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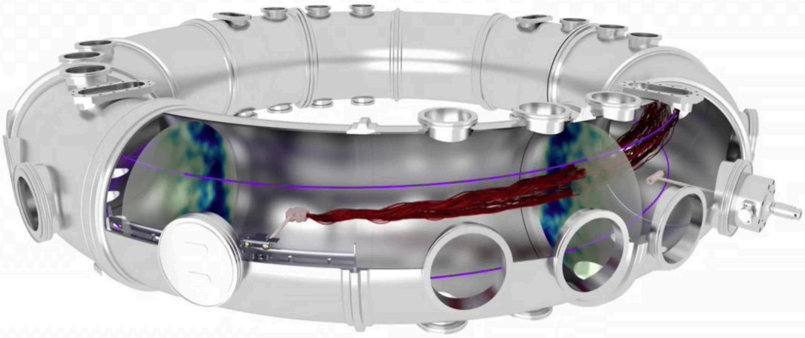
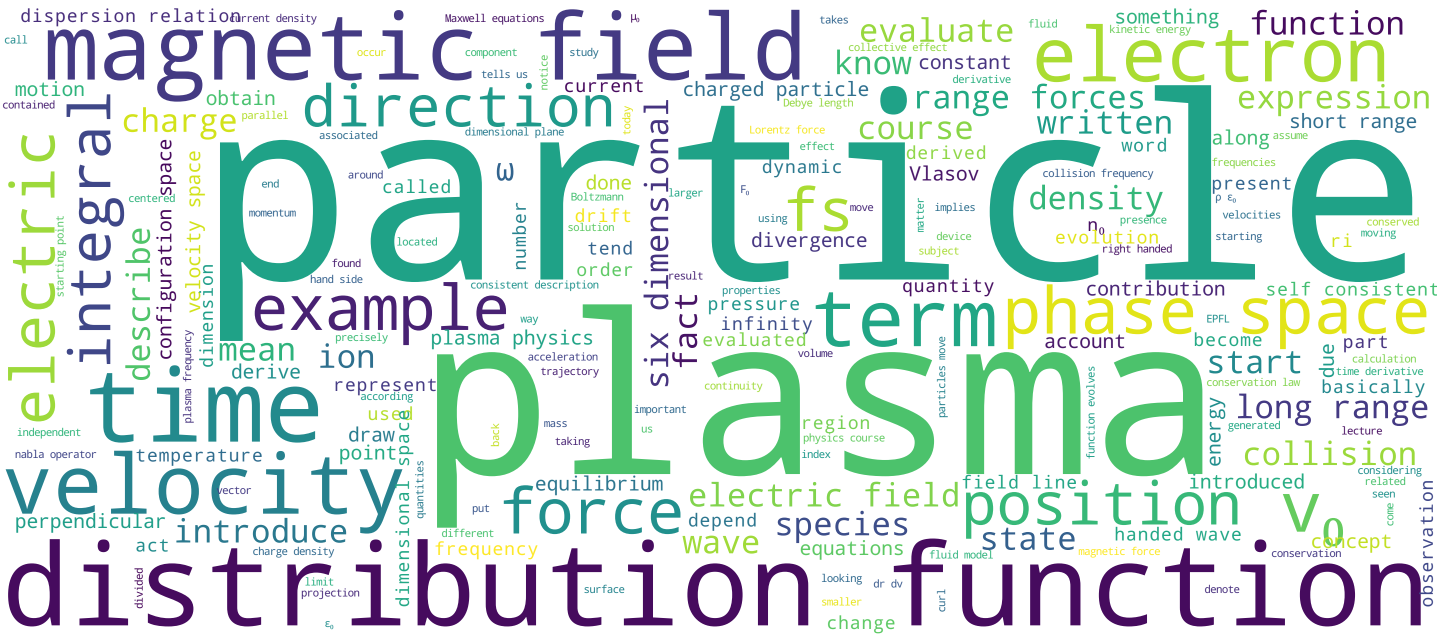


From single particle to kinetic description

Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

Lecture 2a

Paolo Ricci



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Introduction

- Self-consistent description of a plasma
- The distribution function
- Examples of distribution functions
- Conservation in phase space
- The Boltzmann equation

