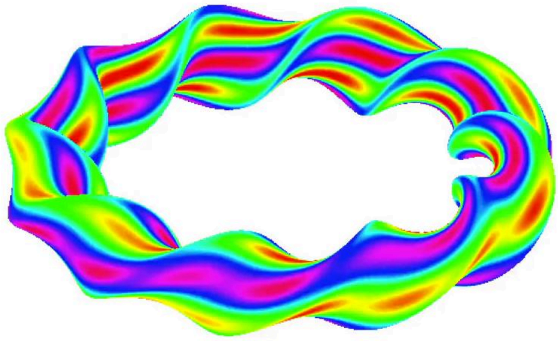
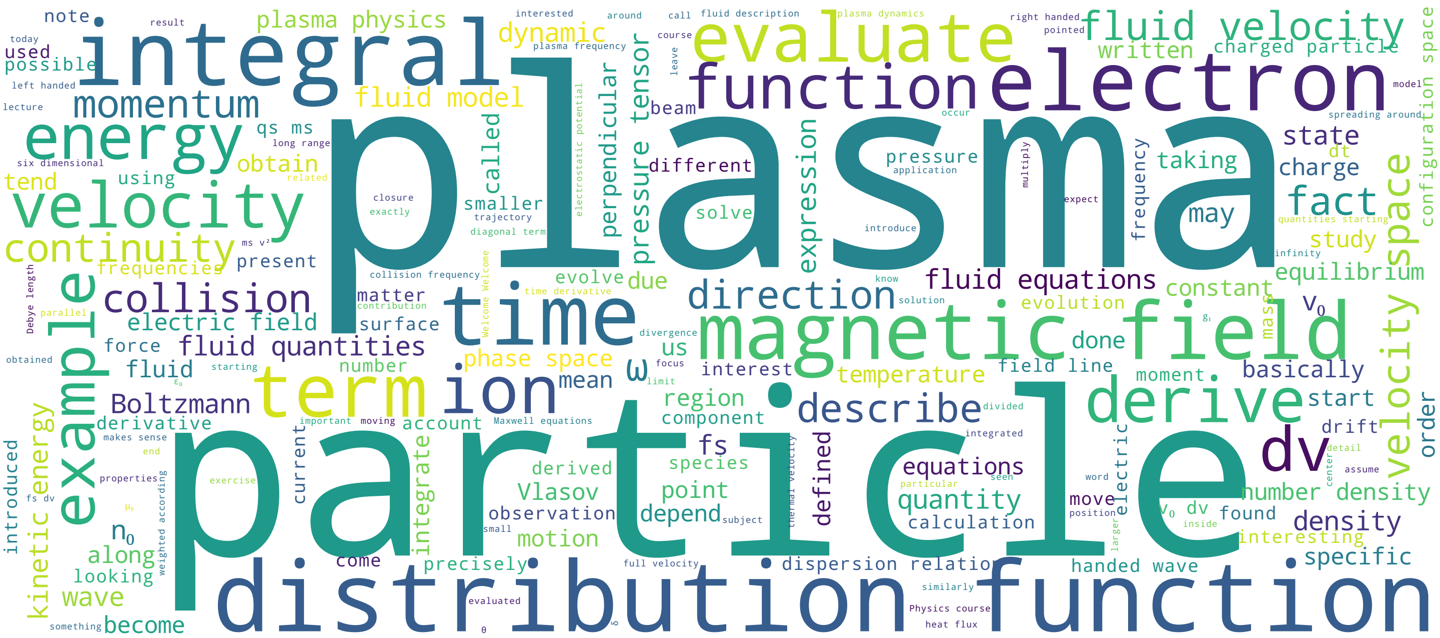


From kinetic to the two-fluid model

Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

Lecture 3a

Paolo Ricci



Search MOOC

Video

Introduction

- Fluid quantities from the distribution function
- Moment of the kinetic equation
- The continuity equation
- The two-fluid model

Plasma

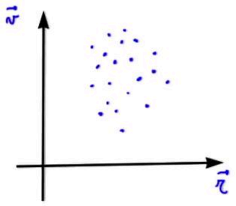
Welcome. Welcome to the Plasma Physics course of the EPFL. Today, we are going to derive the fluid equations starting from the kinetic equations. Why do we do that? The kinetic description of a plasma is extremely detailed and precise. However, to deal with it is extremely complicated both analytically and numerically. It is extremely expensive to solve a kinetic equation, and sometimes it's not even necessary. There is a need to come up with a simpler model like a fluid model to describe the plasma dynamics., This is exactly what we will do today more precisely. Today, we will first derive the fluid quantities from the distribution function, and we'll then take the moments of the kinetic equation to derive a set of fluid equations. With you, I will go through the details of deriving the continuity equation and I will leave to you the derivation of the momentum and energy fluid equations. What I will do at the end of this lecture is state a two-fluid model that can well describe the dynamics of a plasma.

