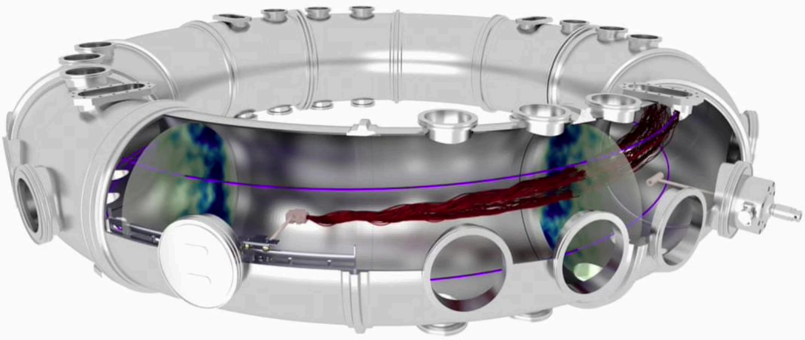
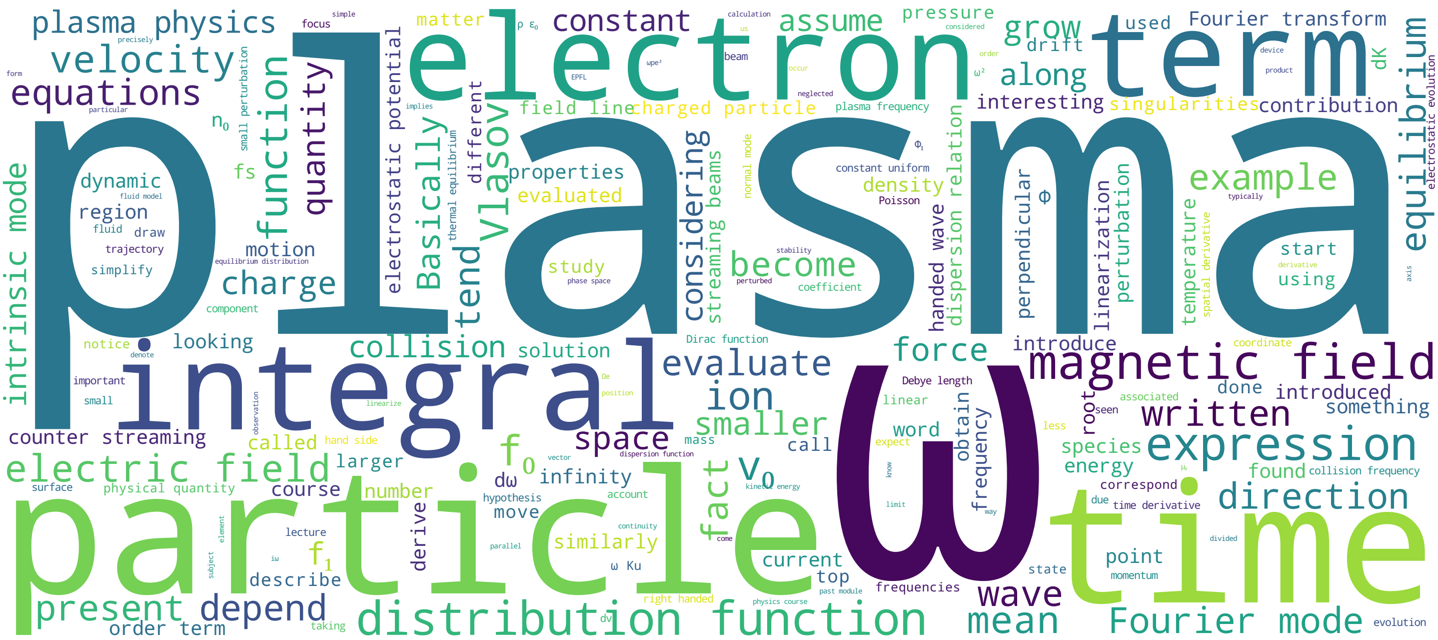


The two-stream instability

Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

Lecture 2e

Paolo Ricci



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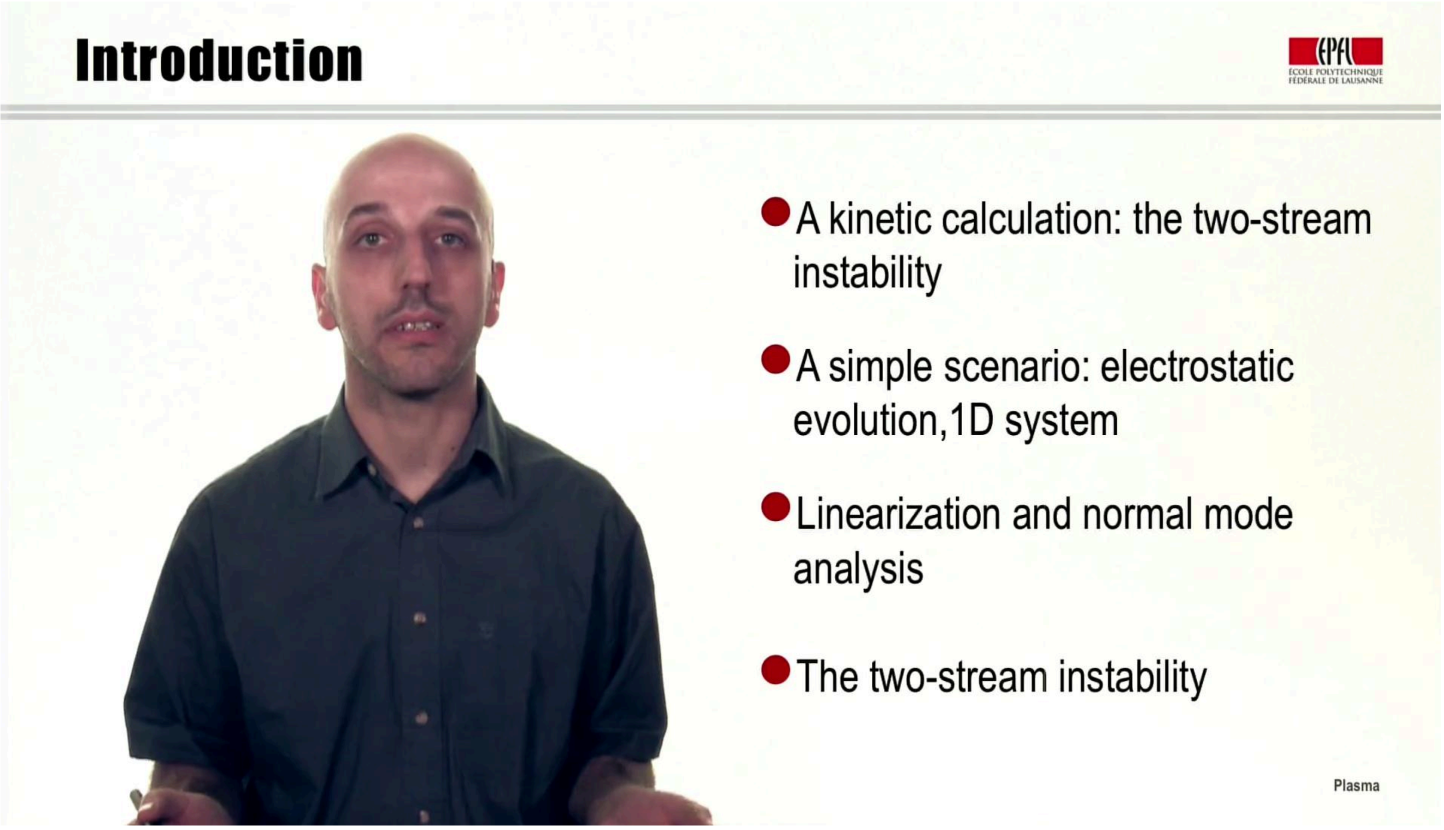


Welcome, welcome to the plasma physics course of EPFL. In the past modules, we have introduced the Vlasov equation. It's now time to see an application. Probably the simplest kinetic calculation that can be done. The one of the stability of two plasma beams that are counter streaming in opposite directions. In order to do this calculation we will first of all simplify the system of equations that we have introduced in the past modules to consider a one-dimensional electrostatic evolution. Then we will introduce some techniques: the one of *linearization* and *normal mode analysis* that will be useful throughout the course. And finally, we will focus on the actual calculation of the stability of these two counter streaming beams.

