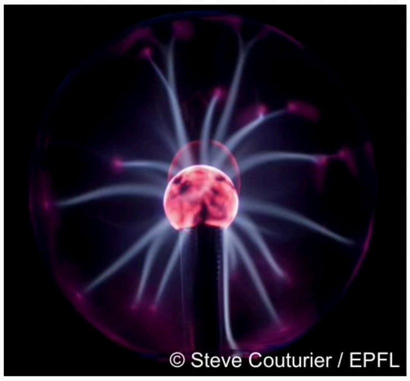
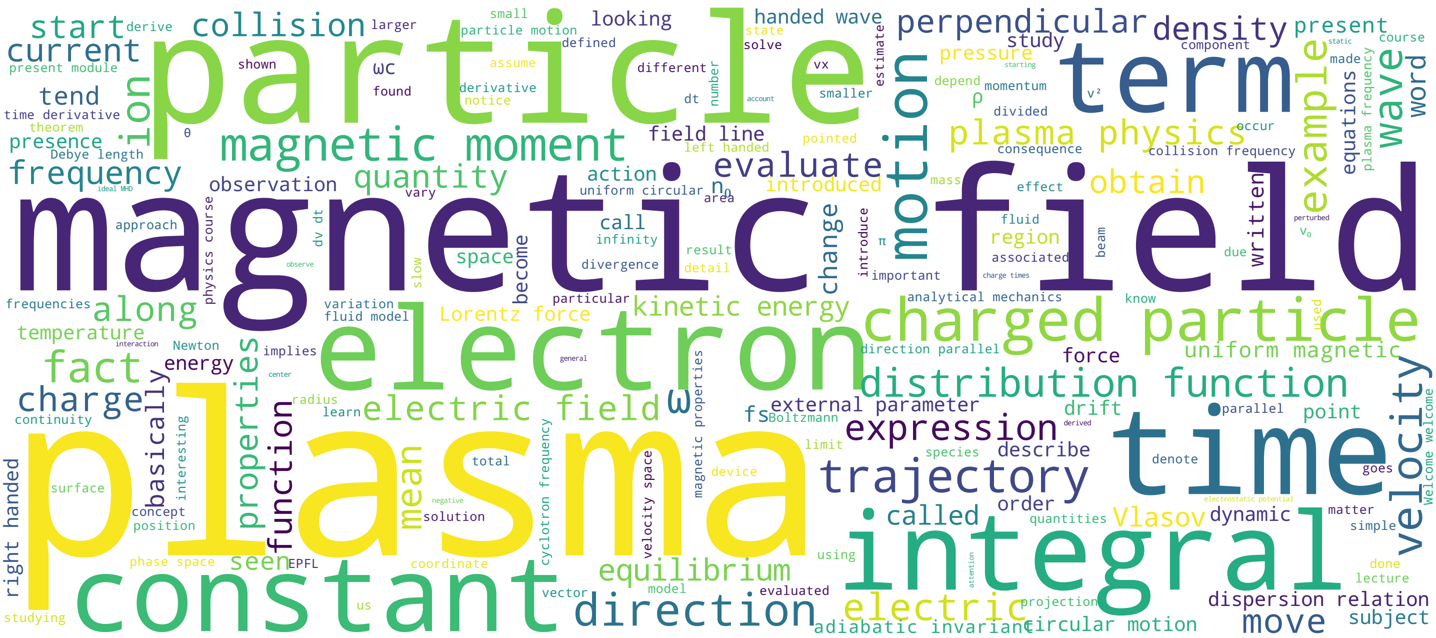


Particle motion in a static uniform magnetic field and plasma magnetic properties

Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

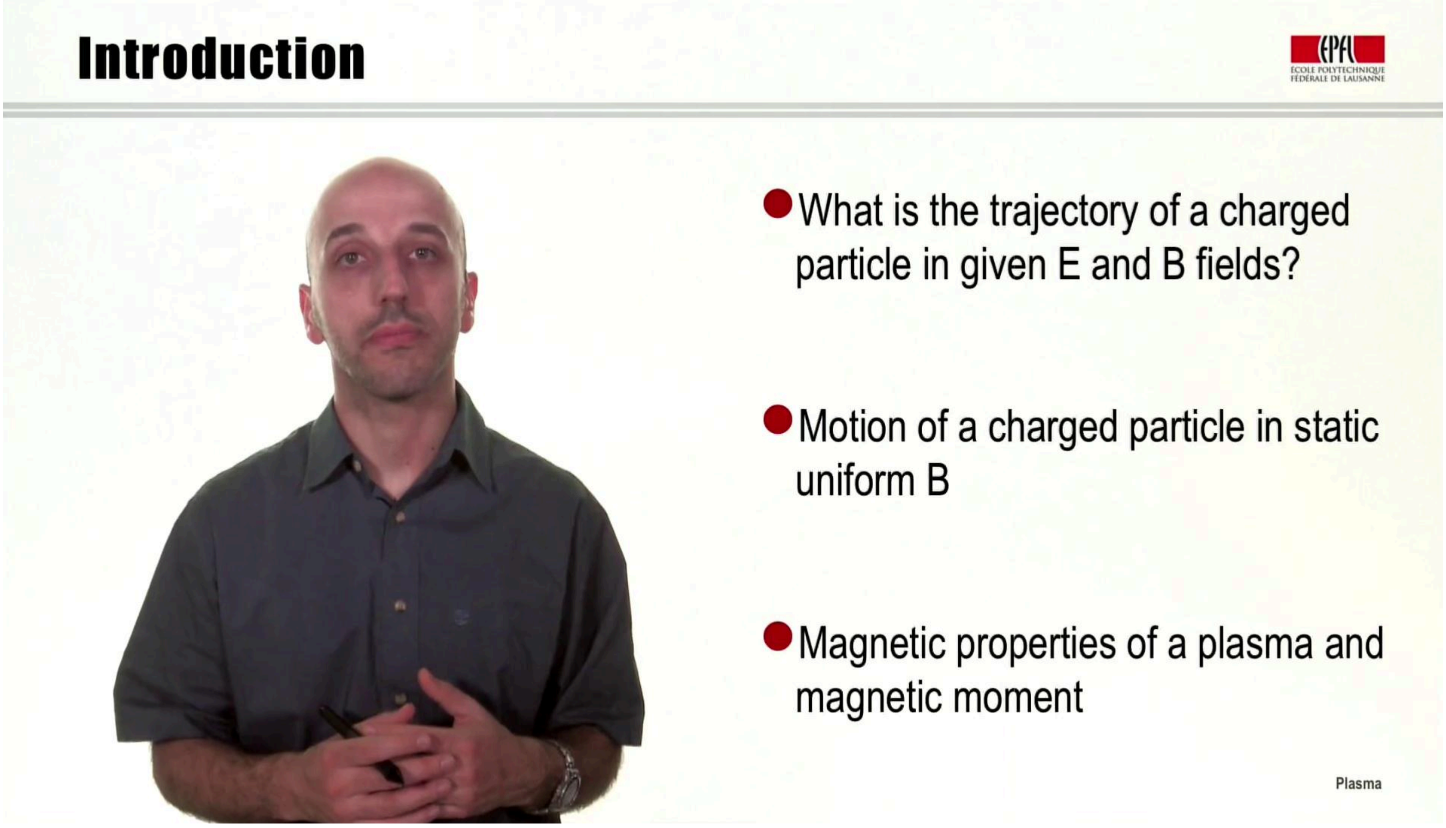
Lecture 1d

Paolo Ricci



Search MOOC

Video



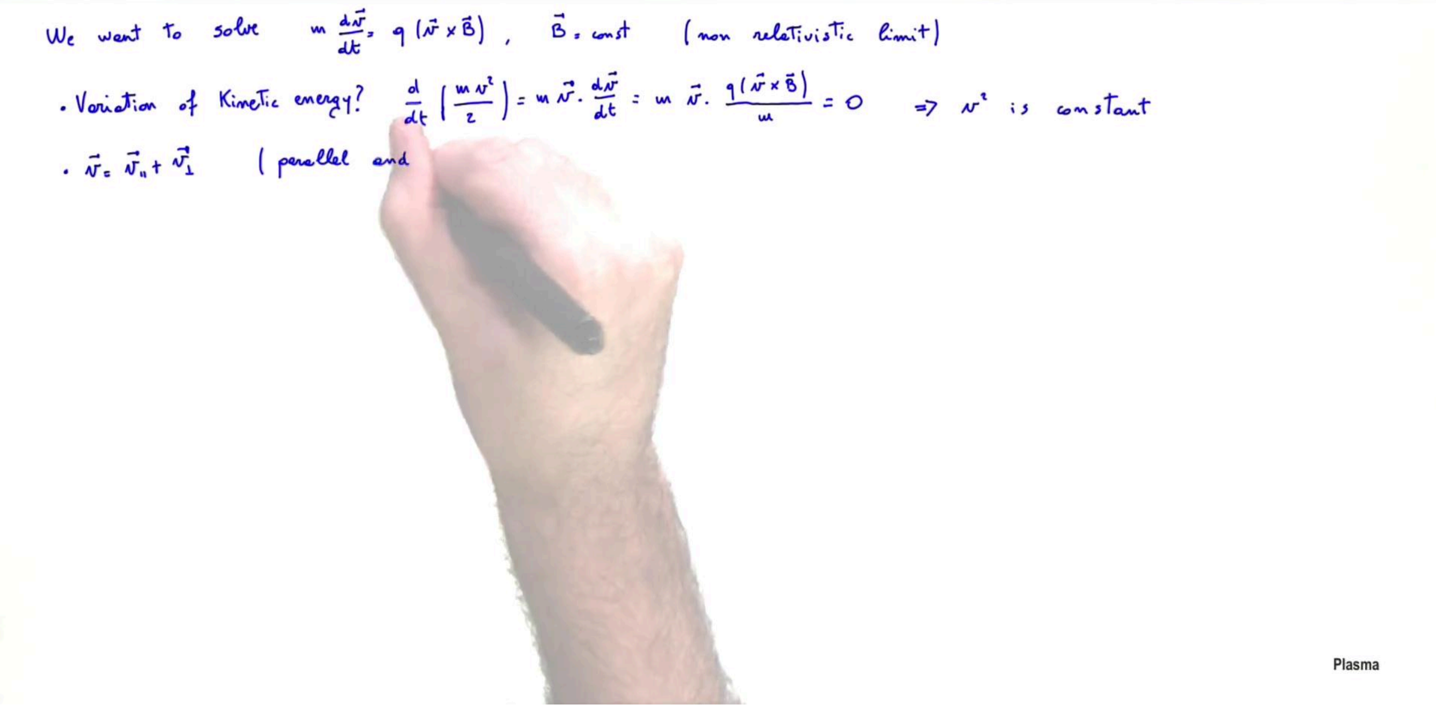
Welcome; welcome to the plasma physics course of the EPFL. In the past module we have given a rigorous definition of plasma. We have now seen how that plasmas are made of electrons and ions, charged particles, charged particles that move under the effect of electric and magnetic field. Now, these electric are magnetic field are actually generated self-consistently by these charged particles that constitute the plasma. Well if we want to describe in a self-consistent way, we have to-- at the same time describe the generation of these electric and magnetic field, and the motion of the particles subject to that. This is a complex nonlinear phenomenon and the analysis of it will be the subject of the future lectures. Here in these first modules, we will start with the simplest study, that is, we will assume that the electric and magnetic fields are given and that the charged particles move under the effect of these [given] electric and magnetic fields. We will study what is the trajectory of these particles. In the present module, we will start from the simplest scenario, that is, of a constant in time, uniform magnetic field. By studying the dynamics -the trajectory- of charged particles in this constant and uniform magnetic field, we will also learn about the magnetic properties of plasmas, and we will have the opportunity to introduce the concept of magnetic moment.

Notes

Summary

0m 05s

Motion of a particle in static and uniform B



So let's start with the simplest case. The motion of a particle in a static and uniform magnetic field. What we basically want to do is to solve the Newton equation with the Lorentz force, basically, with B constant. We will look at the non-relativistic limit. Well, this equation can be solved by using brute force, really by solving all components, all the three components, all the three coupled components of it, or, by studying a bit into detail the properties of the particle motion subject to Lorentz force. Let's take the second approach. The first thing that we want to understand and estimate is what is the variation of kinetic energy. Well we can study the time derivative of the kinetic energy, which is given by mv · dv/dt. dv/dt is given by Newton's equation, that it is given by q (v × B)/m. Now, (v × B) is pendicular to v, so if we take the dot product with v what we obtain is zero. In other words, d/dt of the kinetic energy is constant [≡0] which means that v^2 is constant. The second thing that we have to notice is that the direction parallel to magnetic field [∥] and one perpendicular to it [⊥] are quite different. It's useful to decompose the velocity as the sum of a parallel and a perpendicular component.