Notes

Summary

0m 05s

Fluid quantities from the distribution function



$n_s(\vec{r},t) = \int d\vec{v}\; f_s(\vec{r},\vec{v},t)$, number density

In general, the average value of a generic function, $g(\vec{v})$, is

$$\langle g(\vec{r})\rangle = \frac{1}{n_s}\int g(\vec{v})\, f_s(\vec{r},\vec{v},t)\, d\vec{v}$$

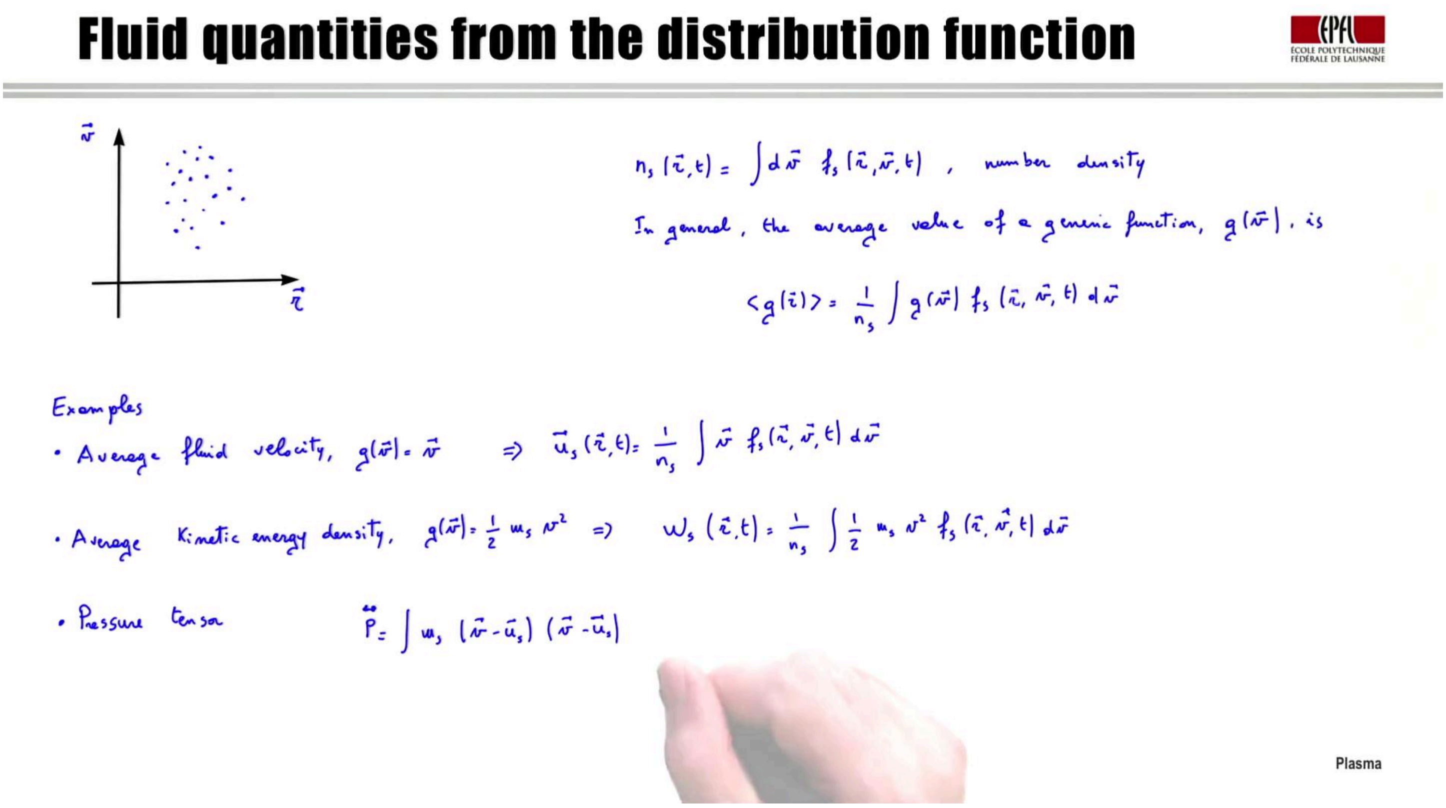
Plasma

Now as we have seen particles, are located in the six-dimensional phase space. Here I am projecting this six-dimensional space on this 2D plane. Here is the projection of the configuration space coordinates, here is the velocity coordinate and particles are scattered all over the phase space. We want to study the dynamics of the plasma by looking at quantities that only depend on the spatial coordinates. Quantities like density, temperature, fluid velocity of the plasma. How can we get this quantities from the distribution function? Well, we have seen already something, for example we have observed that the density of particles can be expressed by integrating over the velocity direction, so over the velocity space, the distribution function. This is what we have called the number density. This is a specific case of a more general way of evaluating fluid quantities starting from a distribution function. In general effect, the average value <g(r)> of a generic function, g(v) that depends on v, is given by the integral of this function, weighted according to the distribution of particles in the velocity space, integrated over full velocity space and this has to be normalized to the number density of our plasma.

Notes

Summary

1m 20s



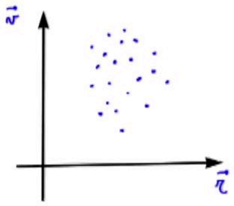
By considering different g functions, we can actually evaluate all the fluid quantities that may be of interest, so let's give a few examples. First we can be interested in the fluid average velocity, and this can be deduced by taking g(v) = v and evaluating the fluid velocity which is a function of space and time that is therefore defined as the weighted average of v fs(r,v,t) integrated over all dv. We may be interested in the average kinetic energy density that is obtained by taking g(v) = 1/2 ms v^2 and which is a function of position and time and is equal to one over the number density, integral of 1/2 ms v^2 [times the] distribution function, integrated over the full velocity. A quantity that may be of great interest, the fluid quantity is the pressure. In plasma physics, we define - as the distribution function may not be isotropic- A *pressure tensor* that is defined as... Here we want to normalize with respect to the number density. We will define the pressure tensor as the integral of the mass of the particles, the velocity of the particles minus the fluid velocity, same thing again. Please note that there is no dot product here this is really a tensor quantity.