Plasma

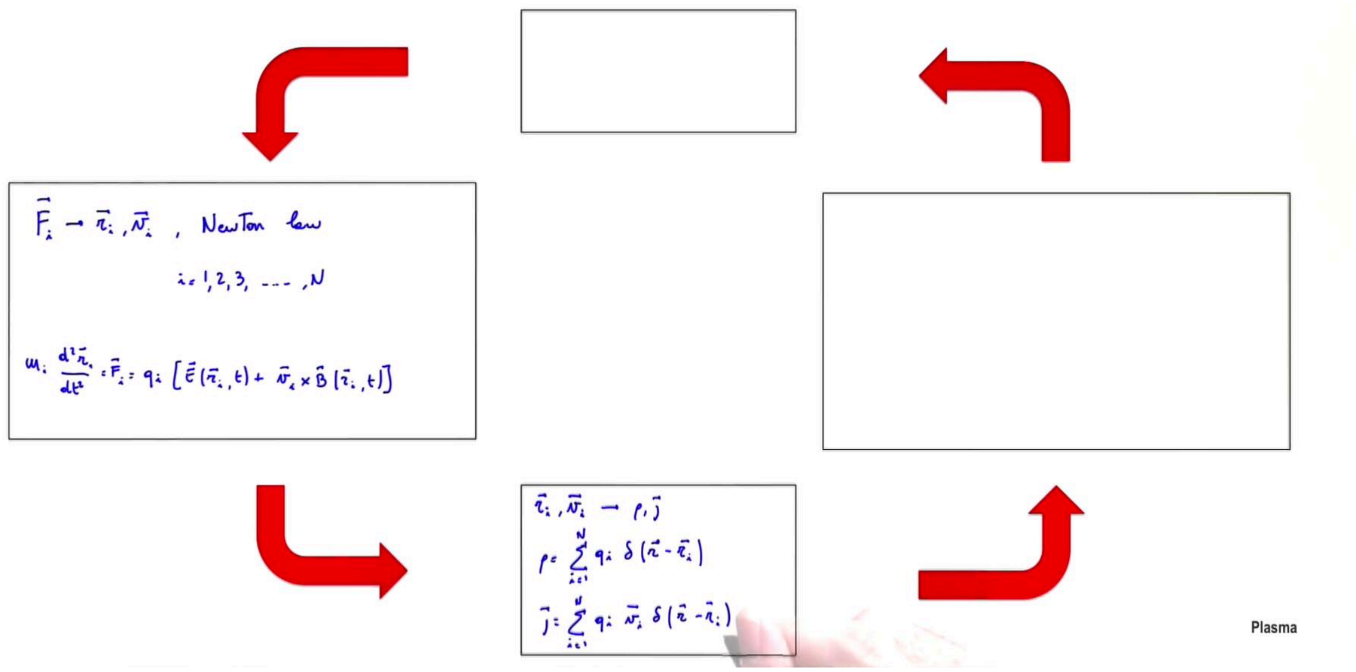
Welcome — welcome to the plasma physics course of EPFL. My name is Paolo Ricci. I am a professor in theoretical plasma phsyics here at EPFL, and today I would like to introduce the concept of kinetic description of a plasma. Throughout the first part of the course, we have studied how charged particles move in given electric and magnetic fields. Today it's time to look at considering a more self-consistent description of a plasma, taking into account that these charged particles move under the effects of the electric and magnetic fields that they have themselves generated. So, as a first step in today's lecture, we will start from a self-consistent description of a plasma — the most complete description we can come up with. Then we will introduce the concept of distribution functions and give some examples, and then starting from the fact that particles are conserved, we will derive an equation, the Boltzmann equation, which actually states how the particle distribution function evolves over time.

