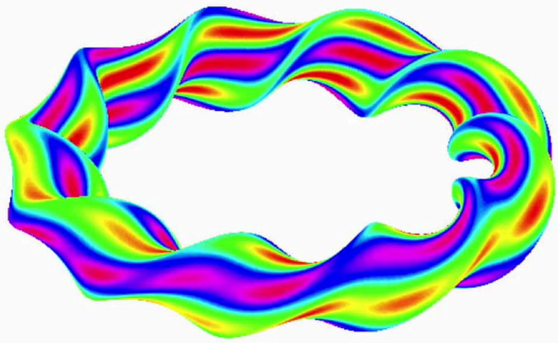
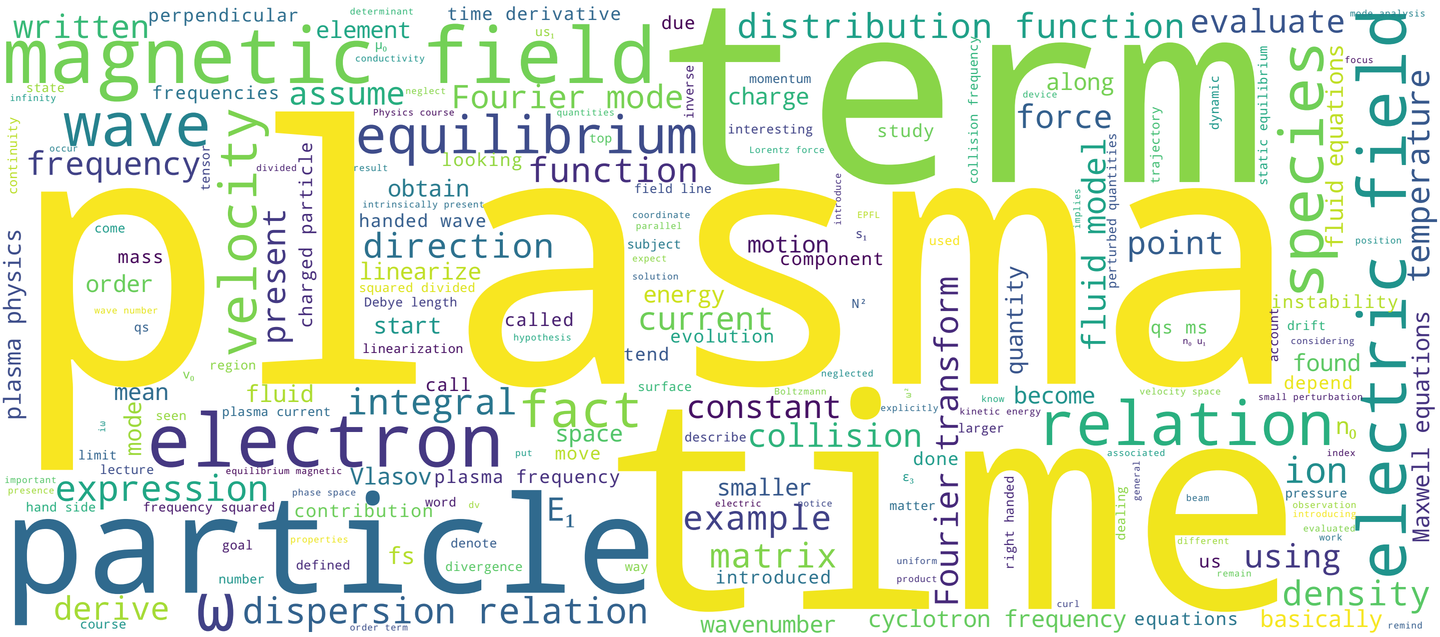