Notes

Summary

3m 28s

Fluid quantities from the distribution function



$$n_s(\vec{r},t) = \int d\vec{v}\, f_s(\vec{r},\vec{v},t) \quad , \quad \text{number density}$$

In general, the average value of a generic function, $g(\vec{v})$, is

$$\langle g(\vec{r})\rangle = \frac{1}{n_s}\int g(\vec{v})\, f_s(\vec{r},\vec{v},t)\, d\vec{v}$$

Examples

- Average fluid velocity, $g(\vec{v}) = \vec{v} \quad \Rightarrow \quad \vec{u}_s(\vec{r},t) = \frac{1}{n_s}\int \vec{v}\, f_s(\vec{r},\vec{v},t)\, d\vec{v}$
- Average Kinetic energy density, $g(\vec{v}) = \frac{1}{2} m_s v^2 \quad \Rightarrow \quad w_s(\vec{r},t) = \frac{1}{n_s}\int \frac{1}{2} m_s v^2 f_s(\vec{r},\vec{v},t)\, d\vec{v}$
- Pressure tensor $\quad \overleftrightarrow{P} = \int m_s (\vec{v}-\vec{u}_s)(\vec{v}-\vec{u}_s)\, f_s(\vec{r},\vec{v},t)\, d\vec{v}$

  where $[(\vec{v}-\vec{u}_s)(\vec{v}-\vec{u}_s)]_{ij} = (\vec{v}-\vec{u}_s)_i\,(\vec{v}-\vec{u}_s)_j$

Plasma

This has to be weighted according to the particle distribution function and integrated as we were saying over the full velocity space. What is this [pressure] tensor? The ij component of this tensor will be given by the i component of the first vector [(v-us)i] multiplied with the j component of the second vector [(v-us)j] So the conclusion of the point I want to make is that basically one can derive the fluid quantities of interest by starting from the distribution function if a proper integral is taken in the velocity space.

Notes

Summary

5m 52s

Examples of fluid averaged quantities

- $f = n_0 \, \delta(\vec{v} - \vec{v}_0)$

$\rightarrow n = \int f \, d\vec{v} = \int n_0 \, \delta(\vec{v} - \vec{v}_0) \, d\vec{v} = n_0$

$\rightarrow \vec{u} = \frac{1}{n} \int \vec{v} f \, d\vec{v} = \frac{1}{n_0} \int n_0 \, \vec{v} \, \delta(\vec{v} - \vec{v}_0) \, d\vec{v} = \vec{v}_0$