Notes

Summary

1m 58s

Motion of a particle in static and uniform B

We want to solve $m\frac{d\vec{v}}{dt} = q(\vec{v}\times\vec{B})$, $\vec{B}$ = const (non relativistic limit)

- Variation of Kinetic energy? $\frac{d}{dt}\left(\frac{mv^2}{2}\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} = m\vec{v}\cdot\frac{q(\vec{v}\times\vec{B})}{m} = 0 \Rightarrow v^2$ is constant
- $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$ (parallel and perpendicular to $\vec{B}$, $\vec{v}_\parallel = (\vec{v}\cdot\vec{B})\frac{\vec{B}}{B^2}$, $\vec{v}_\perp = \vec{v} - \vec{v}_\parallel$)

  → $m\frac{d\vec{v}}{dt}\cdot\frac{\vec{B}}{B} = m\frac{dv_\parallel}{dt} = q(\vec{v}\times\vec{B})\cdot\frac{\vec{B}}{B} = 0 \Rightarrow v_\parallel$ = const

  → $v^2 = v_\perp^2 + v_\parallel^2 \Rightarrow v_\perp$ = const

  → $\frac{d\vec{v}_\perp}{dt}\perp\vec{v}_\perp$, $v_\perp$ = const ⇒ uniform circular motion $\frac{m v_\perp^2}{\rho} = |q| v_\perp B \Rightarrow \begin{cases} \rho = \frac{m v_\perp}{|q| B} & \text{Larmor radius or cyclotron radius} \\ \omega_c = \frac{v_\perp}{\rho} = \frac{|q| B}{m} \end{cases}$

Plasma

Parallel and perpendicular to B. That is, v parallel [v∥] is defined as the projection of v along B, taking account the direction of B, and v perp [v⊥] is equal to v-v∥. We can project the Newton equation along to parallel direction, which gives therefore the time variation of v∥, which will be equal to the projection of the Lorentz force along the B direction and this will be equal to zero, which implies that v∥ is constant. But we also have that $v^2 = v_\perp^2 + v_\parallel^2$ and as both v∥ is constant and the v total is constant then v⊥ is constant. Now what do we have is that the derivative of v⊥ is perpendicular to v⊥. At the same time, the modulus of v⊥ is constant. The only possibility is that the particle has, in the perpendicular plane, a uniform circular motion whose property can be obtained by balancing the centrifugal force with the Lorentz force, where we have introduced two quantities here ρ, That is ρ = (m v⊥) / (|q| B) which is the radius of the circular motion that is also called *Larmor radius* or *cyclotron* radius; and this motion will be associated with the frequency that we will denote with omega c [ωc] That is given by ωc = v⊥ / ρ And the frequency of this uniform circular motion is also called *cyclotron frequency*.

Notes

Summary

4m 00s

Motion of a particle in static and uniform B

We want to solve $m\frac{d\vec{v}}{dt} = q(\vec{v}\times\vec{B})$, $\vec{B} = \text{const}$ (non relativistic limit)

- Variation of Kinetic energy? $\frac{d}{dt}\left(\frac{m v^2}{2}\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} = m\vec{v}\cdot\frac{q(\vec{v}\times\vec{B})}{m} = 0 \Rightarrow v^2$ is constant
- $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$ (parallel and perpendicular to $\vec{B}$, $\vec{v}_\parallel = (\vec{v}\cdot\vec{B})\frac{\vec{B}}{B^2}$, $\vec{v}_\perp = \vec{v} - \vec{v}_\parallel$)

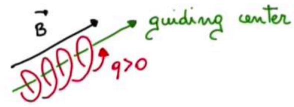
→ $m\frac{d\vec{v}}{dt}\cdot\frac{\vec{B}}{B} = m\frac{dv_\parallel}{dt} = q(\vec{v}\times\vec{B})\cdot\frac{\vec{B}}{B} = 0 \Rightarrow v_\parallel = \text{const}$

→ $v^2 = v_\perp^2 + v_\parallel^2 \Rightarrow v_\perp = \text{const}$

→ $\frac{d\vec{v}_\perp}{dt} \perp \vec{v}_\perp$, $v_\perp = \text{const} \Rightarrow$ uniform circular motion $\frac{m v_\perp^2}{\rho} = |q| v_\perp B \Rightarrow$

$$\begin{cases} \boxed{\rho = \frac{m v_\perp}{|q|B}} & \text{larmor radius or cyclotron radius} \\ \boxed{\omega_c = \frac{v_\perp}{\rho} = \frac{|q|B}{m}} & \text{cyclotron frequency} \end{cases}$$

$\Rightarrow$



Helical Trajectory $\left(\begin{array}{l} q>0: \text{ left-hand rotation with respect to } \vec{B} \\ q<0: \text{ right-hand rotation with respect to } \vec{B} \end{array} \right)$

Istenteneous center of rotation: guiding center - In this simple case, it moves with uniform motion along $\vec{B}$.