# The two-fluid dispersion relation

## Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

**Lecture 3b**

Paolo Ricci

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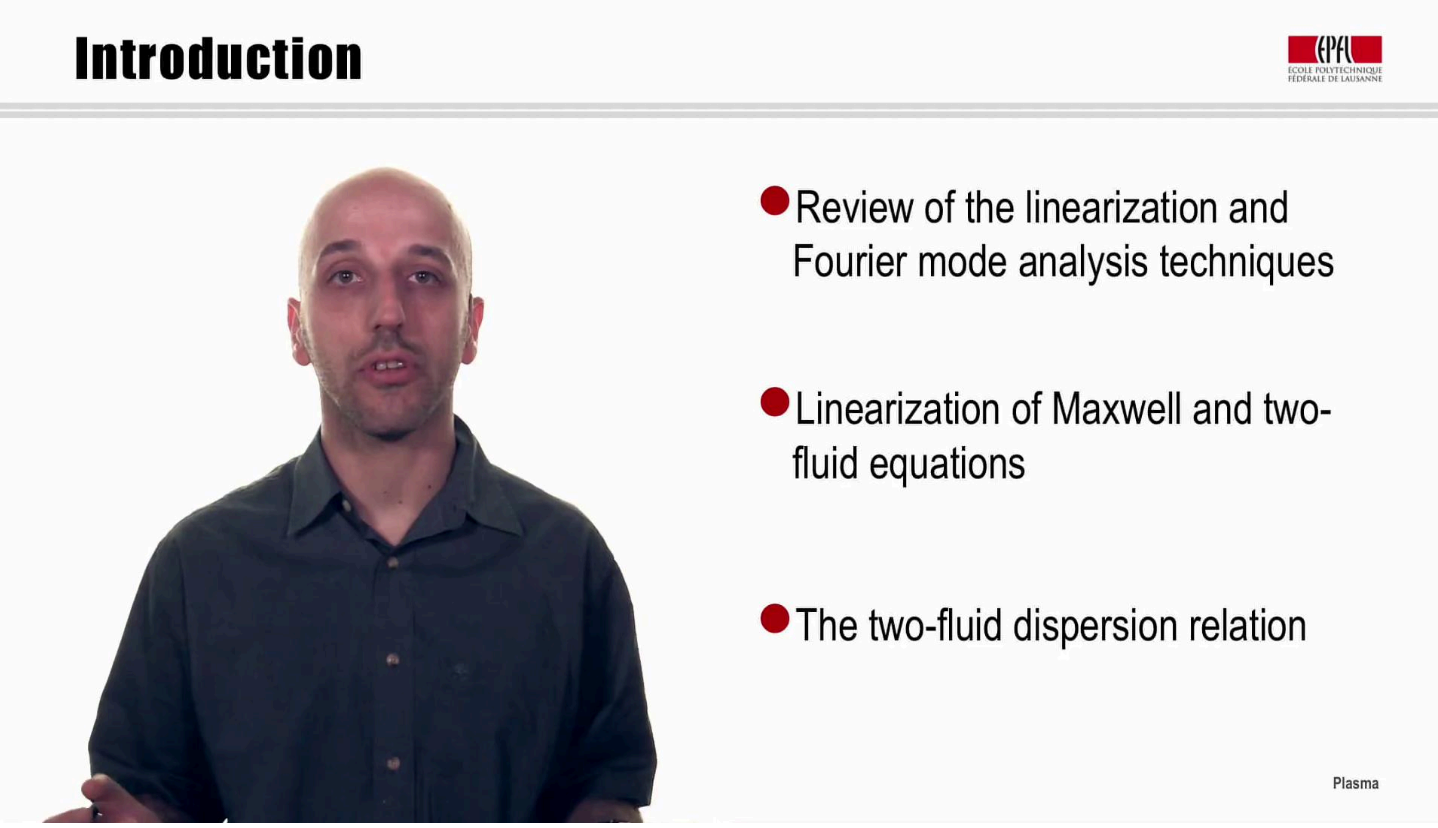
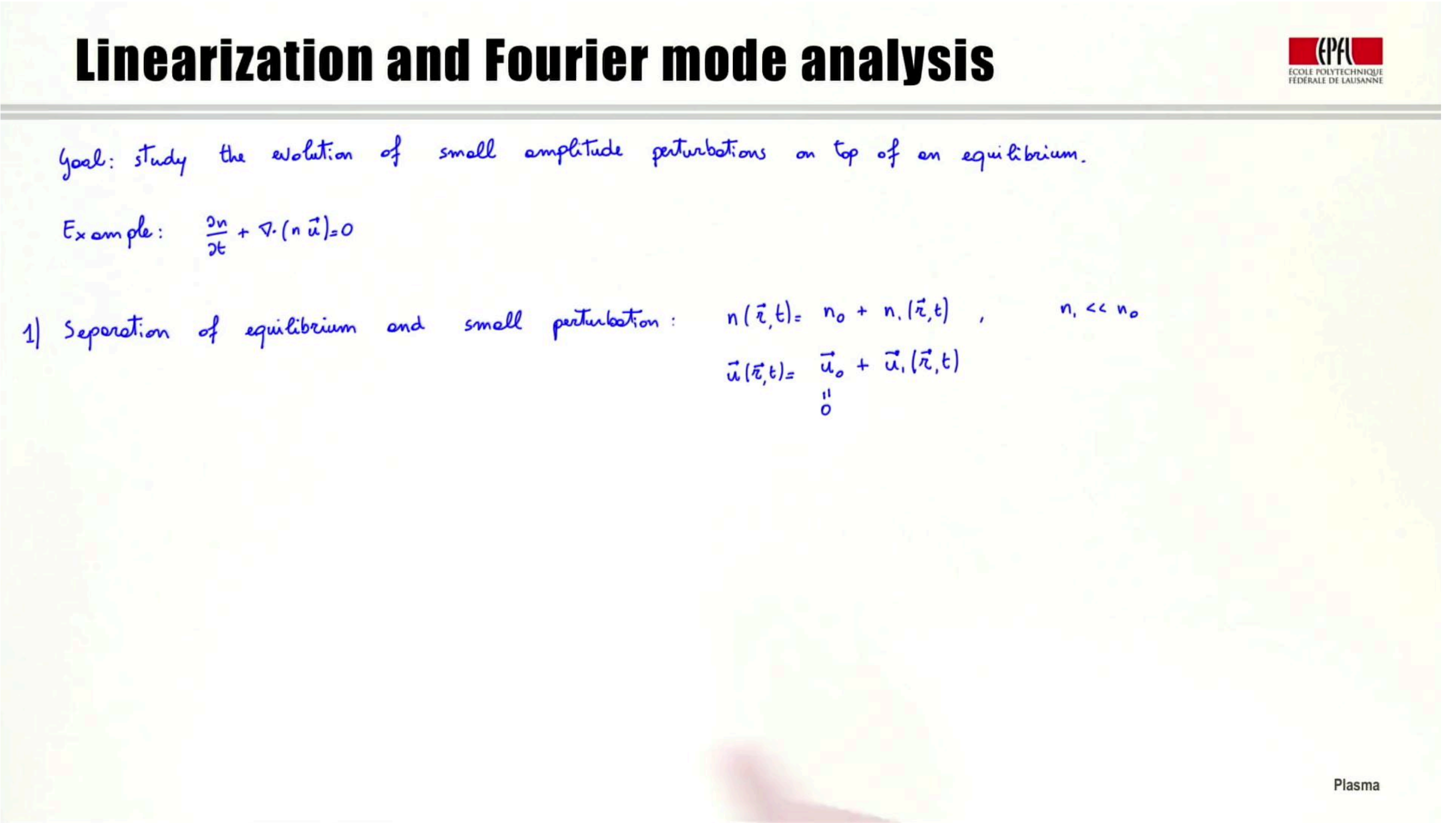