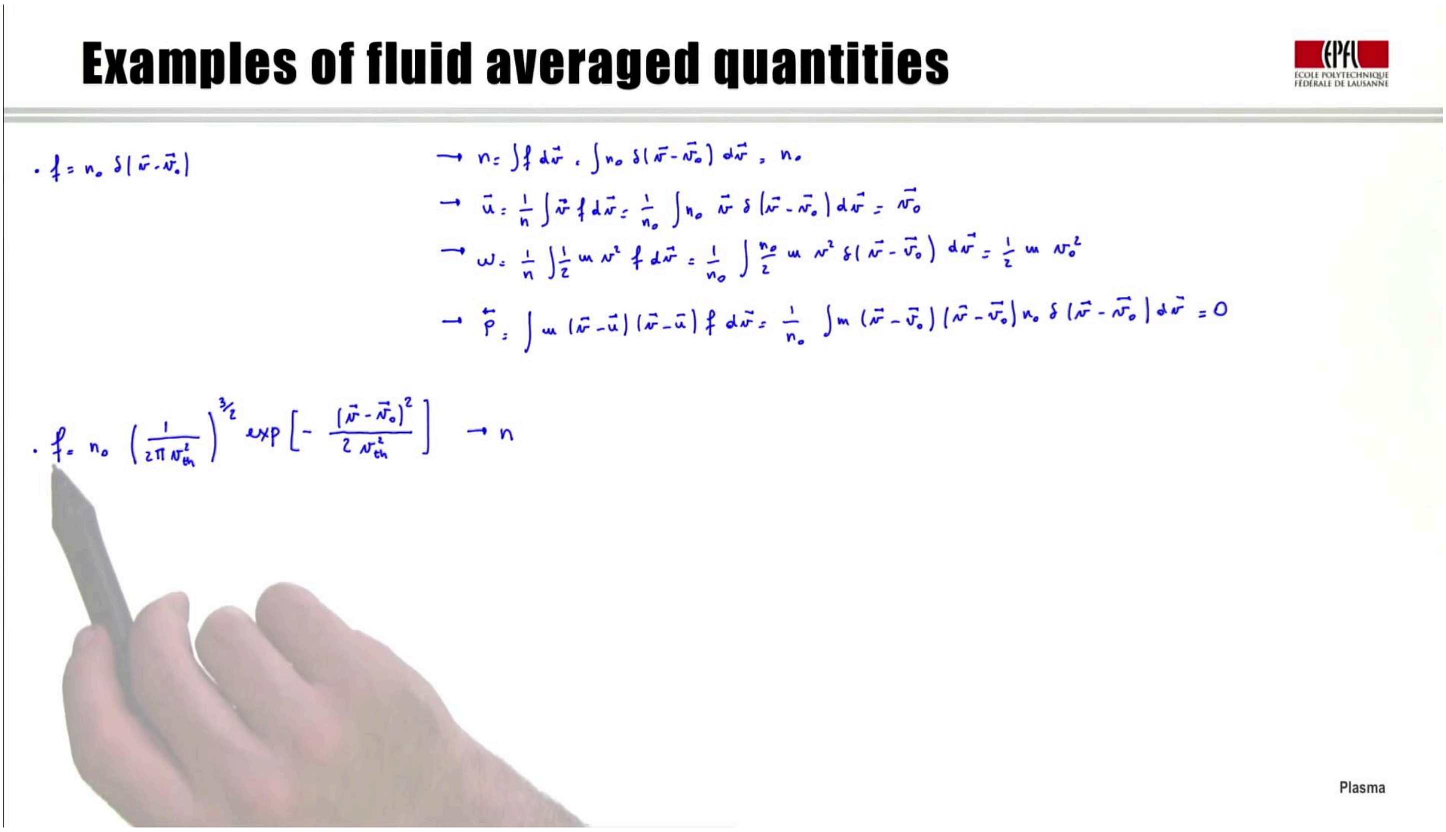
Plasma

To make things more practical let's evaluate a few fluid quantities from some distribution functions that may be of interest. The first distribution function that we can consider is the one of a beam of particles. That moves with the velocity v_0, therefore the distribution function is equal to 0, except when v equal to v_0. This is the reason of this delta [Dirac] function. Now we can evaluate some fluid quantities starting from this distribution function. And we note for example that n, the number density is equal to the integral of f dv is equal to the integral of n_0. In this specific case, $\delta(v-v_0)$ dv and this is basically n_0 that we can take out from the integral, and then integral of the delta function, which is equal to one. So we find that the number density is equal to n_0. We can evaluate the fluid velocity of this beam. * is equal to 1/n integral of v weighted according to the distribution function [times] dv and for the specific case, is $1/n_0 \int n_0 \; v \; \delta(v-v_0)$ dv n_0 and n_0 simplify, integral of v $\delta(v-v_0)$ is equal to v_0 This is exactly what we were expecting also from a fluid point of view: all these particles of the distribution function are moving with velocity v_0.

Notes

Summary

6m 49s



The average fluid velocity should be equal to v_0. Energy?. Energy will be the weighted average 1/n ∫ 1/2 m v² f dv and for this specific case, this will be $1/n_0$ ∫ n_0/2 m v² δ(v-v_0) dv Then this will be equal to 1/2 mv_0^2 Pressure; pressure tensor as we have defined it, will be given by the tensor $1/n_0$ ∫ m (v-u) (v-u) f dv and this will be equal to, for this specific case, $1/n_0$ ∫ m (v-v_0) (v-v_0) n_0 δ(v-v_0) dv If we integrate over dv, we will obtain zero. What does it mean? The pressure tensor gives us an idea of the the dispersion of the velocity, as given by the distribution function around the mean value, and what we found is that this is equal to zero. Basically there is no spreading around the mean velocity and as a matter of fact this function that is very peaked around the average fluid velocity. We may consider another distribution function and the most interesting one is for sure the Maxwellian distribution function. This would be given by n_0 (m/(2 π vth²))^(3/2) [vth = thermal velocity] exp{ -(v-v_0)² / (2 vth²) } [note: no m in the exp...] We can evaluate the number density. If you do the integral of this function you will get 1. And therefore n, the number density is equal to n_0.

Notes

Summary

9m 02s

Examples of fluid averaged quantities

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

- $f = n_0 \, \delta(\vec{v} - \vec{v}_0)$

$$\rightarrow n = \int f \, d\vec{v} = \int n_0 \, \delta(\vec{v} - \vec{v}_0) \, d\vec{v} = n_0$$

$$\rightarrow \vec{u} = \frac{1}{n} \int \vec{v} f \, d\vec{v} = \frac{1}{n_0} \int n_0 \, \vec{v} \, \delta(\vec{v} - \vec{v}_0) \, d\vec{v} = \vec{v}_0$$

$$\rightarrow w = \frac{1}{n} \int \frac{1}{2} m v^2 f \, d\vec{v} = \frac{1}{n_0} \int \frac{n_0}{2} m v^2 \, \delta(\vec{v} - \vec{v}_0) \, d\vec{v} = \frac{1}{2} m v_0^2$$

$$\rightarrow \overleftrightarrow{P} = \int m (\vec{v} - \vec{u})(\vec{v} - \vec{u}) f \, d\vec{v} = \frac{1}{n_0} \int m (\vec{v} - \vec{v}_0)(\vec{v} - \vec{v}_0) n_0 \, \delta(\vec{v} - \vec{v}_0) \, d\vec{v} = 0$$

- $f = n_0 \left(\frac{1}{2\pi v_{th}^2} \right)^{3/2} \exp\left[- \frac{(\vec{v} - \vec{v}_0)^2}{2 v_{th}^2} \right]$

$$\rightarrow n = n_0$$

$$\rightarrow \vec{u} = \vec{v}_0$$

$$\rightarrow w = \frac{1}{2} m v_0^2 + \frac{3}{2} m v_{th}^2 = \frac{1}{2} m v_0^2 + \frac{3}{2} T \qquad \left(v_{th} = \sqrt{\frac{T}{m}} \right)$$

$$\rightarrow \overleftrightarrow{P} = \begin{pmatrix} n_0 T & 0 & 0 \\ 0 & n_0 T & 0 \\ 0 & 0 & n_0 T \end{pmatrix}$$