Notes

Summary

0m 05s

A few simplifications

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\frac{\partial f_s}{\partial \vec{x}} + \frac{q_s}{m_s}\left(\vec{E} + \vec{v}\times\vec{B}\right)\frac{\partial f_s}{\partial \vec{v}} = 0 \qquad s = i, e$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \mu_0\left(\vec{j} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

- Electrostatic evolution, $\vec{B} = 0$

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\frac{\partial f_s}{\partial \vec{x}} + \frac{q_s}{m_s}\vec{E}\cdot\frac{\partial f_s}{\partial \vec{v}} = 0$$

$$\nabla\times\vec{E} = 0, \quad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{E} = -\nabla\phi, \qquad \nabla^2\phi = -\frac{\rho}{\varepsilon_0}$$

- Ions fixed background, density n_0

$$\frac{\partial f_e}{\partial t} + \vec{v}\cdot\frac{\partial f_e}{\partial \vec{x}} - \frac{e\vec{E}}{m_e}\frac{\partial f_e}{\partial \vec{v}} = 0, \quad \nabla^2\phi = \frac{e}{\varepsilon_0}\int f_e\, d\vec{v} - \frac{e}{\varepsilon_0}n_0 \qquad f_e \to f$$

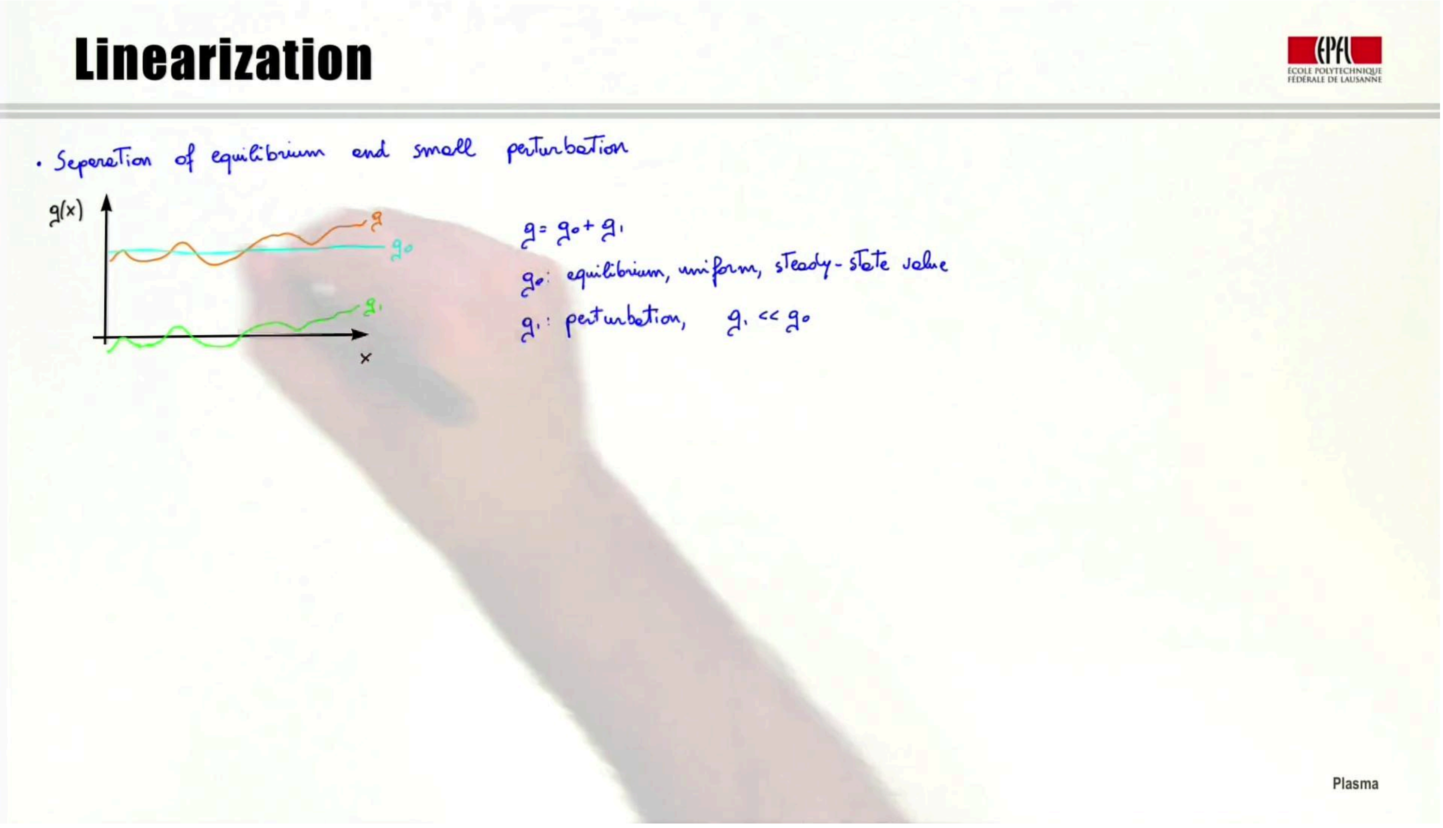
Plasma

Let me first of all recall the system that we are considering. Basically we are considering the Vlasov equation for species s (s = ions and electrons), which has to be coupled to Maxwell's equations: $\nabla\cdot E = \rho/\varepsilon_0$ $\nabla\cdot B = 0$ $\nabla \times E = -\partial B/\partial t$ and $\nabla \times B = \mu_0 (j + \varepsilon_0\, \partial E/\partial t)$ In order to simplify this system we will consider an electrostatic evolution with B = 0. Therefore the Vlasov equation becomes the following one where we have neglected the (v × B) term and the Maxwell system reduces to $\nabla \times E = 0$ and $\nabla\cdot E = \rho/\varepsilon_0$ From which we deduce that $E = -\nabla\Phi$ and $\nabla^2\Phi = -\rho/\varepsilon_0$ [∇^2 = Laplacian square] We will make the further assumption that the ions constitute a fixed the background of constant density n_0. We will have therefore to follow the Vlasov equation for the electron species only with just charge e and mass me, which has to be associated with the Poisson equation for the electrostatic potential, which takes the form of $\nabla^2\Phi = e/\varepsilon_0 \int fe\, dv$ minus the contribution of the ions [$-(e/\varepsilon_0)\, n_0$] As we will consider only the electron species to simplify the notation. Let me replace fe by f. Therefore we will talk about f. We will mean the electron distribution function.

Notes

Summary

1m 06s

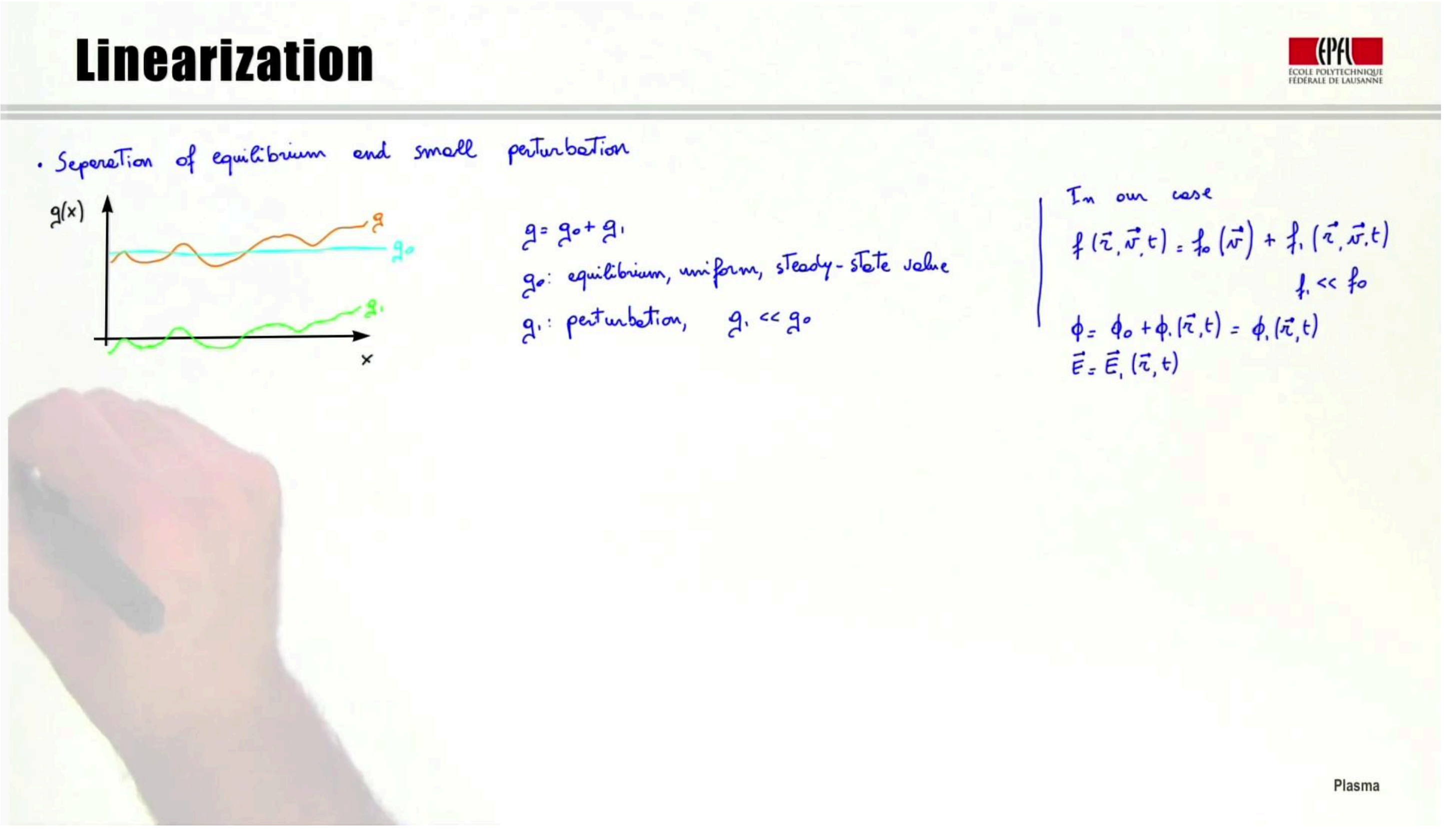


We will now introduce a mathematical technique: the *linearization*, which will be used throughout the rest of the course. What is the main idea? Basically let's assume that we have a physical quantity and this physical quantity can be written as the sum of a constant, uniform background value plus a small perturbation on top of it. We typically are interested in studying the evolution of this small perturbation on top of it and this can be done easily by linearizing the equations. So let's look at the details of what I'm saying here. We want to separate the equilibrium and the small perturbation. Basically, let's assume that we have a physical quantity as a function of a some coordinate x. Physical quantity for example g(x). So, let's consider the g quantity which depends somehow on x. This g quantity can be written as the sum of a constant uniform value plus a perturbation of this uniform background. In other words, g can be written as the sum of $g_0 + g_1$ where g_0 is an equilibrium steady state and uniform value and g_1 the perturbation. As a matter of fact, the linearization, this technique we are talking about is meaningful in the case where g_1 is much much smaller than g_0.