Plasma

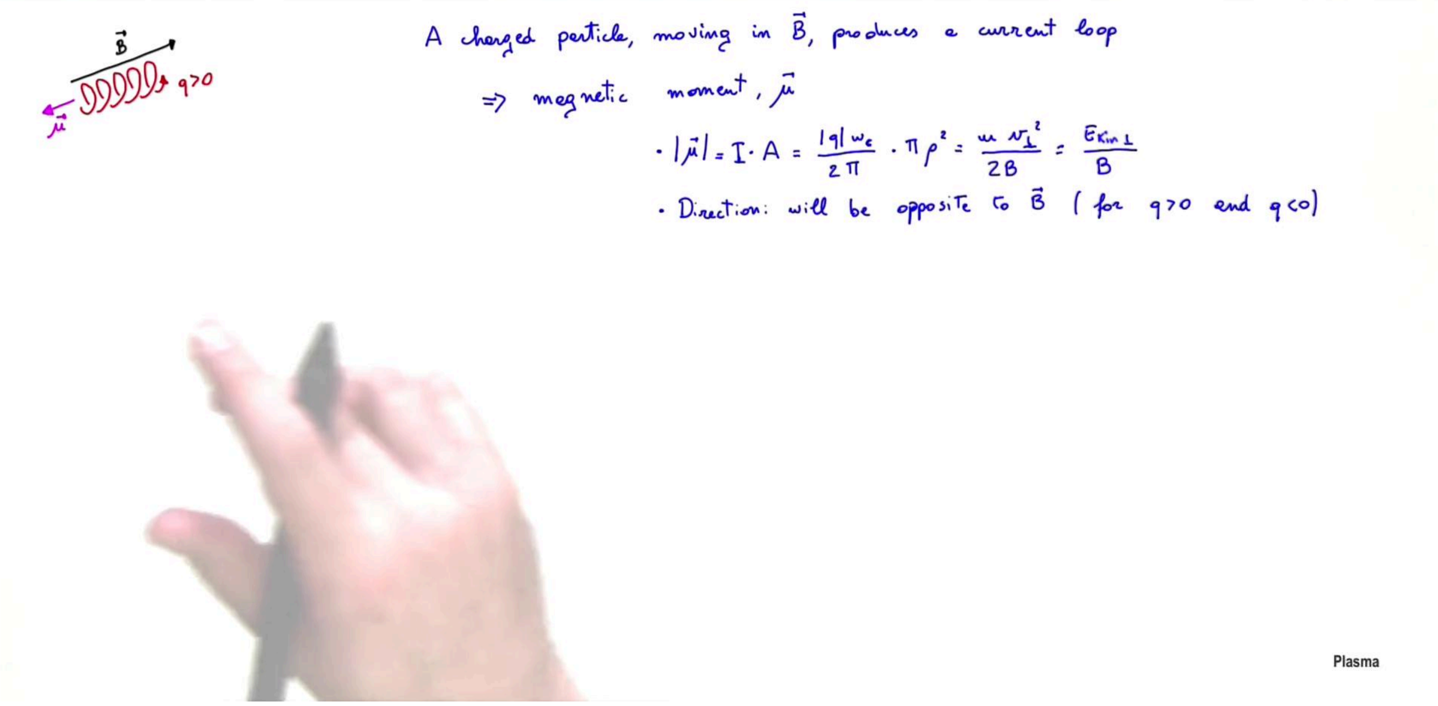
These are two fundamental quantities in plasma physics. What is the consequence of all these observations? Well, that a charged particle, in the presence of a magnetic field, -a uniform magnetic field- moves with a trajectory that is a helical. Basically it's motion in the direction parallel the magnetic field lines is not affected, while in the perpendicular direction it goes through a uniform circular motion. Okay what I have drawn here is the case of a positive charge, and in fact, for q>0 we will have a left-handed rotation with respect to B while if q<0, we will have a right-handed rotation with respect to B. In this particle motion we can observe that there is an instantaneous center of rotation that moves in this simple case, in the direction parallel to the magnetic field, that in this simple case moves with the uniform motion along B.