Plasma

Similarly if you play a bit and you do the integral of v f(v), you get that the particles -as we expect-, move with a fluid velocity that is given by v_0. That's where the Maxwellian is centered. If we evaluate this integral to get the mean average kinetic energy, you obtain that this will be given by 1/2 mv_0^2. That is the kinetic energy associated with the collective fluid motion of the plasma, plus-- there will be some spreading, some kinetic energy related with the spreading around the mean velocity. And this will be given by 3/2 m vth^2 which is equal to 1/2 mv_0^2 and if we introduce the temperature T, we can write this kinetic energy as 3/2 T [note: no Boltzmann cst] times the more precisely what we have found in determining was the temperature that is written as the-- is related to the thermal velocity by the relation vth^2 = T/m [T in eV] The pressure tensor is something that you can also evaluate and what you will find is that as this distribution function is isotropic, then only the diagonal terms of this pressure tensor will be different from zero, all the others will be equal to zero.

Notes

Summary

12m 10s

The moments of the kinetic equation

Boltzmann equation : $\frac{\partial f_s}{\partial t} + \vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s}(\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f_s}{\partial \vec{v}} = \left(\frac{\partial f_s}{\partial t} \right)_c$

Moments of Boltzmann equations: $\int g_i(\vec{v}) \, [\text{Boltzmann}] \, d\vec{v} = 0$

$\rightarrow g_1(\vec{v}) = 1 \Rightarrow$ continuity

$\rightarrow g_2(\vec{v}) = m_s \vec{v} \Rightarrow$ momentum

$\rightarrow g_3(\vec{v}) = m_s \frac{v^2}{2} \Rightarrow$ energy

Plasma

And more precisely, what you will find is that they will be equal to n_0 T [Note: No Boltzmann's cst] for the diagonal terms, and they will be equal to zero for all the off-diagonal terms. We have now introduced the fluid quantities starting from the distribution function. It's now time to see how they evolve, how we can actually derive an equation that states how these fluid quantities such as density, fluid, velocity, pressure evolve as a function of time. To do that we start it from Boltzmann equation. Boltzmann equation states that the derivative with respect of time of the distribution function is due to the streaming in configuration space, plus the streaming in velocity space due to long range interactions, plus a term that takes into account of collisions. In order to derive the fluid equation, what we will do is take the moment of the Boltzmann equation. More precisely, we will consider the Boltzmann equation, we will multiply it by a function $g_i(v)$ and we will integrate this with respect to the velocity space. By properly choosing the function $g_i(v)$ we will get the continuity, the momentum and the energy fluid equations. More precisely we will consider $g_1(v) = 1$ and in this case we will get the continuity equation.

Notes

Summary

14m 16s

The continuity equation

$$\int \left[\frac{\partial f_s}{\partial t} + \vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s} \left(\vec{E} + \vec{v} \times \vec{B} \right) \cdot \frac{\partial f_s}{\partial \vec{v}} - \left(\frac{\partial f_s}{\partial t} \right)_c \right] d\vec{v} = 0$$

(1) (2) (3) (4)

(1) $\int \frac{\partial f_s}{\partial t} d\vec{v} = \frac{\partial}{\partial t} \int f_s d\vec{v} = \frac{\partial}{\partial t} n_s$

(2) $\int \vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}} d\vec{v} = \frac{\partial}{\partial \vec{r}} \cdot \int \vec{v} f_s d\vec{v} = \frac{\partial}{\partial \vec{r}} \cdot (n_s \vec{u}_s)$

(3)

Plasma

We will then consider $g_2(v)$ = ms v and get the momentum equation, and finally if we consider $g_3(v)$ = (ms/2) v² we will get an energy equation. I will derive carefully with you the continuity equation and I will leave to you the derivation of the momentum and energy equation. So let's try to derive together the continuity equation. We have to take the Boltzmann equation multiply it by $g_1(v)$, that is by 1, and then integrate over the velocity space. Let's see what we obtain. So, we consider the Boltzman equation taking into account also the collision term. We multiply this equation by 1, that is by $g_1(v)$. We integrate it with respect to velocity. This will have to be equal to zero. So let's consider the four integrals that we have to perform separately. Let's start from the first one. What we have to evaluate is the integral of ∂fs/∂t dv that is ∂/∂t of the integral of fs with respect to dv. That is equal to the time partial derivative of the particle density ∂ns/∂t. Second term. We have to evaluate the integral of v·∂fs/∂r dv Now, you can bring the v inside ∂/∂r as v and r are independent phase space coordinates, and then you can also take the ∂/∂r out of the integral as the integral is with respect to v.

