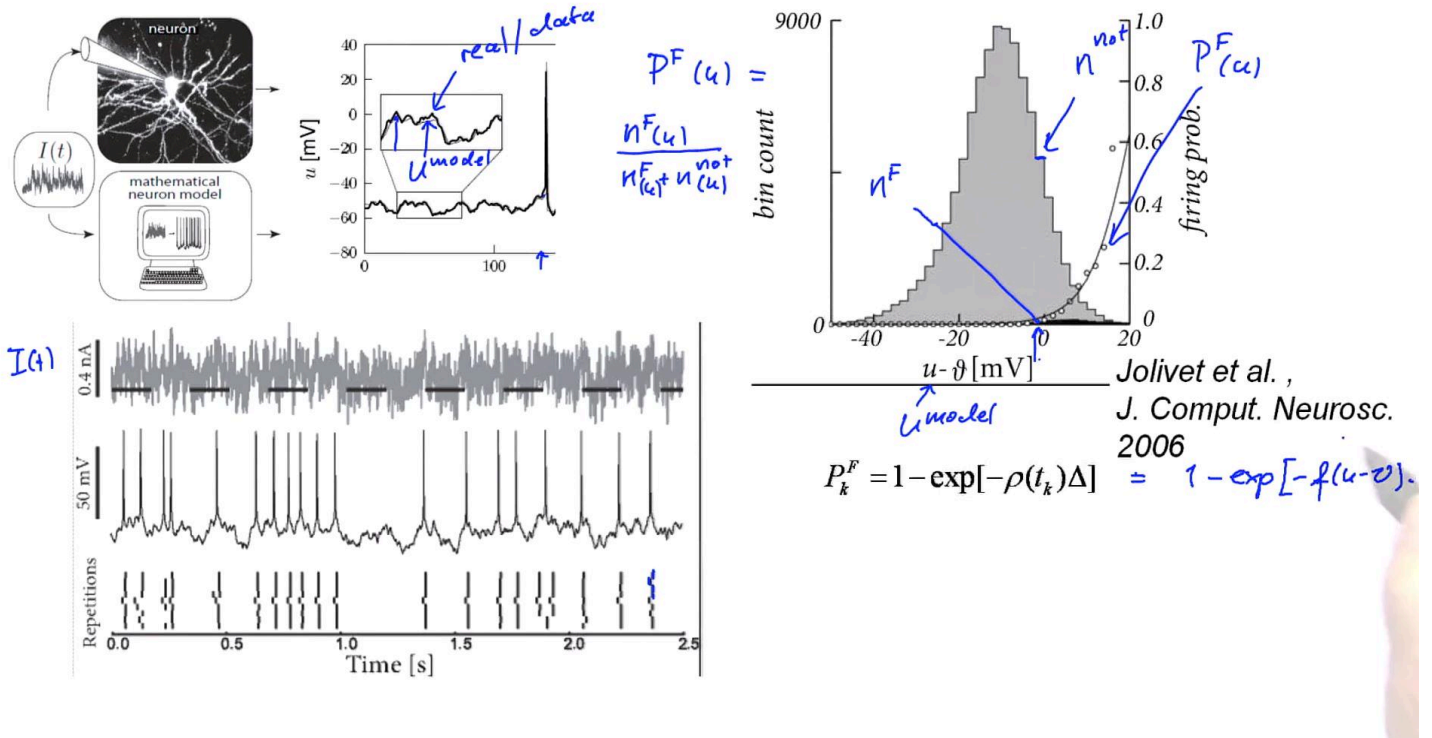
So let's come back to our situation where we inject a time-dependent current into a real neuron, and also into the mathematical neuron model. The mathematical neuron model could be your integrated fire model, and you can use this time-dependent current, the known signal, to calculate the membrane potential of our neuron model. This model membrane potential is the thin line here. You also do recordings from the real neuron, and the real neuron will be the thick line. This is the data. Now from time to time, the real neuron emits a spike. Now what we find is that if you look at these kind of recordings, there's several repetitions of the same time-dependent input current, the same signal I of t . If you look at several recordings, then the membrane potential is slightly different, and moreover, the spike times are slightly different between one trial and the next. So let's use this to our advantage, so as to find out what the intrinsic noise sources of the neuron are, of this intrinsic escape rate of the neuron is. The way we do this is the following. Let's look at some value, say, a point here. This point here. This is a certain value, U , of my model. And this is my U model, which is just below theta.