Notes

Summary

3m 24s

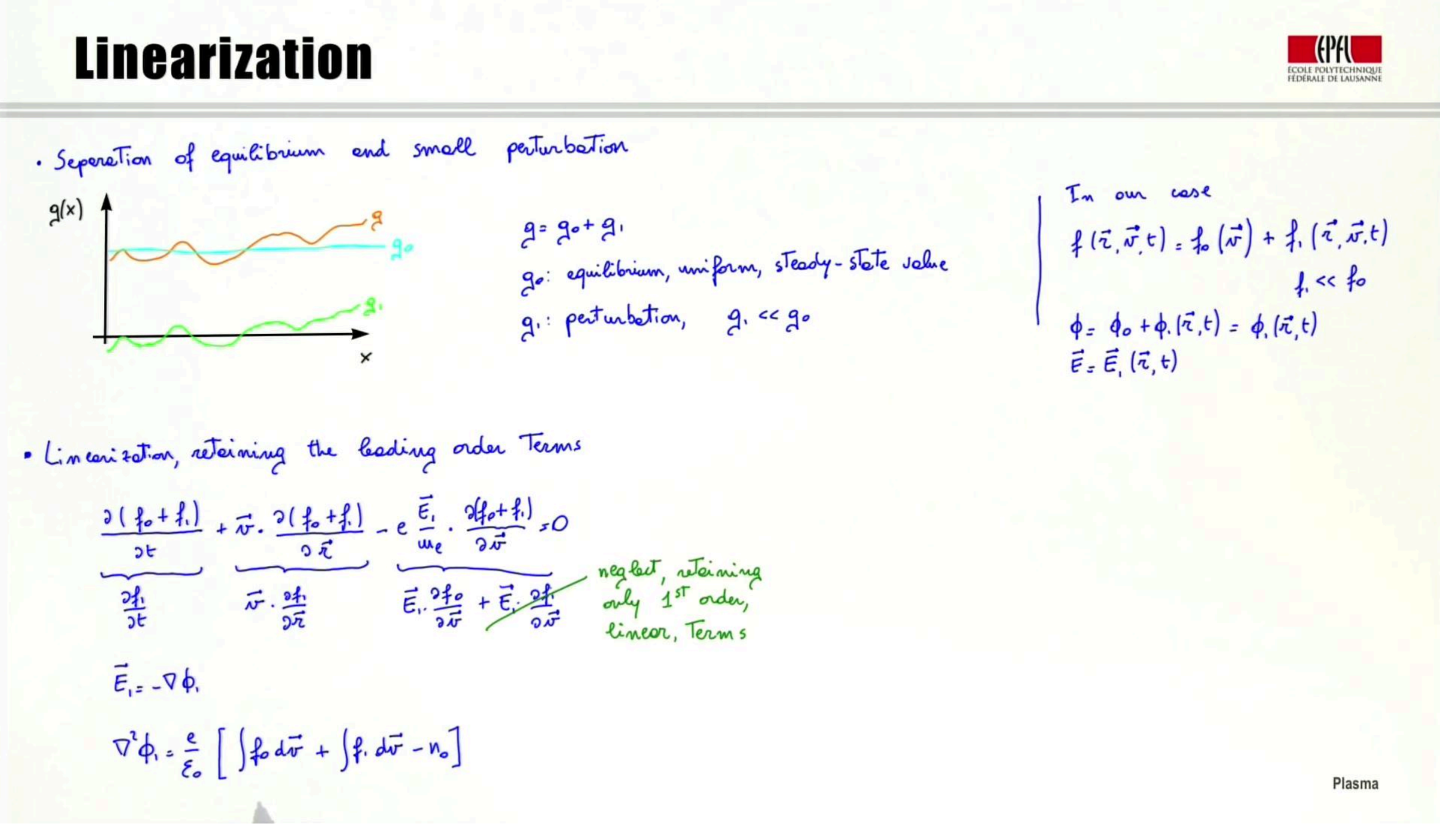


Basically the background constant uniform value is perturbed by a small amount. In our case, in the case of the Vlasov equation that we are considering, we will assume that the distribution function of the electrons will be written as the background value $f_0(v)$, which we will allow only to depend on v plus a perturbation $f_1(r,v,t)$, much much smaller than f_0 and for the electrostatic potential we will write it as $\Phi = \Phi_0 + \Phi_1(r,t)$ that depends on space and time and we will assume that the constant background value, $\Phi_0 = 0$ and therefore that the electrostatic potential is basically equal to the perturbation $\Phi_1(r,t)$. And similarly we will have the electric field will be [only paterbed] electric field. What is the advantage that I've introduced in Linearization, well as the paterbation of much smaller than the equilibrium quantities when writing down the equation we can neglect terms that comes from the product of paterb quantity with other paterb quantities. Basically what we can do, we can linearize the equation into the paterb quantities neglecting higher order terms and retaining only the leading one. Let's see this practically in one example.

Notes

Summary

5m 28s

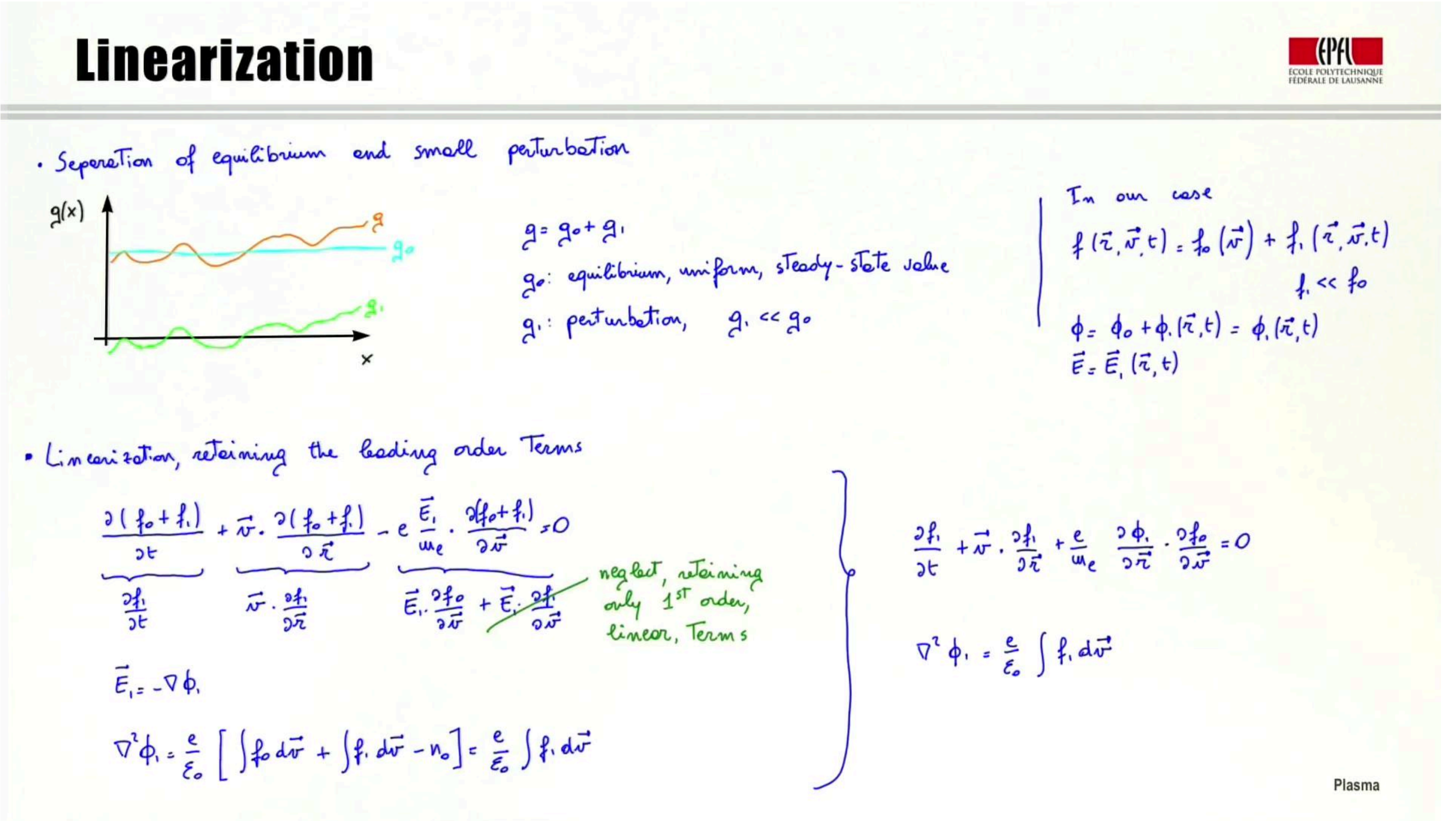


So what we want to do, is to linearize the equations, in particular the Vlasov-Poisson system that are we are considering, retaining only the leading order terms. Let's consider first of all the case of the Vlasov equation. We will write f as $f_0(v) + f_1(r,v,t)$ in all terms, in all expressions and we will introduce only the perturbed electric field E_1 into the force term. Okay, f_0 does not depend on time, therefore this expression reduces to $\partial f_1/\partial t$. Here we have that $f_0(v)$ does not depend on r, therefore this will become $\partial f_1/\partial r$ and this expression, well both-- f_0 and f_1 depend on v, therefore we will have that the system is given by the sum of two terms: $E_1 \cdot \partial f_0/\partial v + E_1 \cdot \partial f_1/\partial v$ Now f_0 is much larger than f_1 therefore this term is much smaller than the first one. We can neglect it. Retaining only the first order term in the perturbation, If you want only terms that are linear in the perturbation and neglecting the quadratic terms in the perturbation. We will retain only the leading order term. How about the equation for the electric field and electrostatic potential? Well we have that $E_1 = -\nabla\Phi_1$ and for the Poisson equation, we will have the contribution from the background charge of the electrons, the perturbed charge of the electrons, minus the charge of the ions.