Notes

Summary

16m 38s

The continuity equation

$$\int \left[\underbrace{\frac{\partial f_s}{\partial t}}_{①} + \underbrace{\vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}}}_{②} + \underbrace{\frac{q_s}{m_s} (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f_s}{\partial \vec{v}}}_{③} - \underbrace{\left(\frac{\partial f_s}{\partial t}\right)_c}_{④} \right] d\vec{v} = 0$$

① $\int \frac{\partial f_s}{\partial t} d\vec{v} = \frac{\partial}{\partial t} \int f_s \, d\vec{v} = \frac{\partial}{\partial t} n_s$

② $\int \vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}} d\vec{v} = \frac{\partial}{\partial \vec{r}} \cdot \int \vec{v} f_s \, d\vec{v} = \frac{\partial}{\partial \vec{r}} \cdot (n_s \vec{u}_s)$

③ $\int \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f_s}{\partial \vec{v}} = \frac{q_s}{m_s} \int \frac{\partial}{\partial \vec{v}} \cdot \left[(\vec{E} + \vec{v} \times \vec{B}) f_s \right] d\vec{v} = \frac{q_s}{m_s} \int_{S_v \to +\infty} (\vec{E} + \vec{v} \times \vec{B}) f_s \cdot d\vec{S}_v = 0$

④ $\int \left(\frac{\partial f_s}{\partial t}\right)_c d\vec{v} = 0$ collisions do not create nor destroy particles

$$\Rightarrow \boxed{\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (n_s \vec{u}_s) = 0}$$ Continuity equation

Plasma

Therefore, what you will have is ∂/∂r · ∫ v fs dv and if you look at this quantity, this quantity is the fluid velocity [us] times the particle density [ns]. Let's focus on the third integral. This is qs/ms ∫ (E + v × B) · ∂fs/∂v dv and as E does not depend on v and v × B is perpendicular to v, we can write this as qs/ms ∫ ∂/∂v ·{(E + v × B) fs} dv This is equal to qs/ms... -and then here you can use the divergence theorem and this becomes the integral over a surface that contains the velocity space and this surface tends to infinity- of (E + v × B) fs dotted with the surface vector [· dSv] Since there are no particles that have an infinite speed, fs will tend to zero [on the surface] and the surface integral has to be equal to zero. Finally, the fourth integral: we have to evaluate the integral of the collision operator. Now this will be equal to zero as a matter of fact because collisions do not create nor destroy particles.

Notes

Summary

19m 38s

The two-fluid model

- $\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (n_s \vec{u}_s) = 0$

- $m_s n_s \frac{d\vec{u}_s}{dt} = q_s n_s (\vec{E} + \vec{u}_s \times \vec{B}) - \frac{\partial}{\partial \vec{r}} \cdot \overleftrightarrow{P}_s + \vec{R}_s$ $\qquad \frac{d}{dt} = \frac{\partial}{\partial t} + u_s \cdot \frac{\partial}{\partial \vec{r}}$, $\vec{R} = \int m_s (\vec{v} - \vec{u}_s) \left(\frac{\partial f}{\partial t} \right)_c dv$

If we put ①, ②, ③, ④ together, what do we obtain? We obtain ∂ns/∂t + ∂/∂r·(ns us) = 0 What we have just derived is the *fluid continuity equation*. We have derived together the continuity equation. Following exactly the same steps and I leave that to you. One can derive the momentum and energy equation. It's enough to take the Boltzmann equation, weighted with the correct g function and carry out the integration. I will now summarize the results of this exercise, if you integrate the Boltzmann equation with these different weights. What we come up with is a fluid model, a fluid model that separates each species present in our plasma. As there are typically ions and electrons in the plasma, therefore two species, the model that we will describe that I will write down now is the so-called *two-fluid model*. In fact, for the electrons and ions, you will have a continuity equation that we have just derived. Again, for ions and electrons we will have a momentum equation... ... that states that the derivative of the velocity we have indicated with d/dt, the convective derivative.

Notes

Summary

22m 17s

The two-fluid model

- $\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (n_s \vec{u}_s) = 0$
- $m_s n_s \frac{d\vec{u}_s}{dt} = q_s n_s (\vec{E} + \vec{u}_s \times \vec{B}) - \frac{\partial}{\partial \vec{r}} \cdot \overleftrightarrow{P}_s + \vec{R}_s$ $\quad \frac{d}{dt} = \frac{\partial}{\partial t} + u_s \cdot \frac{\partial}{\partial \vec{r}}$, $\vec{R} = \int m_s (\vec{v} - \vec{u}_s) \left(\frac{\partial f}{\partial t}\right)_c d\vec{v}$
- $\frac{3}{2} n_s \frac{dT_s}{dt} + \overleftrightarrow{P}_s : \frac{\partial}{\partial \vec{r}} \vec{u}_s + \frac{\partial}{\partial \vec{r}} \cdot \vec{q}_s = Q_s$ $\quad T_s = \frac{1}{3n_s} \int m_s (\vec{v} - \vec{u}_s)^2 f_s d\vec{v}$, $\vec{q}_s = \frac{1}{2} m_s \int (\vec{v} - \vec{u}_s)^2 (\vec{v} - \vec{u}_s) f_s d\vec{v}$

$$Q_s = \int \frac{1}{2} m_s (\vec{v} - \vec{u}_s)^2 \left(\frac{\partial f}{\partial t}\right)_c d\vec{v}$$

Observations

- Closure problem
- Coupling with Maxwell equations , $\rho = \sum_s q_s n_s$, $\vec{j} = \sum_s q_s n_s \vec{u}_s$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Plasma