Notes

Summary

0m 05s

Self-consistent description of a plasma



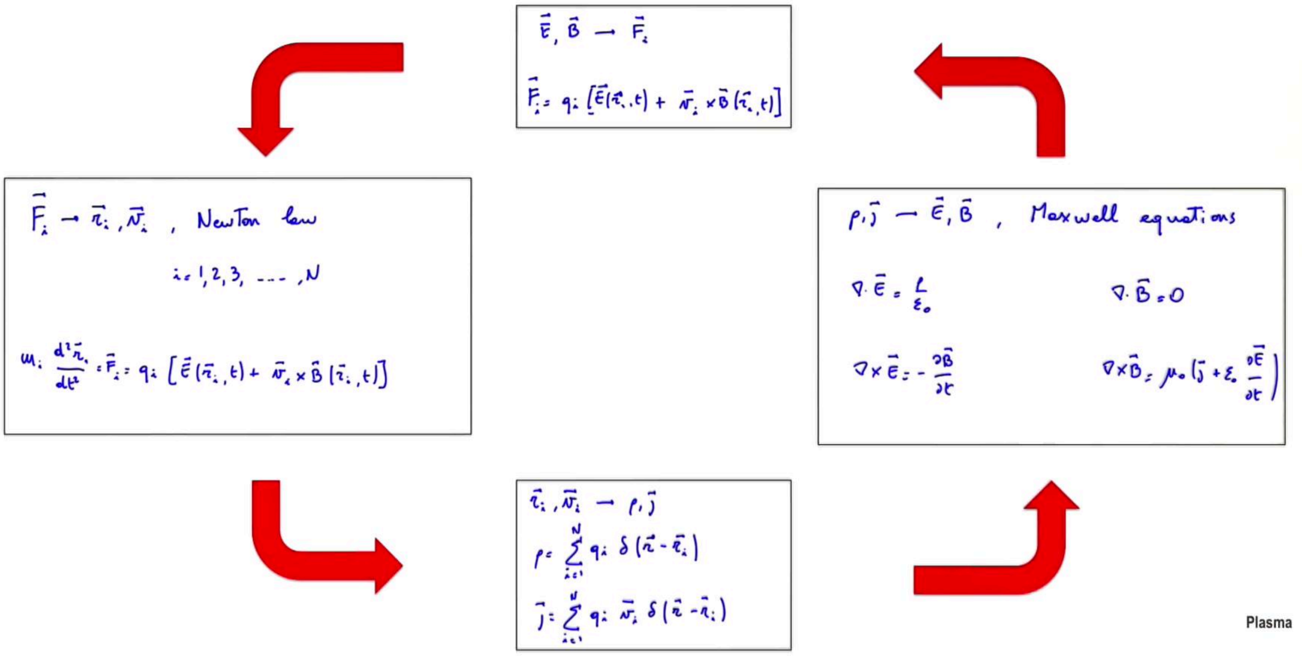
The self-consistent description of a plasma starts by noticing that once we know the forces that act on each particle, we can know the position and velocity of all particles by using Newton's law, for all particles from one, two, three, to the last one, N. More precisely, what we want to do is to consider the mass times the acceleration equal to the forces that are acting on each particle. As we are dealing with electrically charged particles will be given by qi { E (ri, t) + vi × B (ri, t) } Once we know the position and the velocity of all particles we can evaluate the charge density and the current density. In fact, the [charge] density will be equal to the sum over all particles, of their charge [qi] times a delta function [δ(r-ri)] that takes into account that the particles are located at the position ri, and therefore the contribution of the charge will be equal to zero everywhere else but at their own position. Similarly, for the current density we will have that this will be equal to the sum -over all particles-, [of the] charge of all particles, then their velocity [vi] times the delta function that takes into account that particles are located ri.

Notes

Summary

1m 26s

Self-consistent description of a plasma



Once we know the charge density and the current density, we can evaluate the electric and magnetic fields. This can be done by using Maxwell's equations, which state that the divergence of E is equal to ρ/ε_0, [$\nabla \cdot E = \rho/\varepsilon_0$] the curl of E is equal to $-\partial B/\partial t$. [$\nabla \times E = -\partial B/\partial t$] Divergence of B is equal to zero [$\nabla \cdot B = 0$] and the curl of B is equal to $\mu_0 (j + \varepsilon_0\, \partial E/\partial t)$. [$\nabla \times B = \mu_0 (j + \varepsilon_0\, \partial E/\partial t)$] Once we have evaluated the electric and magnetic fields by solving Maxwell's equations, we can actually evaluate the force that acts on each particle. In fact, the force that acts on each particle will be given by the electric force with the electric field evaluated at ri plus the Lorentz force. The system of equations that I have just described actually is able to represent carefully and in the most complete way the dynamics of the plasma. It's able to describe the evolution and the motion of all the particles that constitute our system in the electric and magnetic fields that are self-consistently generated by their presence.

Notes

Summary

3m 17s

The distribution function

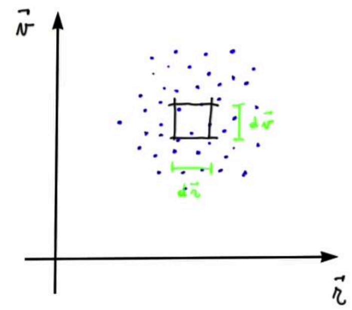
The description that we've given of a plasma is very complete, very accurate, however it's not practical. It wants to describe the evolution of a huge number of particles. Just to give you an example, there are about 10^{21} particles in a fusion experiment. It's way too many to deal both numerically and analytically. We need a different approach, a statistical approach. This is why we are going to introduce the distribution function, which is a function that tells us how particles are distributed, -are located- in phase space. Phase space is a six-dimensional space: 3 dimensions in velocity, 3 dimensions in configuration space. It's something that I cannot represent on the screen here, on the two-dimensional screen. Therefore I will do a projection of the six-dimensional plane on the two-dimensional plane that we can look at. Here is the projection of this six-dimensional plane for what concerns the configuration space position and for what concerns the velocity position. Particles are distributed in this phase space and I will draw a few of them. They are these points that I am representing here.