Notes

Summary

7m 24s

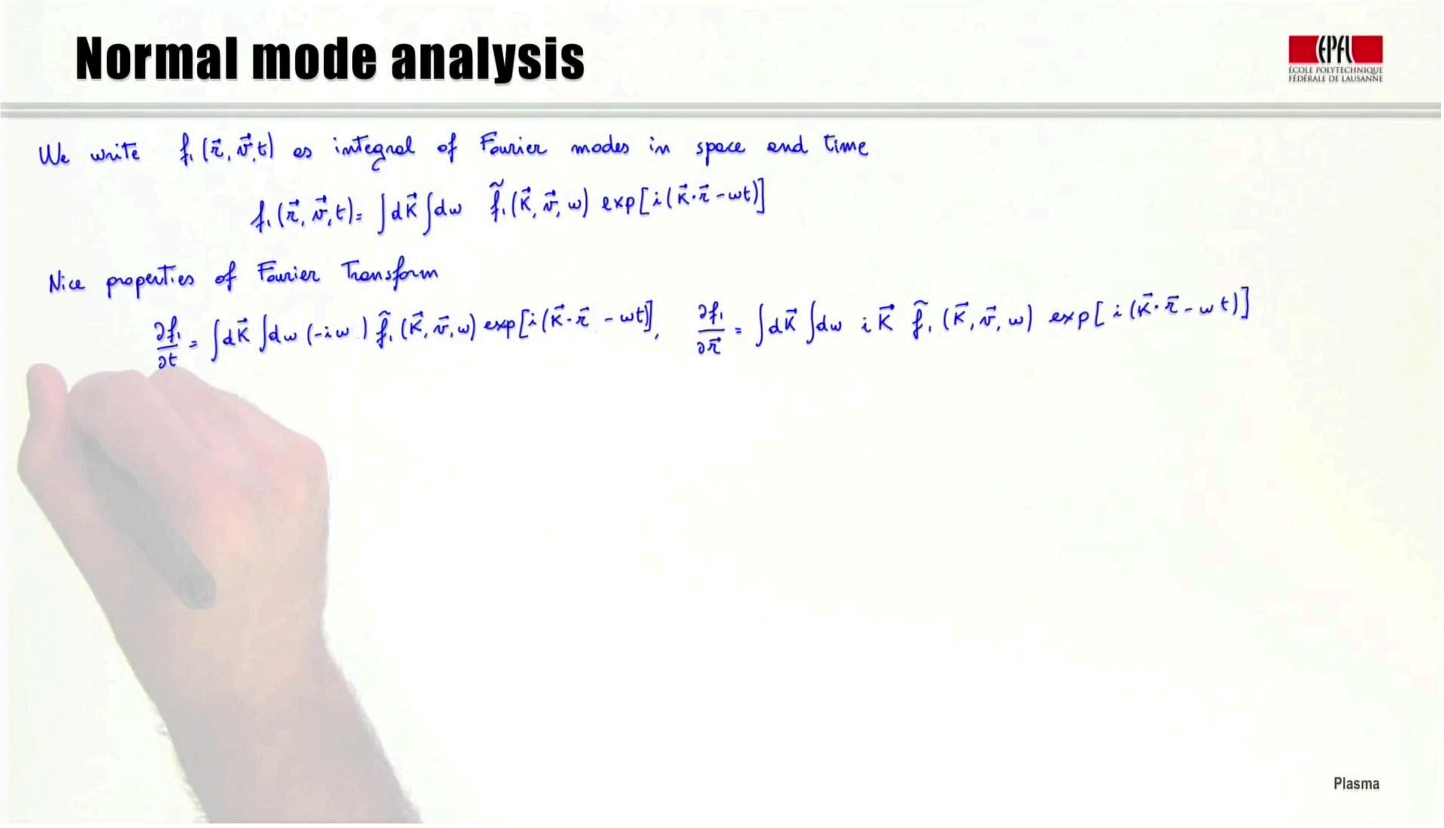


Now we expect that at the equilibrium the electron equilibrium distribution function will have a density that is equal to the one of the ions and therefore that these two terms will cancel. We will obtain therefore $\nabla^2\Phi_1 = e/\varepsilon_0 \int f_1\, dv$. Therefore, our linearized system will be given by the linearized Vlasov equation, where we can actually replace E_1 with Φ_1 [$E_1 = E_1 = -\partial\Phi_1/\partial r$], and lineralized Poisson equation. Now let me point out that this set of equations, that we have obtained is much much simpler than the one that we had considered at the beginning, as the perturbed quantities, the ones that evolve in time, f_1 and Φ_1 are actually containing expressions that are linear with respect to them.

Notes

Summary

9m 56s



Having linearized the equations, now we can use the normal mode of analysis. Basically we can decompose f_1 and Φ_1 in Fourier modes. And then look at the dynamic evolution of each single Fourier mode. We will write the f_1(r,v,t) as the integral for the mode in space and time. We will write the distribution function, f_1(r,v,t) as the integral over all the Fourier modes in space so over dK, the integral over dω of the Fourier modes $\tilde{f}_1$(K,v,ω) which we will denote with a tilde, of (K,v,ω). So the Fourier modes $\tilde{f}_1$ times exp {i (K·r - ωt) } Why do we do that? It's because that the Fourier transform has some very nice properties that we will use. The first one is if we want to evaluate the time derivative of f_1, whichit will be given by the integral over dK and dω and then we will have to derive this expression and it will give -iω by deriving with respect to time and then $\tilde{f}_1$(K,v,ω) and the exponential, and similarly the spatial derivative will become the integral over dK and dω, and when we take the spatial derivative, we get iK $\tilde{f}_1$(K,v,ω). If you want: ∂/∂t ⇒ -iω and ∂/∂r ⇒ iK Starting from the Vlasov equation...

Notes

Summary

11m 05s

Normal mode analysis

We write $f_1(\vec{r},\vec{v},t)$ as integral of Fourier modes in space and time

$$f_1(\vec{r},\vec{v},t)=\int d\vec{K}\int d\omega\ \tilde{f}_1(\vec{K},\vec{v},\omega)\exp[i(\vec{K}\cdot\vec{r}-\omega t)]$$

Nice properties of Fourier Transform

$$\frac{\partial f_1}{\partial t}=\int d\vec{K}\int d\omega\,(-i\omega)\,\tilde{f}_1(\vec{K},\vec{v},\omega)\exp[i(\vec{K}\cdot\vec{r}-\omega t)],\qquad \frac{\partial f_1}{\partial \vec{r}}=\int d\vec{K}\int d\omega\ i\vec{K}\ \tilde{f}_1(\vec{K},\vec{v},\omega)\exp[i(\vec{K}\cdot\vec{r}-\omega t)]$$

Starting from Vlasov equation

$$\frac{\partial f_1}{\partial t}+\vec{v}\cdot\frac{\partial f_1}{\partial \vec{r}}+\frac{e}{m_e}\frac{\partial \Phi_1}{\partial \vec{r}}\frac{\partial f_0}{\partial \vec{v}}=0 \Rightarrow \int d\vec{K}\int d\omega\left[(-i\omega+i\vec{K}\cdot\vec{v})\tilde{f}_1+\frac{ie\tilde{\Phi}_1}{m_e}\vec{K}\cdot\frac{\partial f_0}{\partial \vec{v}}\right]\exp[i(\vec{K}\cdot\vec{r}-\omega t)]=0$$

$$(-i\omega+i\vec{K}\cdot\vec{v})\tilde{f}_1+\frac{ie\tilde{\Phi}_1}{m_e}\cdot\frac{\partial f_0}{\partial \vec{v}}=0 \Rightarrow \tilde{f}_1=\frac{e}{m_e}\frac{\tilde{\Phi}_1}{(\omega-\vec{K}\cdot\vec{v})}\vec{K}\cdot\frac{\partial f_0}{\partial \vec{v}}$$