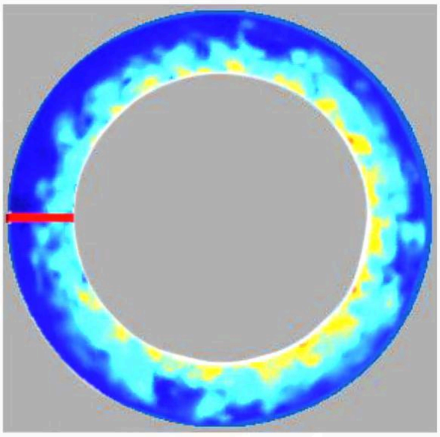
d/dt = ∂/∂t + us·∂/∂r The acceleration of the fluid, will be due to the electric and magnetic force plus the contribution of the pressure and a viscous force that is due to the collisions that plasma does. And it's defined as you see. One gets also an energy equation that states that the variation of the energy of the system where we have defined with Ts the integral of the kinetic energy due to the spreading around the fluid velocity, properly averaged, is due to the viscous heat here. We note that this is a double dot product plus the heat that is transported by the heat flux vector qs [not to be confused with charge!] is defined as qs = ms/2 ∫ (v-us)² (v-us) fs dv therefore the kinetic energy that is transported with the velocity of the particles, And then the heat [Qs] generated because of the viscous forces. Observations: The first observation that I would like to make is that this system of equation is not closed. As a matter of fact, in order to solve the energy equation, what you need to have is an expression for the heat flux. And this requires to know the distribution function. What we have is a problem of closure. How to derive a good expression of the heat flux without having to evolve the full distribution function ? This is a very active area of research in plasma physics. And at the same problem is for the pressure tensor, the collisional forces.

Notes

Summary

24m 29s

Two-fluid simulation of plasma turbulence



Turbulence simulation in the periphery of magnetic fusion devices

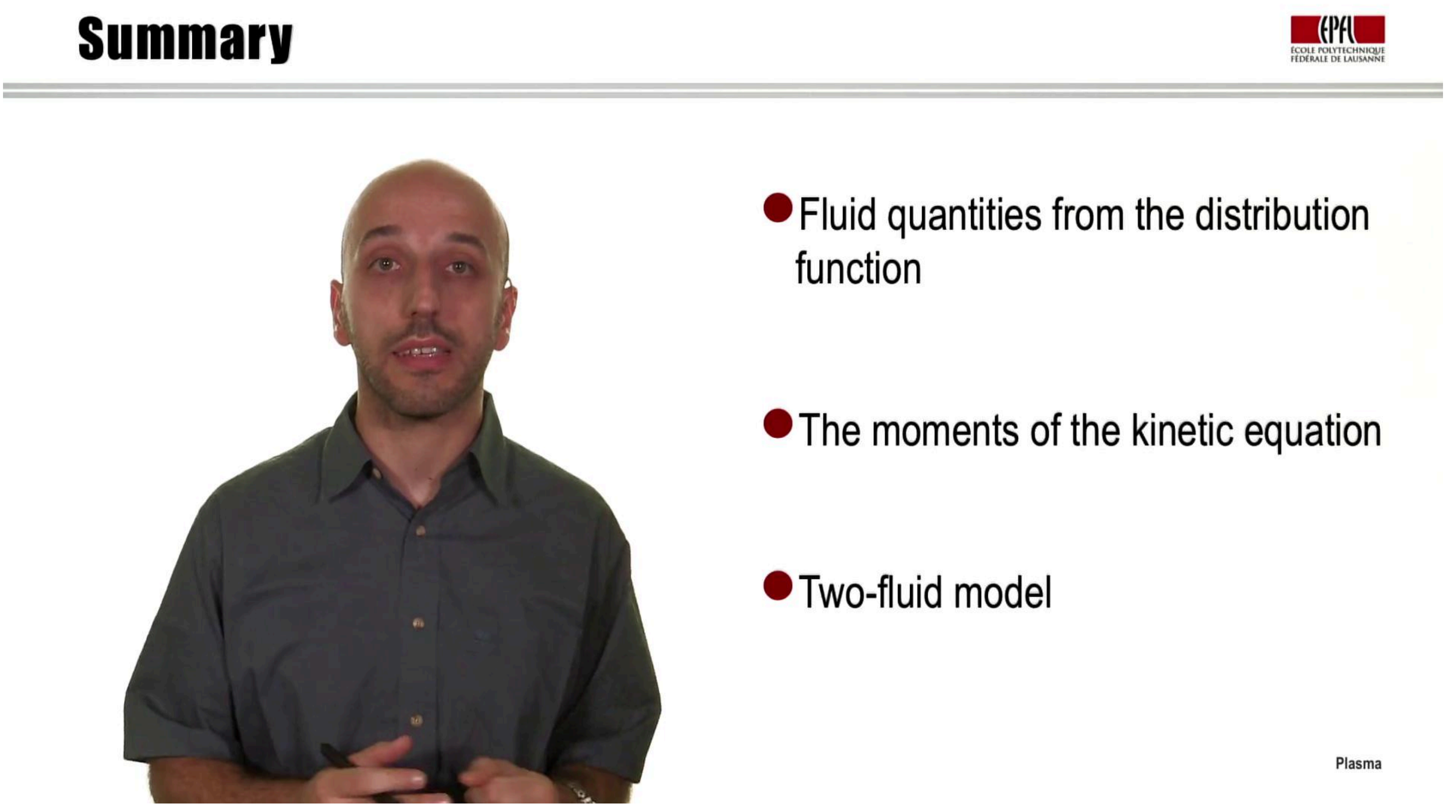
Plasma

The second observation is that this fluid model has to be coupled to the Maxwell equations, and this can be done because the electric charge present in our system can obtained by summing the electric charge of ions and electrons and similarly for the current density, and then the charge and current densities can then be injected in the Maxwell equations that therefore read: $\nabla\cdot E = \rho/\varepsilon_0$ $\nabla \times E = -\partial B/\partial t$ $\nabla\cdot B = 0$ and $\nabla \times B = \mu_0 (j + \varepsilon_0\, \partial E/\partial t)$ Where does it actually makes sense to use a fluid description like the one I have just given, the two-fluid description to evolve and describe the dynamics of a plasma? It definitely makes sense where it is possible. It's easy, or relatively easy to find the good closure for the set of fluid equations. And this is where the plasma is fairly collisional. There in this region, you expect that the distribution function will be close to a Maxwellian, and therefore somehow one can make progress in evaluating the closure of the set of fluid equations.