Notes

Summary

5m 07s

The distribution function



$f(\vec{r}, \vec{v}, t)\, d\vec{r}\, d\vec{v}$ = # of particles at time t, in the phase space volume $d\vec{r}\, d\vec{v}$ centered in $\vec{r}$ and $\vec{v}$

For each species, we have $f_s(\vec{r}, \vec{v}, t)$

Some interesting quantities:

- Total number of particles

$$N_s = \int d\vec{r} \int d\vec{v}\, f_s(\vec{r}, \vec{v}, t)$$

- number density

Plasma

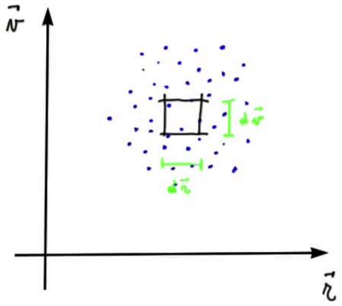
Let's consider a volume — a volume that is large from a microscopic point of view, so that many particles are contained in it, but which is still small by looking at it macroscopically. This volume will have an extension that is given by dv along the v direction and dr along the r direction. What we do: we define a function, the distribution function, f(r,v,t) which is a function of position, velocity and time, such that when the value of this function, is multiplied by dr dv, what we obtain is the number of particles at time t that are contained the phase space volume, dr dv, centered in r and v. Actually there is a distribution function for each species and therefore, for each species we will have a distribution function that we will call fs(r,v,t) -s is the index of the species- that will depend, of course, on position, v, and t. Some interesting quantities: first the total number of particles for species s that we'll call Ns, and this will be the integral over all positions, over all velocities of the distribution function. [Ns = ∫ dr ∫ dv fs(r,v,t)] Anotherquantity of interest: the number density, which represents the density of particles in configuration space independently of their distribution in velocity space.

Notes

Summary

6m 39s

The distribution function



$f(\vec{r}, \vec{v}, t)\, d\vec{r}\, d\vec{v}$ = # of particles at time t, in the phase space volume $d\vec{r}\, d\vec{v}$ centered in $\vec{r}$ and $\vec{v}$

For each species, we have $f_s(\vec{r}, \vec{v}, t)$

Some interesting quantities:

- Total number of particles $\quad N_s = \int d\vec{r} \int d\vec{v}\, f_s(\vec{r}, \vec{v}, t)$
- number density $\quad n_s(\vec{r}, t) = \int d\vec{v}\, f_s(\vec{r}, \vec{v}, t)$
- average velocity $\quad \vec{u}_s(\vec{r}, t) = \frac{1}{n_s} \int d\vec{v}\, \vec{v}\, f_s(\vec{r}, \vec{v}, t)$

Plasma

This will be a function of position [and] time and will be given by the integral over all velocities of the distribution function [ns = ∫ dv fs(r,v,t)] and finally the average velocity which is a function of position and which is given by the average of the velocity of all particles, weighted according to their distribution function. [us = 1/ ns ∫ dv v fs(r,v,t)] We have introduced the concept of distribution function.

Notes

Summary

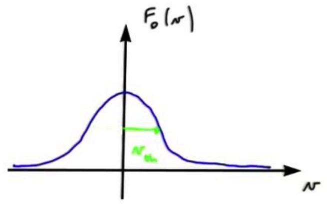
8m 46s

Examples of distribution functions

- Maxwell-Boltzmann distribution function

$$F_0(\vec{v}) = n_0 \left(\frac{1}{2\pi v_{th}} \right)^{3/2} \exp\left(-\frac{v^2}{2 v_{th}^2} \right)$$

in 1D $\Rightarrow$ $F_0(v) = n_0 \left(\frac{1}{2\pi v_{th}} \right)^{1/2} \exp\left(-\frac{v^2}{2 v_{th}^2} \right)$



- Monoenergetic beam in 1D

$$F_0(v) = n_0\, \delta(v - v_0)$$

Let's give some examples. We are going to consider distribution functions that are independent of the spatial coordinates. We will in other words consider a uniform plasma. We will consider a distribution function that also does not depend on time. Probably the most known example of a distribution function is the Maxwell-Boltzmann distribution function. This is the distribution function of the plasma particles once the thermal equilibrium is attained. It states that particles are distributed according to a Gaussian distribution function. If we specify this distribution function in one dimension, we have that $F_0(v)$ - this time it is not the modulus, it is really the one-dimensional distribution function - is equal to n_0 1/(2 π vth) [vth = v_thermal] - this time to the power 1/2 - exp {-v² / (2 vth²) }. We can draw this distribution function for the one-dimensional case as a function of the velocity and what we have is a curve that peaks at v = 0. The spreading of this curve is given by vth. Another interesting distribution function is a monoenergetic beam in one dimension and this will be given by $F_0(v)$ = n_0 δ(v-v_0), [where] v_0 is the velocity of the beam. Basically all particles have a velocity that is equal to v_0 and there are no other particles that have a velocity different from v_0.