Notes

Summary



Neuronal Dynamics – 6.2. Escape noise - experiments



Say it's this value here. Now for this value here of my model, I see that the neuron here has not fired. Now, I look at all the cases here I have the same value of U model, and I count the number of bins where the neuron does not fire. So this is the number of counts where it's not firing. But from time to time, for the same voltage of the model, say a point here, the neuron actually fires. So I also have instances where, for this very same voltage the neuron fires. And that's the black part here. So there are very few instances, but there are. So this black thing would be the n firing at the same voltage. Now, let's compare the number of bins where the neuron fired, to the total number of bins that are found that this voltage. Well, the total number of bins are those where the neuron fired, plus those, where the neuron did not fire. Now, if I do this for a fixed voltage u , then I find my firing probability for this voltage. Experimental firing probability, P^F of u , are these data points. On the other hand, from my escape noise model, formalism, I know that this probability P^F of firing at a time bin, a time bin where the escape rate row of t was given by u minus θ and then times Δ .

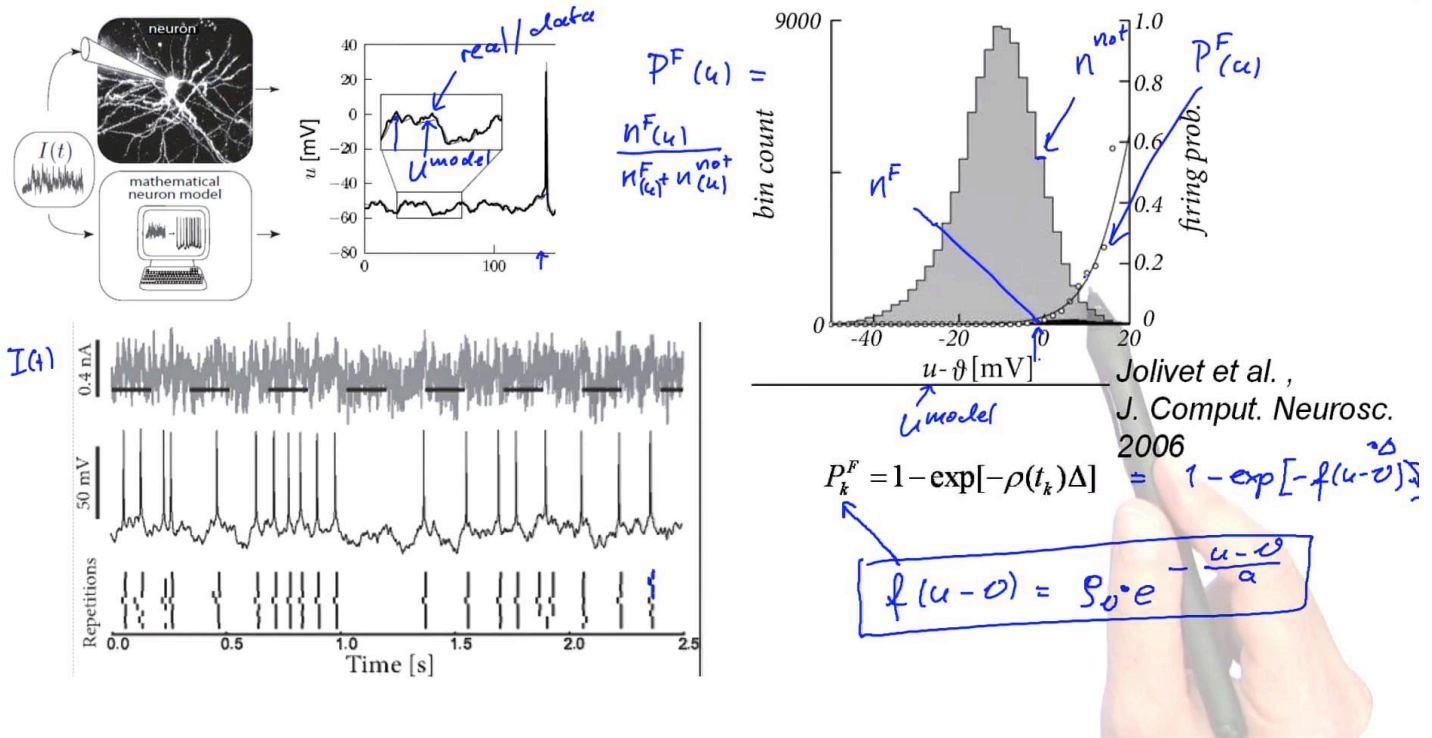
Notes

Summary



23m 44s

Neuronal Dynamics – 6.2. Escape noise - experiments



Let me put the delta up here. So this is my firing probability for this voltage, with a certain escape rate. I can use to extract the escape rate. For example, let's say my escape rate, f of u minus θ , is the exponential one, e to the minus, u minus θ over some parameter α . And that would be a rate constant in front. This is a fixed parameter that characterizes defined rate at the threshold, if u is equal to θ . Now, if you use this exponential function F in this discrete time formula, P^F , then we get this solid line. So we see that there's an excellent fit of the exponential escape rate and the data. In other words, the exponential escape rate that I introduced earlier is a reasonable or in a very good approximation to the intrinsic noise sources in neuronal membranes--intrinsic noise sources that contribute to the firing of the neuron.