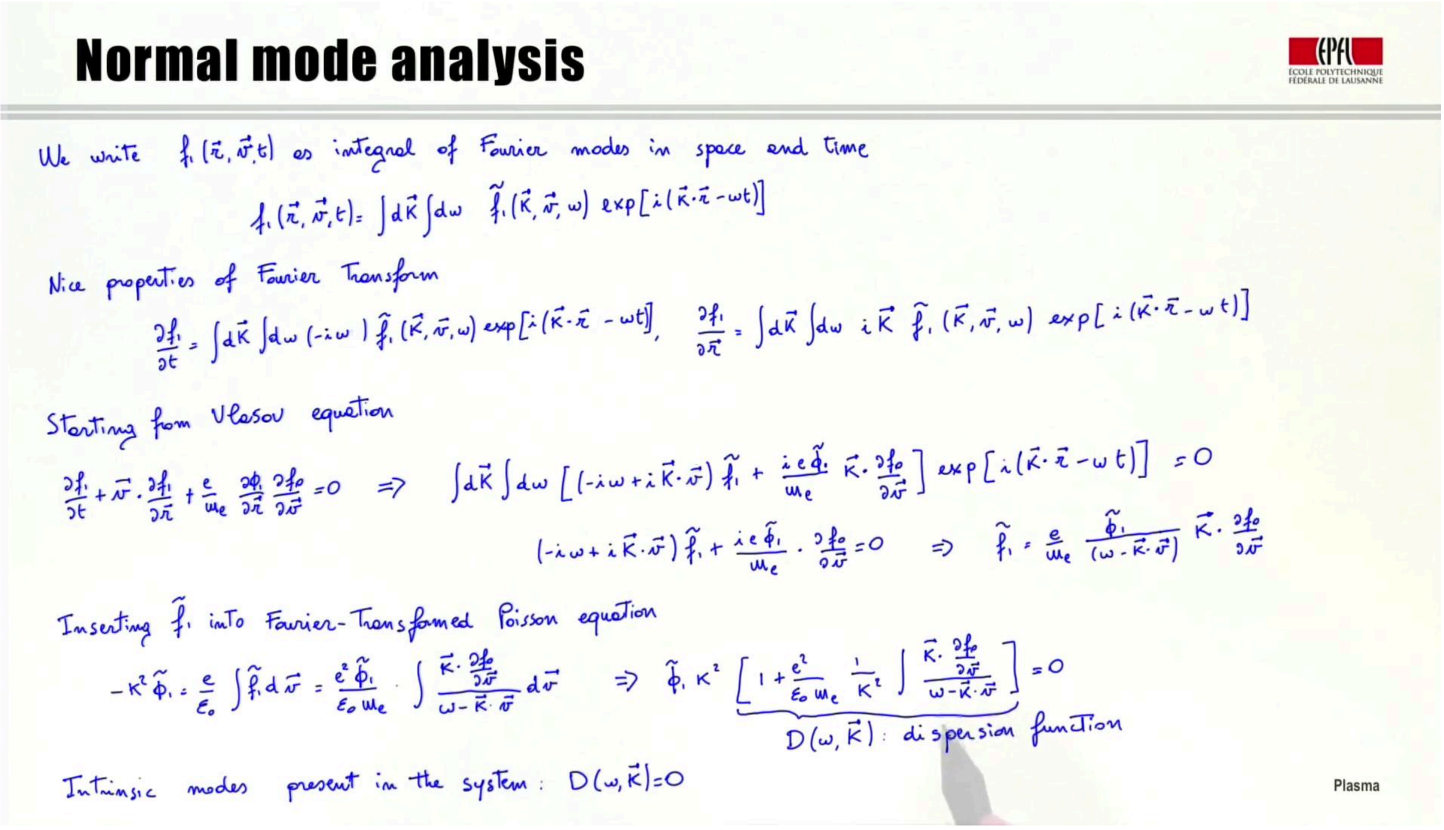
Plasma

introducing the expression for f_1 that we have written here the Vlasov equation can be written as the integral over dK and dω and then taking into account the fact that there are ∂/∂t and ∂/∂r so a (-iω + iK·v) $\tilde{f}_1$ term and then Fourier transform of Φ_1. We notice that $\partial f_0/\partial v$ is constant with respect to space and time and therefore can be brought inside the integral over dK and dω. It's just a constant with respect to space and time. And here we are using again the properties of the spatial derivative. Everything has to be multiplied by the exponential term, and this has to be equal to zero. One property of the Fourier modes is that there are all orthogonal to each other so the only way that this integral is equal to zero is that all the coefficients of Fourier modes are equal to zero, which means that for every ω and K we need to have that (-iω + iK·v) $\tilde{f}_1$ + i (e/me) $\tilde{\Phi}_1\cdot\partial f_0/\partial v$ has to be equal to zero, which also implies that $\tilde{f}_1$, the Fourier transform of f_1 has to be equal to - extracting it from this expression-, an expression that is linear in $\tilde{\Phi}_1$ and which depends on ω, K and the equilibrium distribution function.

Notes

Summary

13m 30s



We can then insert this $\tilde{f}_1$ in the Fourier transform of the Poisson equation and $\nabla^2\tilde{\Phi}_1$ will become $-K^2\tilde{\Phi}_1$, equal to -basically- the integral of $\tilde{f}_1$ over dv. and now we can use the expression for $\tilde{f}_1$ to obtain an expression that depends on $\tilde{\Phi}_1$. Now we can actually collect in this expression $\tilde{\Phi}_1$ and K^2 and we will have left-hand side minus the right-hand side, changing both signs in this integral. This function that multiplies $\tilde{\Phi}_1$ will be denoted as D(ω,K). It's a function that depends essentially on ω and K and is called the *dispersion function*. So what are the possibilities here? Well it can be that $\tilde{\Phi}_1$ is equal to zero. It means that the Fourier mode is not present in the system and this is the less interesting case or more interesting it could be that D(ω,K) = 0 and in this case we could have $\tilde{\Phi}_1 \neq 0$. If you want, with D(ω,K) = 0 we will find the intrinsic modes that can be present in the system. In other words, by looking at the solutions, i.e. the values of ω and K such that D(ω,K) = 0 we find the modes that can be sitting in our plasma without, being driven from outside. Let me note one more thing before we proceed. In this integral, there is actually a singularity when ω becomes equal to K·v.

Notes

Summary

15m 38s

Normal mode analysis

We write $f_1(\vec{r},\vec{v},t)$ as integral of Fourier modes in space and time

$$f_1(\vec{r},\vec{v},t) = \int d\vec{K} \int d\omega \; \tilde{f}_1(\vec{K},\vec{v},\omega) \exp[i(\vec{K}\cdot\vec{r}-\omega t)]$$

Nice properties of Fourier Transform

$$\frac{\partial f_1}{\partial t} = \int d\vec{K}\int d\omega \,(-i\omega)\, \tilde{f}_1(\vec{K},\vec{v},\omega) \exp[i(\vec{K}\cdot\vec{r}-\omega t)], \quad \frac{\partial f_1}{\partial \vec{r}} = \int d\vec{K}\int d\omega \; i\vec{K}\, \tilde{f}_1(\vec{K},\vec{v},\omega) \exp[i(\vec{K}\cdot\vec{r}-\omega t)]$$

Starting from Vlasov equation

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\frac{\partial f_1}{\partial \vec{r}} + \frac{e}{m_e}\frac{\partial \phi_1}{\partial \vec{r}}\frac{\partial f_0}{\partial \vec{v}} = 0 \Rightarrow \int d\vec{K}\int d\omega \left[(-i\omega + i\vec{K}\cdot\vec{v})\tilde{f}_1 + \frac{ie\tilde{\phi}_1}{m_e}\vec{K}\cdot\frac{\partial f_0}{\partial \vec{v}}\right]\exp[i(\vec{K}\cdot\vec{r}-\omega t)] = 0$$

$$(-i\omega + i\vec{K}\cdot\vec{v})\tilde{f}_1 + \frac{ie\tilde{\phi}_1}{m_e}\cdot\frac{\partial f_0}{\partial \vec{v}} = 0 \Rightarrow \tilde{f}_1 = \frac{e}{m_e}\frac{\tilde{\phi}_1}{(\omega - \vec{K}\cdot\vec{v})}\vec{K}\cdot\frac{\partial f_0}{\partial \vec{v}}$$

Inserting $\tilde{f}_1$ into Fourier-Transformed Poisson equation

$$-K^2\tilde{\phi}_1 = \frac{e}{\varepsilon_0}\int \tilde{f}_1 d\vec{v} = \frac{e^2\tilde{\phi}_1}{\varepsilon_0 m_e}\cdot\int \frac{\vec{K}\cdot\frac{\partial f_0}{\partial \vec{v}}}{\omega - \vec{K}\cdot\vec{v}} d\vec{v} \Rightarrow \tilde{\phi}_1 K^2 \underbrace{\left[1 + \frac{e^2}{\varepsilon_0 m_e}\frac{1}{K^2}\int \frac{\vec{K}\cdot\frac{\partial f_0}{\partial \vec{v}}}{\omega - \vec{K}\cdot\vec{v}}\right]}_{D(\omega,\vec{K}):\ \text{dispersion function}} = 0$$