Notes

Summary

9m 25s

Examples of distribution functions

- Maxwell-Boltzmann distribution function

$$F_0(\vec{v}) = n_0 \left(\frac{1}{2\pi v_{th}}\right)^{3/2} \exp\left(-\frac{v^2}{2 v_{th}^2}\right)$$

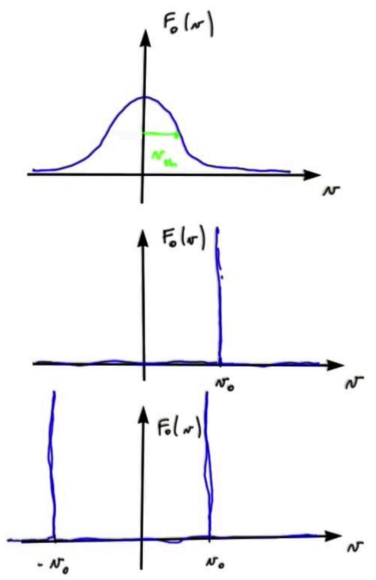
in 1D $\Rightarrow$ $F_0(v) = n_0 \left(\frac{1}{2\pi v_{th}}\right)^{1/2} \exp\left(-\frac{v^2}{2 v_{th}^2}\right)$

- Monoenergetic beam in 1D

$$F_0(v) = n_0\, \delta(v - v_0)$$

- Two counterstreaming beams in 1D

$$F_0(v) = \frac{n_0}{2}\left[\delta(v - v_0) + \delta(v + v_0)\right]$$



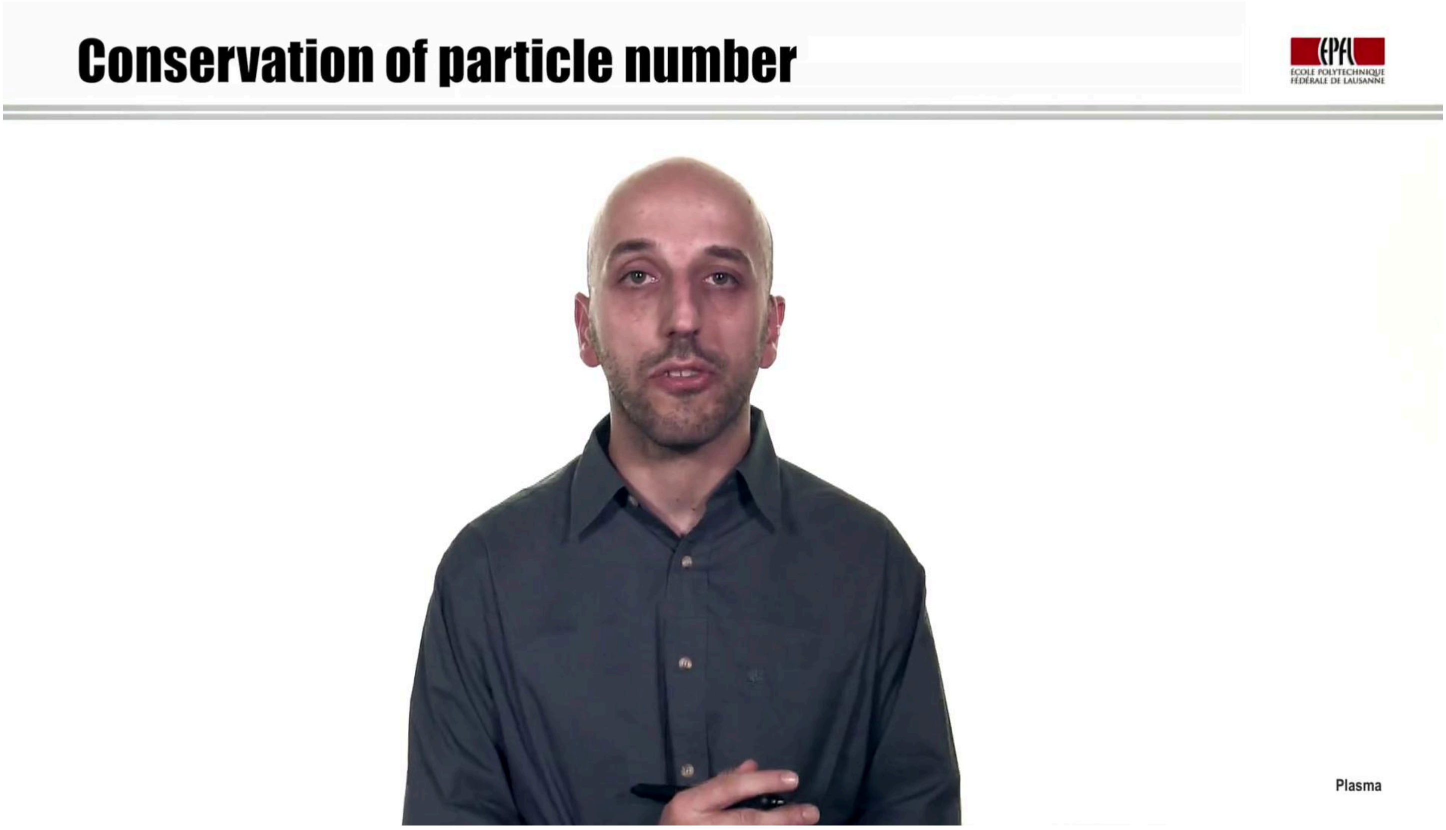
Plasma

If we try to draw this function, it's very difficult to draw a Dirac distribution, but at least intuitively we can give an idea of how this distribution function looks. In fact, the distribution function will be equal to zero except at $v = v_0$. It will be something very close to zero,... ... and then around $v = v_0$ it will go up to infinity, - of course I cannot go out from the screen-, then it will fall back to zero. The last example of distribution function that I would like to give is two counter-streaming