Notes

Summary



Neuronal Dynamics – 6.2. Renewal process, firing probability

Escape noise = stochastic intensity

-Renewal theory

- hazard function
- survivor function
- interval distribution

-time-dependent renewal theory

-discrete-time firing probability

-Link to experiments

→ basis for modern methods of neuron model fitting (week 7)



So let me summarize. In the first lecture today, I have introduced the notion of stochastic processes, the notion of escape rates. Now we have seen here that we can translate this continuous time escape rate into a discrete time firing probability. And we can match this firing probability to data. Moreover, the formalism of stochastic point processes allows us to calculate things like interval distributions. In the case of a constant external input and integrating fire model with resets at each moment of firing, you are in the framework of standard stationary renewal theory. In this case, the escape rate corresponds to the hazard of the renewal model. But we also have seen that the formalism is much more general and can account for arbitrary time-dependent input. And this leads to the notion of arbitrary time-dependent interval distributions. And we can calculate this. Now this is important because the methods that are described in this lecture now are the basis for the likelihood of a spike train, which you will see in the next lecture. But also the basis for modern methods of neuron model fitting, that we will see next week.

Notes

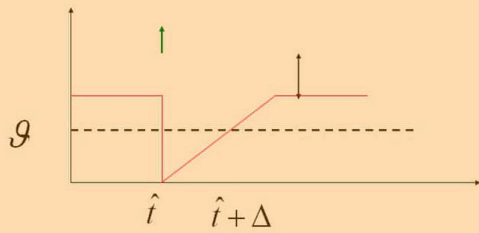
Summary



27m 55s

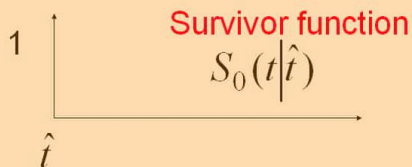
Neuronal Dynamics – Homework assignment 6.1

neuron with relative refractoriness, constant input

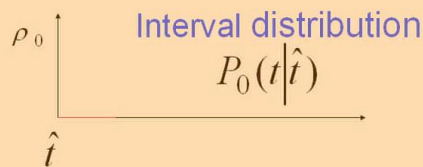


escape rate

$$\rho(t) = \rho_0 \frac{u}{g} \text{ for } u > g$$



$$S_0(t | \hat{t}) = \begin{cases} 1 & t < \hat{t} \\ 0 & t \geq \hat{t} \end{cases}$$



$$P_0(t | \hat{t}) = \begin{cases} 0 & t < \hat{t} \\ \rho_0 & t \geq \hat{t} \end{cases}$$

Please have a look and take some time with the homework assignments so that you get acquainted with the notions of time-dependent, interval distribution, hazard, escape rate, and so forth.

Notes

Summary



29m 37s