Welcome. Welcome to the Plasma Physics course of EPFL. In the past module, we have deduced the two-fluid equation. In the present one, we will deduce the two-fluid dispersion relation. That is, the relation between the frequency and the wave number of the waves and instability that are described by the two-fluid model. In order to do that, the first thing that I will do with you is to recall the main elements of the linearization and the Fourier mode analysis. Then we will linearize and Fourier transform both Maxwell equations and the two-fluid equations. And at that point, we will deduce together the two-fluid dispersion relation.

Notes

Summary

0m 05s

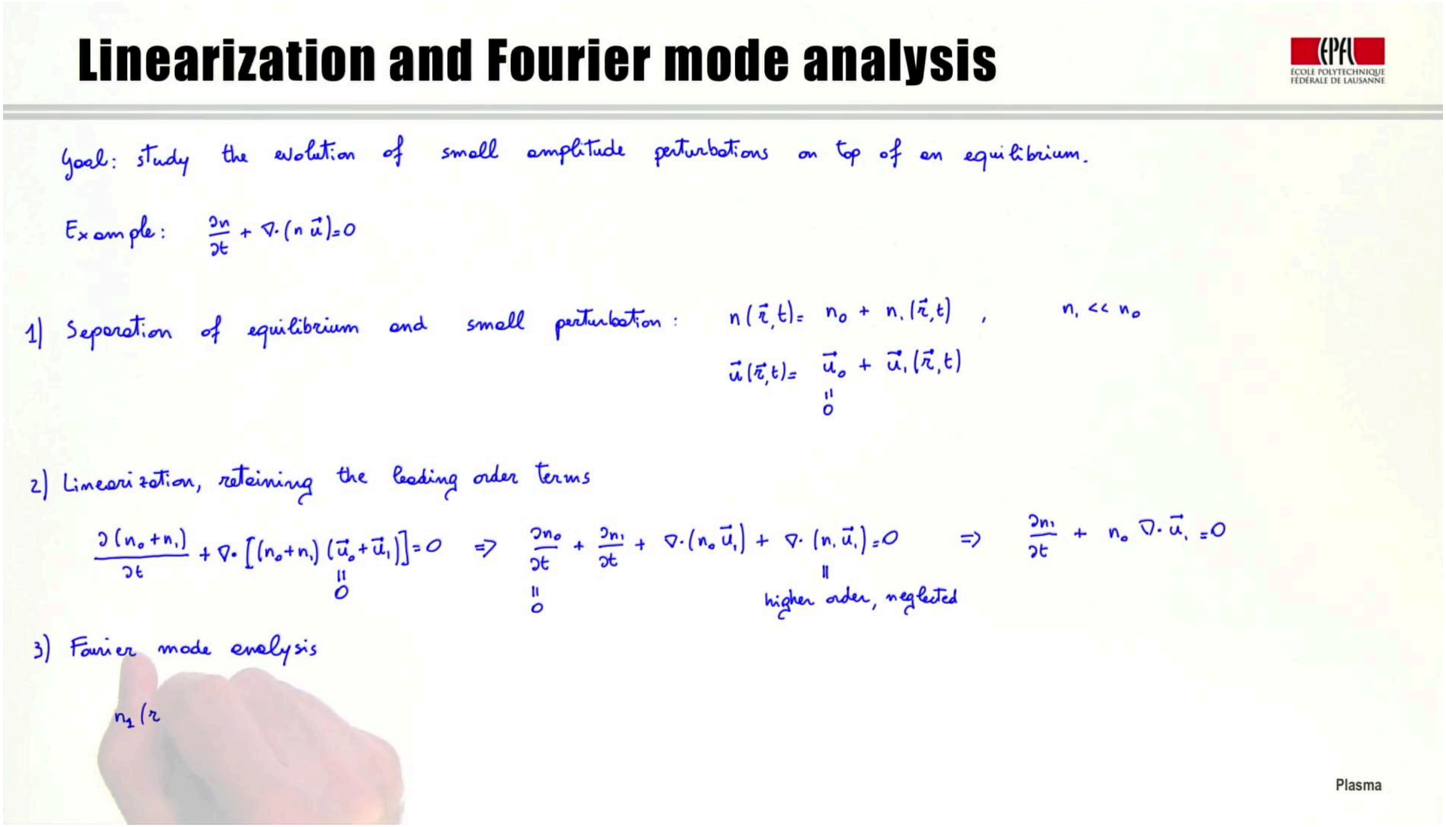



So let's recall together very briefly, the main elements of linearization Fourier mode analysis, something that we have already applied within the kinetic model of plasmas. Our goal is to study the evolution of small amplitude perturbations on top of an equilibrium. Let's take an example to one of the continuity equation. This says that the time derivative of the density is due to the divergence of the flux. In order to study the evolution of small amplitude perturbations on top of the equilibrium, the first thing that we have to do is to separate the equilibrium quantities and the small perturbations. Basically, we will write the density, for example, which is a function of space and time, as the sum of $n_0$, a constant which does not depend on space nor time, plus a perturbation $n_1(r,t)$ and as we are dealing with small perturbations, we will assume that $n_1(r,t)$ is much, much smaller than $n_0$. Similarly for the velocity, we will say that this is given by an equilibrium quantity $u_0$ plus the perturbation $u_1(r,t)$, Now, if we look at the case of static equilibrium, we will have that this $u_0$ will be equal to zero. And therefore, u will only be given by this small perturbed velocity, $u_1(r,t)$.