beams in one dimension. In that case, the distribution function is given by the sum of two Dirac functions, one centered at v_0, and one centered at $-v_0$, for example. The distribution function in this case will be equal to zero everywhere except at $-v_0$ and at $+v_0$. Therefore equal to zero up to $-v_0$, where it will go up to infinity— of course here I cannot represent that, then go down back to zero, up to $v = v_0$, where I will go up to infinity again, and then it will come back [to zero].

Notes

Summary

11m 36s

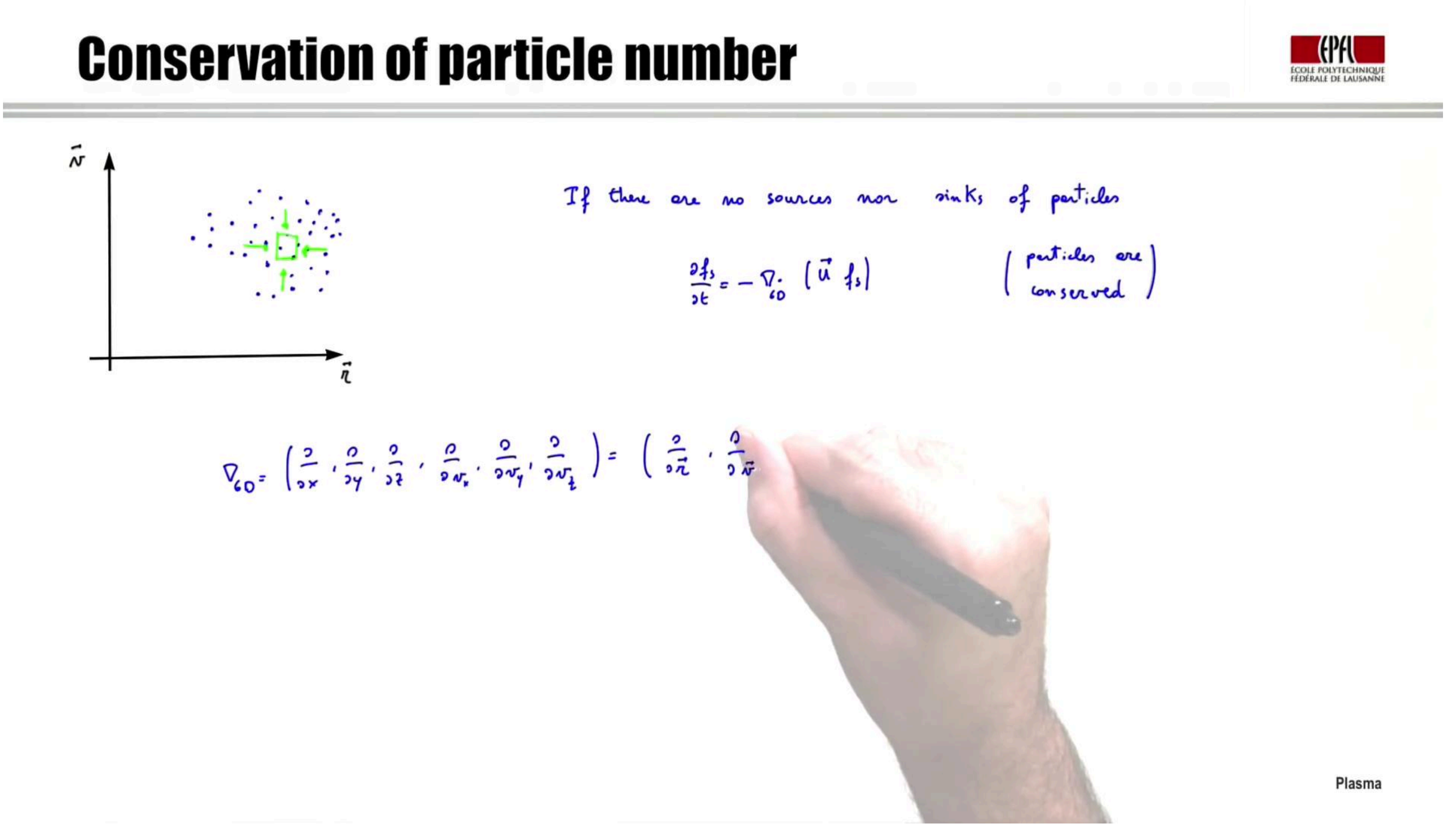


Now that we have introduced the concept of distribution function and we have given a few examples of distribution functions, we can start to derive an equation, an equation that shows how the distribution function evolves as a function of time. To do that, the starting point is noticing, observing that particles are not created nor destroyed in phase space. So the starting point of this equation will be the observation that particles are conserved in phase space. Let's try to translate it into mathematical expressions.