Intrinsic modes present in the system: $D(\omega,\vec{K}) = 0$

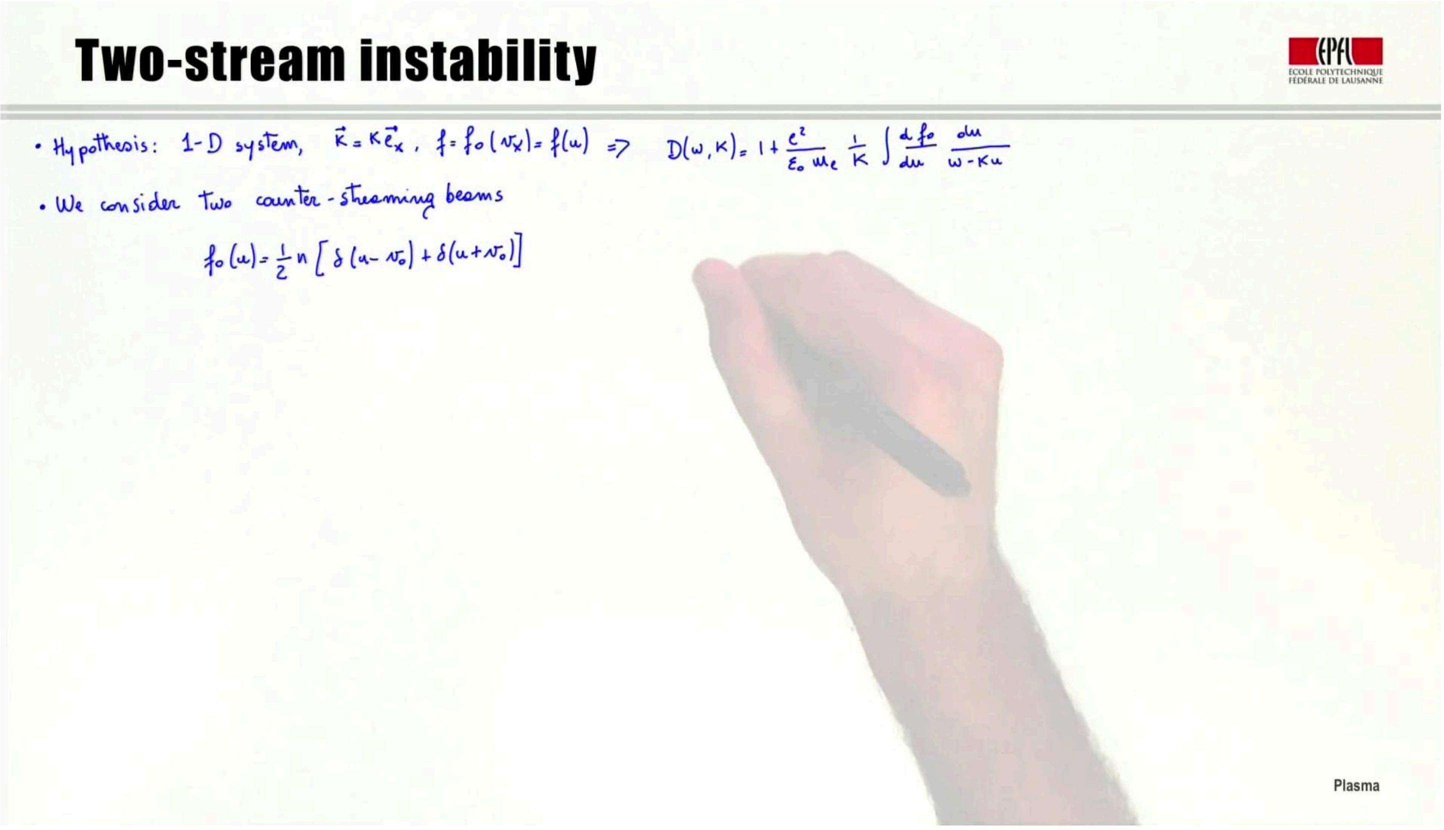
Plasma

This is actually something very interesting in plasma physics. This singularity occurs when the velocity of the particles match the phase velocity of the wave. Basically when particles are *resonant* with the wave. Dealing with these singularities is actually very difficult and we will avoid having to explain how to deal with these singularities in the present course by considering a case where they are not present because f_0 is equal to zero when we consider particles with which $\omega = K\cdot v$.

Notes

Summary

18m 03s

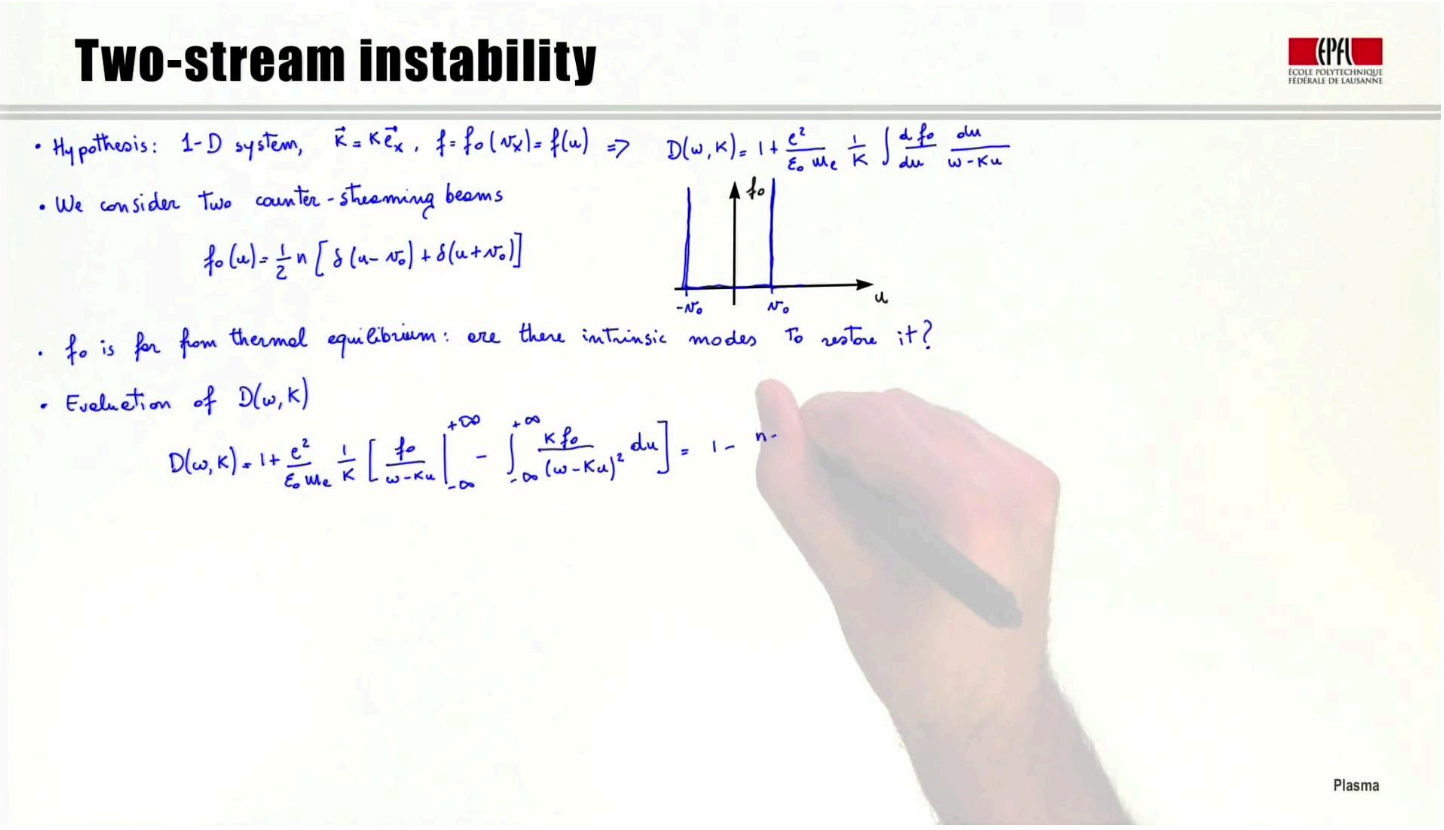


Now we have all the elements to look and study the dynamics of the system that we want to focus on. The system of two plasma beams, shot one against each other, two counter streaming beams. We want to understand what are the intrinsic modes present in this system as a matter of fact two counter streaming beams is a system that is very far from thermodynamic equilibrium. So, are there intrinsic modes in the system that can grow and try to restore the thermodynamic equilibrium ? This is what we will do now. To furthermore simplify system we are looking at, we will make the hypothesis of a 1-D system, therefore where **K** is only along the x-direction [**K** = K **ex**] and f_0, d the equilibrium function, depends only on vx and for simplicity let me call this vx u. [f = f_0(vx) = f_0(u)] So I won't have to carry the index x. In the hypothesis of a 1-D system we have that the dispersion function becomes equal to one plus this term that has been simplified by considering only a one-dimensional distribution function. We are now considering two counter streaming beams. Basically the equilibrium distribution function f_0 is given by the sum of two beams: one at v_0 and the other with particles flowing at -v_0.