Notes

Summary

6m 26s

Plasma magnetic properties and magnetic moment



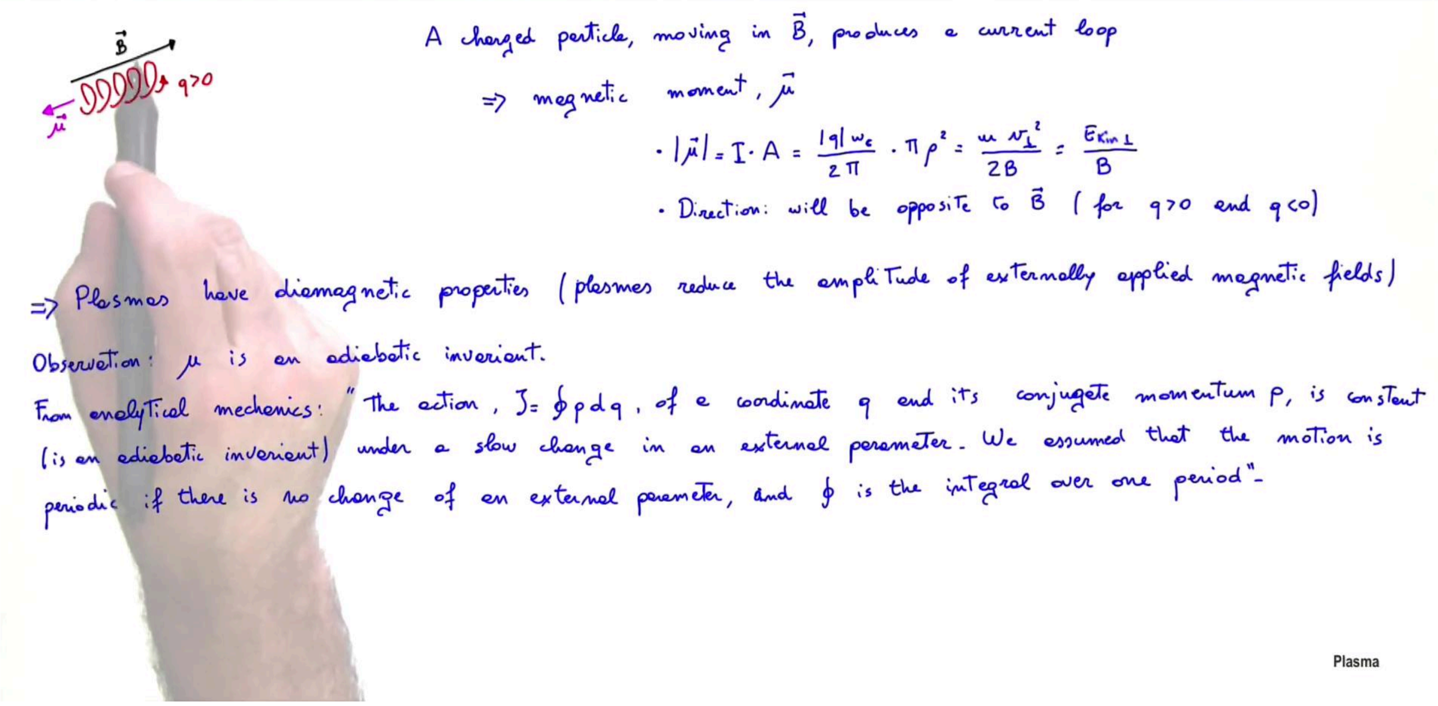
Just by looking at this simple case, the motion of this charged particle in the magnetic field, we can infer the properties, the magnetic properties of a plasma. So as we've seen in the presence of a magnetic field, - this is case of a positively charged particle - and now a charged particle therefore produces a current loop. As we know, there is a magnetic moment [μ] associated with a magnetic loop. What are the properties of this vector magnetic moment [μ] ? Its intensity is given by the current times the area of the loop. In the present case, the current will be given by the charge times the cyclotron frequency divided by 2π, multiplied by the area which is π [times] ρ². If we substitute the correct expressions for ωc and for ρ we obtain that this quantity will be equal to the mass times the perpendicular velocity squared divided by 2B, which is the perpendicular kinetic energy divided by B. The direction will be opposite to B. Both for q>0 and q<0. In other words, this magnetic moment vector will be something directed like that. So what this is consequence of all this ? That actually, because of the motion of both negative and positive particles, the magnetic field imposed - applied-externally will be reduced by the [perpendicular] motion of the charged particles in a plasma.

Notes

Summary

8m 09s

Plasma magnetic properties and magnetic moment



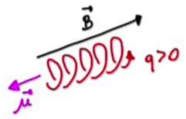
In other words, plasmas have *diamagnetic* properties. In other words, the plasma tends to reduce the amplitude of externally applied magnetic fields. Let me make an observation about the magnetic moment. In fact, the magnetic moment is an adiabatic invariant. What do I mean with this? Let me just remind a result that is given by analytical mechanics. In fact, the analytical mechanics books tell us that the action, let's call it J that is given by the integral of p dq of a coordinate q and it's conjugate momentum p, well, J -this action- is constant, or better, we will say it's an *adiabatic invariant* under a slow change of an external parameter. This is valid under the assumption that the motion is periodic if there is no change of an external parameter and we have indicated with this integral, the integral over one period. What does it mean? Let's suppose that we have a motion of a particle that is periodic, for example, the motion of this particle in the presence of a magnetic field. Then we evaluate this integral, and we find that this integral is constant, even when you change slowly an external parameter that could be, for example, the magnetic field.