Notes

Summary

13m 14s



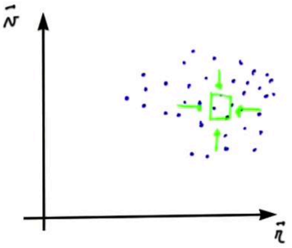
Let's start from our phase space — the six-dimensional phase space that here I have to project over two dimensions, and where the particles are located. If there are no sources, nor sinks of particles, then if we look at a small volume present in our system, what we will observe is that the variation of the particles contained in this small volume will be due to the flux of particles inflowing or outflowing from this volume. In other words, it will be given by minus the divergence, divergence that has to be evaluated in a six-dimensional space [$\nabla_6 \cdot$], the six-dimensional phase space — of u, which is the velocity at which particles are moving in the phase space, times fs. This is something that tells us that particles are conserved. They are not created nor destroyed. What do I mean with this nabla in six dimensions? [∇_6] This is the generalization of the nabla operator that we know from three-dimensional space, extended to velocity space. So this will be equal to $\nabla_6 \cdot = (\partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial vx, \partial/\partial vy, \partial/\partial vz)$ which is also equal, and we will shorten this as the nabla operator in the configuration space that I will write as ($\partial/\partial \mathbf{r}$) and the nabla operator in velocity space, that I will write as ($\partial/\partial \mathbf{v}$).

Notes

Summary

13m 51s

Conservation of particle number



If there are no sources nor sinks of particles

$$\frac{\partial f_s}{\partial t} = -\nabla_{6D}\cdot\left(\vec u\, f_s\right) \qquad \left(\text{particles are conserved}\right)$$

$$\nabla_{6D} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right) = \left(\frac{\partial}{\partial \vec r}, \frac{\partial}{\partial \vec v}\right)$$

$$\vec u = \left(\frac{d\vec r}{dt}, \frac{d\vec v}{dt}\right) = \left(\vec v, \frac{\vec F}{m_s}\right) = \left(\vec v, \frac{\vec F^{l.r.} + \vec F^{s.r.}}{m_s}\right)$$

l.r. → long range

s.r. → short range

$$\frac{\partial f_s}{\partial t} = -\frac{\partial}{\partial \vec r}\cdot\left(\vec v f_s\right) - \frac{\partial}{\partial \vec v}\cdot\left[\left(\frac{\vec F^{l.r.} + \vec F^{s.r.}}{m_s}\right) f_s\right]$$

Plasma

Now **u** is the derivative of **r** with respect to time and of **v** with respect to time. Particles change their position in time and this change is equal to the velocity and they change their velocity according to their acceleration, which is given by the force divided by the mass of the particles according to Newton's Law. This in other terms is equal to **v**, and then we have to divide the forces in two. As we know from the first part of the course, there will be long-range forces [F^{lr}] that will act on our system plus short-range forces [F^{sr}] The long-range forces are the forces due to collective effects that occur to a global evolution of our system, while the short-range forces are due to collisions, basically one-to-one particle interactions. Just to be clear, [the superscript] *l.r.* will stand for *long-range* forces, the ones due to collective effects, while with *s.r.* we will denote *short-range* forces. If we introduce this expression in the continuity equation in the six-dimensional space, what we obtain is that ∂fs/∂t is equal to the part of the divergence that is the one in configuration space, so it's -∂/∂**r** · (**v** fs) minus ∂/∂**v**... of the contribution of both long-range and short-range forces. Here we have translated this six-dimensional operator in an operator in real space and in velocity space.