Notes

Summary

18m 50s

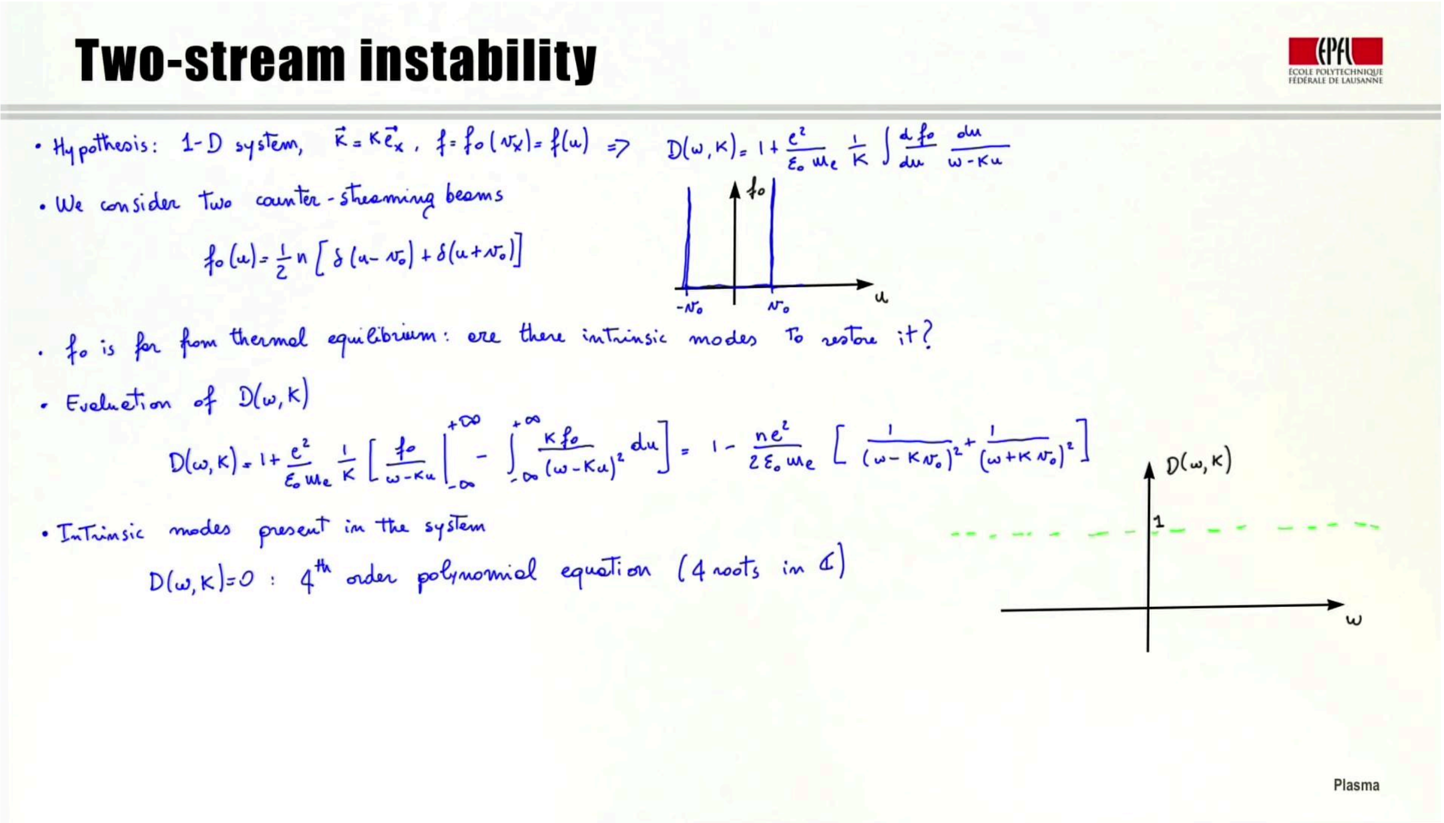


The distribution function that we have already seen in a previous module, is centered between $-v_0$ and v_0. It's always difficult to draw Dirac functions that will be something-- really peaked around $-v_0$ and then equal to zero and then again at v_0 we will have the Dirac function. Now as I was saying, f_0 is very far from thermal equilibrium. Are there intrinsic modes to try to restore thermal equilibrium? What do we have to do is to evaluate the dispersion function. So this function with f_0 given by this expression, the nice thing of this formula here is that it avoids [having] to consider [the singularity at] ω - Ku = 0, as the distribution function is equal to zero, except at two precise points. And this makes this integral a standard integral without the singularities to consider and therefore we have that D(ω,K) will be given by this integral. Now we can actually integrate this expression by parts and it will give f_0/ (ω - Ku) that has to be evaluated from u = -∞ to u = ∞ minus the integral of K f_0 and then the derivative of 1/(ω - Ku) and this is equal to one minus...

Notes

Summary

20m 55s

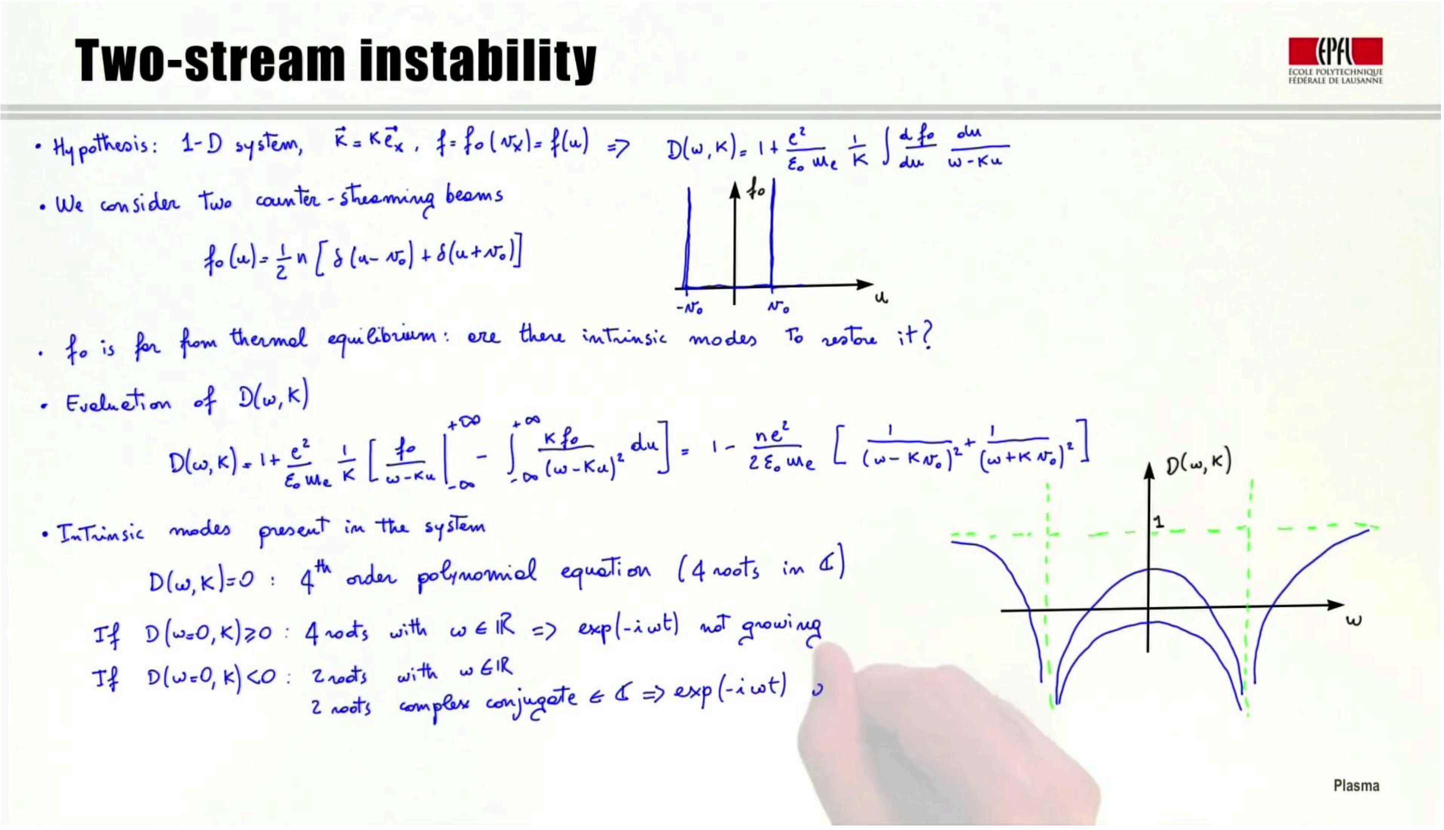


-we have some coefficient here- and then, well this integral is actually simple-- I mean first of all the evaluation of $f_0/(\omega - Ku)$ from with u taken at $-\infty$ and ∞ is easy because f_0 is equal to zero for $u = -\infty$ and $u = \infty$ and this integral is also simply done because it's the sum of two Dirac functions, and therefore there will be only two contributions [at $u=-v_0$ and $u=v_0$]. Carrying out the integral will give... ... we will have to replace u with $-v_0$ and v_0. Now we have to evaluate the intrinsic mode present in the system. Basically, $D(\omega,K) = 0$ If we look at the expression of $D(\omega,K)$ as a function of ω^2, we find that this is a fourth order polynomial equation. Therefore it will have four roots that are complex conjugates, in fact, the coefficients are real and they belong to the complex plane. Now, we can actually draw the function, $D(\omega,K)$, as a function of ω. What we observe is that for ω going to plus and minus infinity, this term tends to zero and therefore D tends to 1. There is in other words, an asymptote that is at D=1 and we can also see that there will be two singularities when ω is equal to plus and minus $K v_0$, where this will tend to infinity and therefore $D(\omega,K)$ will tend to minus infinity.