Notes

Summary

10m 38s

Plasma magnetic properties and magnetic moment



A charged particle, moving in $\vec{B}$, produces a current loop

$\Rightarrow$ magnetic moment, $\vec{\mu}$

- $|\vec{\mu}| = I \cdot A = \frac{|q|\,\omega_c}{2\pi} \cdot \pi \rho^2 = \frac{m v_\perp^2}{2B} = \frac{E_{kin\perp}}{B}$
- Direction: will be opposite to $\vec{B}$ (for $q>0$ and $q<0$)

$\Rightarrow$ Plasmas have diamagnetic properties (plasmas reduce the amplitude of externally applied magnetic fields)

Observation: μ is an adiabatic invariant.

From analytical mechanics: "The action, $J = \oint p\,dq$, of a coordinate q and its conjugate momentum p, is constant (is an adiabatic invariant) under a slow change in an external parameter. We assumed that the motion is periodic if there is no change of an external parameter, and $\oint$ is the integral over one period".

In the case of μ, $q = x$ (direction $\perp$ to $\vec{B}$), $p = m v_x$

$$J = \oint m v_x\,dx = \oint m v_x \frac{dx}{dt}\,dt = \oint m v_x^2\,dt = \int_0^{\frac{2\pi}{\omega_c}} m v_\perp^2 \sin^2(\omega_c t + \varphi)\,dt = \frac{\pi m v_\perp^2}{\omega_c}$$

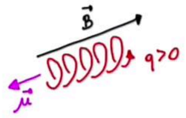
Plasma

Even when you change this external parameter slowly, then what you find is that although the trajectory of the particles can change because the magnetic field is changing, then this parameter [the action] will remain constant. What we say is that that is an adiabatic invariant in the sense that it will be constant if the change of the external parameter is slow in comparison to the period, to the frequency of the motion. Okay, in the case of the magnetic moment, let's see how we can apply this theorem, that the analytical mechanics gives us. We can take q equal to x and we will take one direction perpendicular to B and the conjugate moment will be given by m vx. Now if we evaluate the action of this coordinate x and it's conjugate moment m vx, what do we have? This will be given by m vx, this expression here, we can divide and multiply by dt which is therefore equal to m vx^2 dt, and the integral over a gyro-period can be written as the integral from time zero to time 2π / ωc and then we can write vx^2 as $v\perp^2$... and then with the oscillating component ωc t plus a gyro-phase [φ] this will need to be integrated over time, and by doing this integral, what do we obtain? It's (π m $v\perp^2$) / ωc.

Notes

Summary

13m 05s



A charged particle, moving in $\vec{B}$, produces a current loop

⇒ magnetic moment, $\vec{\mu}$

- $|\vec{\mu}| = I \cdot A = \frac{|q|\omega_c}{2\pi} \cdot \pi \rho^2 = \frac{m v_\perp^2}{2B} = \frac{E_{kin\perp}}{B}$
- Direction: will be opposite to $\vec{B}$ (for $q>0$ and $q<0$)

⇒ Plasmas have diamagnetic properties (plasmas reduce the amplitude of externally applied magnetic fields)

Observation: μ is an adiabatic invariant.

From analytical mechanics: "The action, $J = \oint p\, dq$, of a coordinate q and its conjugate momentum p, is constant (is an adiabatic invariant) under a slow change in an external parameter. We assumed that the motion is periodic if there is no change of an external parameter, and $\oint$ is the integral over one period".

In the case of μ, $q = x$ (direction $\perp$ to $\vec{B}$), $p = m v_x$

$$J = \oint m v_x\, dx = \oint m v_x \frac{dx}{dt} dt = \oint m v_x^2\, dt = \int_0^{\frac{2\pi}{\omega_c}} m v_\perp^2 \sin^2(\omega_c t + \varphi)\, dt = \frac{\pi m v_\perp^2}{\omega_c} = \frac{2\pi m}{|q|}\left(\frac{m v_\perp^2}{2B}\right) = \frac{2\pi m}{|q|}\mu$$