Notes

Summary

15m 55s

$$\frac{\partial f_s}{\partial t} = -\frac{\partial}{\partial \vec{r}} \cdot (\vec{v} f_s) - \frac{\partial}{\partial \vec{v}} \cdot \left[\left(\frac{\vec{F}^{l.r.} + \vec{F}^{s.r.}}{m_s} \right) f_s \right]$$

Observations

- $\vec{v}$ is independent of $\vec{r}$, $\frac{\partial}{\partial \vec{r}} \cdot (\vec{v} f_s) = \vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}}$
- $\vec{F}^{l.r.} = q_s (\vec{E}^{l.r.} + \vec{v} \times \vec{B}^{l.r.})$ $\Rightarrow$ $\frac{\partial}{\partial \vec{v}} \cdot (\vec{F}^{l.r.} f_s) = \vec{F}^{l.r.} \cdot \frac{\partial f_s}{\partial \vec{v}}$

independent of $\vec{v}$

$\vec{v} \times \vec{B}^{l.r.} \perp \vec{v}$

Plasma

The conservation law that we have just derived can be put in a more usual form — the so-called Boltzmann equation, which is the equation that can be used to describe how the distribution function evolves as a function of time. The starting point is the conservation law that we have just written. Let's make a few observations. The first thing that we have to notice is that as v is independent of r, as **v** and **r** are independent phase-space variables, then we will have that ∂/∂**r** · (**v** fs) is equal to **v** · ∂fs/∂**r**. The second thing that we notice is that the forces that we have written are actually electric and magnetic forces. So F long-range [F^{lr}], for example, will be equal to the charge of the particles multiplied by the long-range electric field [E^{lr}] plus the Lorentz force [$\mathbf{v} \times \mathbf{B}^{lr}$] Now the electric field is independent of **v**. The Lorentz force is perpendicular to **v**. This implies that the divergence with respect to **v** of ($\mathbf{F}^{lr}$ fs), is equal to the force long-range dot ∂fs/∂**v** [$\mathbf{F}^{lr} \cdot \partial fs/\partial \mathbf{v}$] What does it mean?

Notes

Summary

18m 22s

The Boltzmann equation

$$\frac{\partial f_s}{\partial t} = -\frac{\partial}{\partial \vec{r}}\cdot\left(\vec{v} f_s\right) - \frac{\partial}{\partial \vec{v}}\cdot\left[\left(\frac{\vec{F}^{l.r.}+\vec{F}^{s.r.}}{m_s}\right) f_s\right]$$

Observations

- $\vec{v}$ is independent of $\vec{r}$, $\frac{\partial}{\partial \vec{r}}\cdot(\vec{v} f_s) = \vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}}$
- $\vec{F}^{l.r.} = q_s\left(\vec{E}^{l.r.} + \vec{v}\times\vec{B}^{l.r.}\right)$

independent of $\vec{v}$ $\qquad$ $\vec{v}\times\vec{B}^{l.r.} \perp \vec{v}$ $\quad \Rightarrow \quad \frac{\partial}{\partial \vec{v}}\cdot\left(\vec{F}^{l.r.} f_s\right) = \vec{F}^{l.r.}\cdot\frac{\partial f_s}{\partial \vec{v}}$

$$\Rightarrow \quad \frac{\partial f_s}{\partial t} = -\vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}} - \frac{\vec{F}^{l.r.}}{m_s}\cdot\frac{\partial f_s}{\partial \vec{v}} - \frac{\partial}{\partial \vec{v}}\cdot\left(\frac{\vec{F}^{s.r.}}{m_s} f_s\right)$$

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}} + \frac{\vec{F}^{l.r.}}{m_s}\cdot\frac{\partial f_s}{\partial \vec{v}} = \underbrace{-\frac{\partial}{\partial \vec{v}}\cdot\left(\frac{\vec{F}^{s.r.}}{m_s} f_s\right)}_{\left(\frac{\partial f}{\partial t}\right)_c} \quad \Rightarrow \quad \boxed{\frac{\partial f_s}{\partial t} + \vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\vec{E}^{l.r.} + \vec{v}\times\vec{B}^{l.r.}\right)\cdot\frac{\partial f_s}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_c}$$

Boltzmann equation

Plasma

If we use this expression in the calculation, in the conservation law that we have written, we have that ∂fs/∂t will be equal to **-v** · ∂fs/∂**r** minus — let's separate the contribution of long-range and short-range forces, so it will be the long-range forces dot ∂fs/∂**v** [$\mathbf{F}^{lr}$/ms · ∂fs/∂**v**] minus the contribution of the short-range forces. We can rewrite these in a more usual form by bringing these terms on the left-hand side and leaving the short-range forces on the right-hand side. This term here is actually a term that is related to collisions. It gives the evolution of the distribution function due to collisions. It is something that is usually called the collision operator and is denoted as (∂f/∂t)c with an index c to say that this is due to collisions. From this we get an equation where I explicitly write the forces as the decomposition of the electric and magnetic force which is known as the *Boltzmann equation*.

Notes

Summary

20m 16s

So here we got to the end of today's lecture. What have we done? We have approached a self-consistent description of a plasma by taking into account that particles move under the electric and magnetic fields that they have themselves generated. Therefore what we've done is to introduce the tools to describe the system of many particles. We have introduced the concept of distribution function, and then we have derived, by starting from the conservation of particles, an equation — the Boltzmann equation — which states how this distribution function evolves in time in the phase space.

Notes

Summary

22m 20s