Notes

Summary

28m 05s

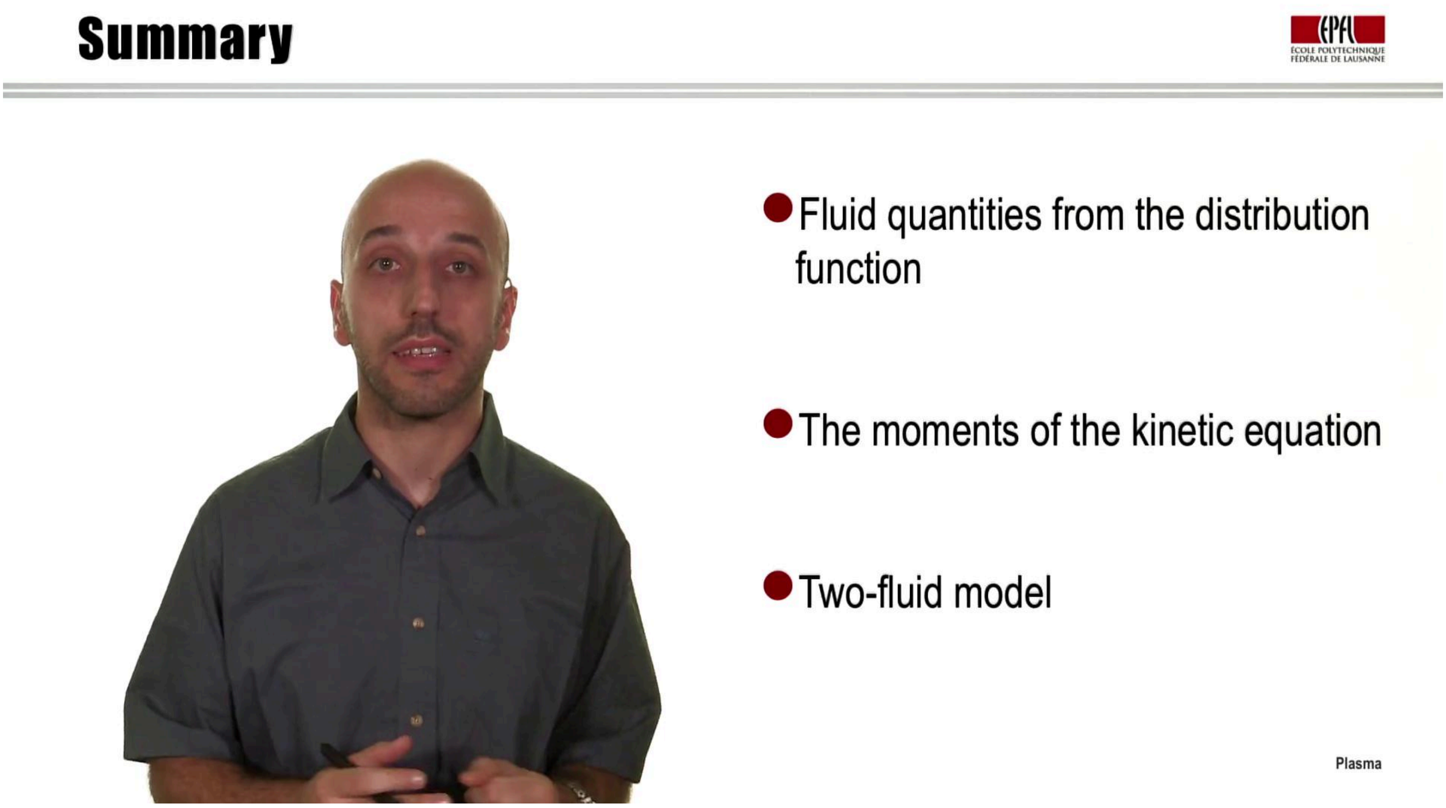


In particular, two-fluid simulations of plasma turbulence are done in the periphery of magnetic fusion devices. In this region, the plasma is relatively cold and one can evolve the dynamics of the plasma, by looking, by evolving as set of fluid equations that describe the evolution of density for electrons and ions of temperature, of velocity, of the plasma contained in this region. What did we do today? What we did was to see how it is possible to derive from the distribution function, fluid quantities that may be of interest to describe our plasma. For example the density, the pressure, the fluid velocity. We pointed out then how it's possible to derive a set of fluid equations. A set of equations that is able to describe the evolution of these fluid quantities. We have done this by taking the moments of the kinetic equation and by using these moments, I have derived a two-fluid model. I have in particular carefully done with you the calculation of the continuity equation, and I have left to you the calculation of the momentum and energy equations. The fluid equations that we came up with describe separately the dynamics of ions and electrons.

Notes

Summary

29m 27s



This is why it's typically called the two-fluid system of equations and it's actually used for a number of applications to study the dynamics of a plasma. What we have just briefly mentioned is the study of the dynamics of the turbulent dynamics of a plasma in the periphery of fusion devices.

Notes

Summary

31m 05s