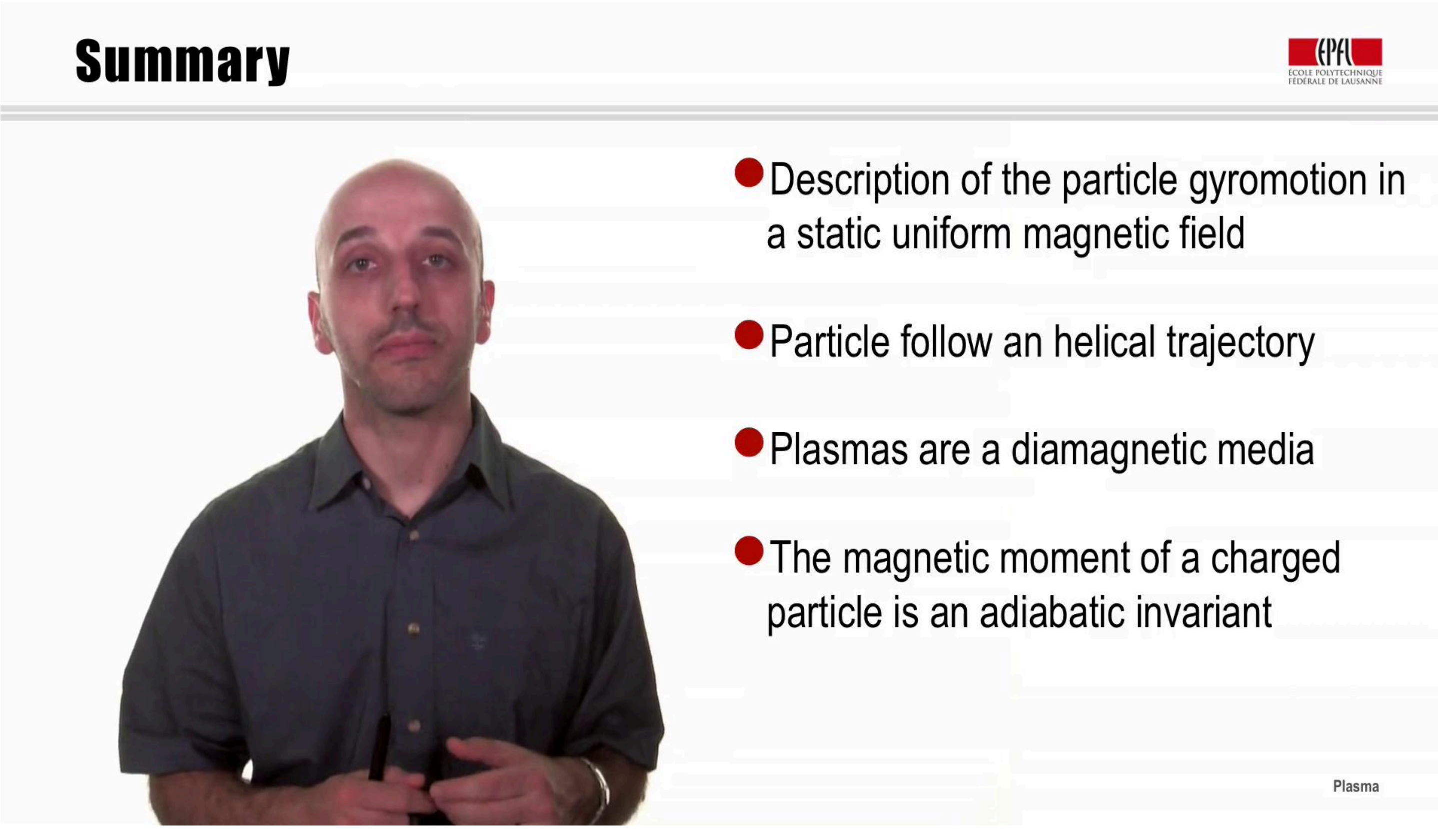
Plasma

Which is 2π m / |q| by expliciting the cytron frequency, and this is equal to 2π m divided by the absolute value of the charge times μ, which is exactly the magnetic moment besides a constant factor. Therefore, what do we learn from this theorem of analytical mechanics? That the magnetic moment remains constant. This quantity remains constant here, even if the magnetic field can vary in time, it can vary in space, then if this variation is slow with respect to cyclotron period, and despite the fact that the trajectory of the particle may change, this quantity will remain constant.

Notes

Summary

15m 33s



In the present module, we have focused our attention on the trajectory of a particle in a fairly simple scenario. The one of a constant uniform magnetic field. We've pointed out that in this case the trajectory of the particle is helical. And by looking at the trajectory of this particle, we've pointed out what are the magnetic properties of a plasma. In particular we have shown that the plasmas are diamagnetic media and we also talked about the magnetic moments. We've shown that magnetic moments are actually adiabatic invariants of the particle motion. In the next module, we will consider more complicated electric and magnetic fields. We will look at what is the trajectory of the charge particles in this.

Notes

Summary

16m 43s