Notes

Summary

23m 08s

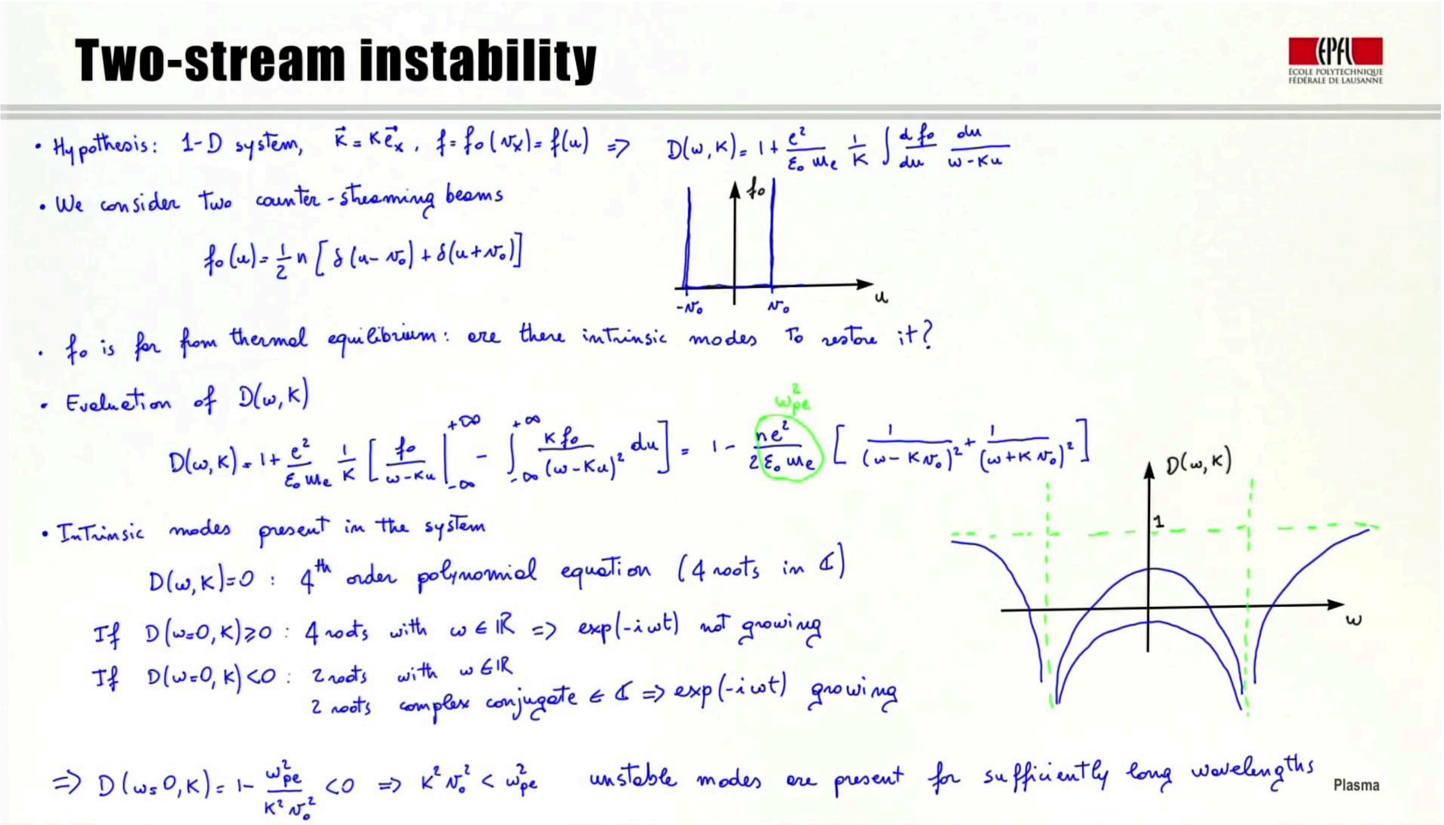


So, two vertical asymptotes. The function will tend to 1 with ω going to infinity and similarly here and then in this region it will depend a bit on the parameters ω, K and v_0. This is an even function, so we expect that it will do something like that or something like that. And the number of roots will be different if the D(ω,K) crosses or not the axis. More precisely, if D(ω,K) evaluated at ω = 0 is greater than zero, then there are four roots. This corresponds to the case that I have drawn here. In this case. Therefore if we look at the time dependence of the Fourier modes, which will be given by exp{-iωt} as their ω belong to the real axis we will have modes that are oscillatory, and not growing. There are no intrinsic growing modes that will tend to destroy the initial distribution function trying therefore to recover thermal equilibrium. But if D(ω= 0,K) is less than zero, we will have two roots with ω belonging to the real axis, which correspond to these two roots and at here we will look at this case where there are no other solutions and therefore we will have two roots, complex conjugates, that belong to ℂ, Therefore if we look at the time dependence of the Fourier modes, we will have that these Fouriermodes will contain two imaginary parts with opposite signs because they are conjugates and therefore one of the solutions will be growing.

Notes

Summary

25m 38s

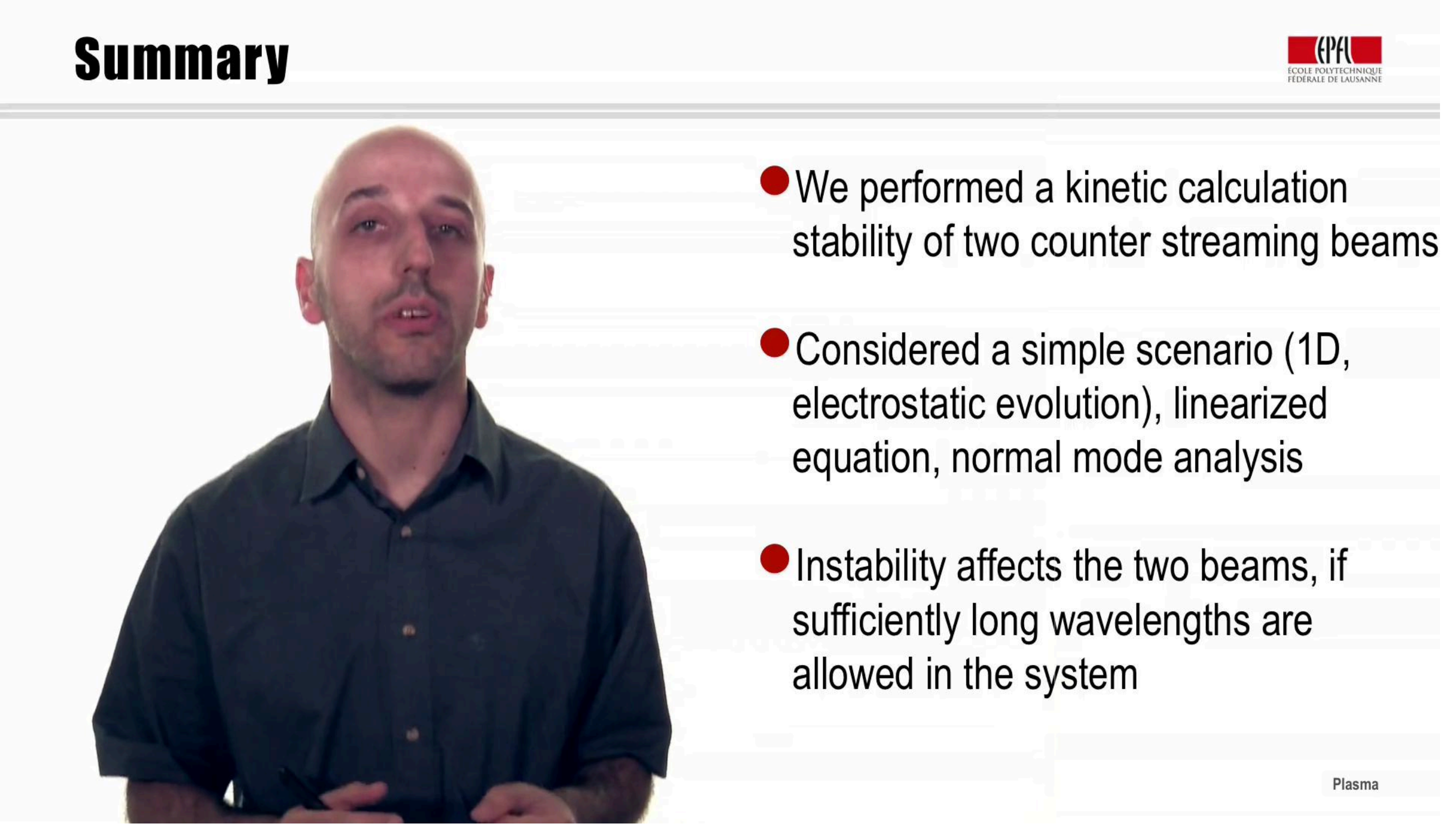


There will be an unstable mode that can grow in the system. What does it mean in practice? If we evaluate D(ω= 0,K) -that is this expression here evaluated at ω = 0- we get one minus-- let me notice that this quantity here, this is sort of interesting is that the plasma frequency. So we can write this as 1 - ω_{pe}^2 / $(K\ v_0)^2$ and this has to be less than zero to have unstable stable modes, which means that $(K\ v_0)^2$ has to be less than ω_{pe}^2. Well this means that unstable modes are present at *sufficiently* low K, which corresponds to *sufficiently* long wave length. So if the system is large enough, therefore if small K's are allowed to grow and to be present in the system, you will have modes that will grow on top of this equilibrium.

Notes

Summary

28m 29s



To summarize the contents of the present lecture, let me say that we have considered one of the simplest kinetic calculations. The one of two beams that are shot to one against each other, two counter streaming beams, we have simplified the Vlasov Maxwell system to look at the evolution of a 1D one dimensional system that displays electrostatic evolution. We have also used and applied mathematical techniques that we will use throughout this course: linearization and normal mode analysis. And what have we found? We've found that there are intrinsic modes present in the system that can grow. They can grow if the system is sufficiently large, if long enough wavelengths are allowed to grow in the system. Analytically this is the only conclusion we can draw. If we want to really look at the dynamics, the temporal evolution of these two beams, we have to use numerical techniques and this will be the goal of the next module.

Notes

Summary

30m 00s