Notes

Summary

0m 52s

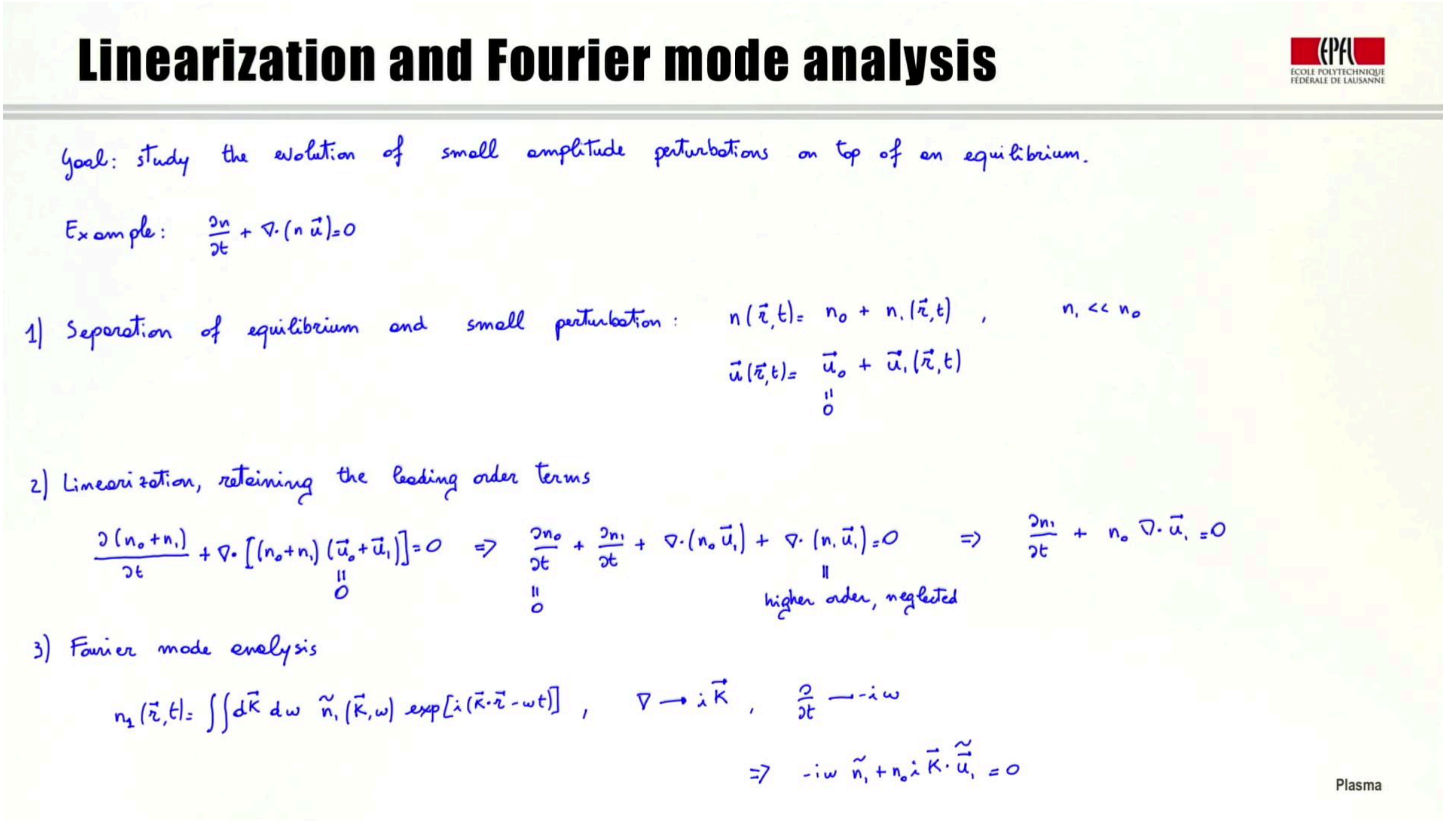



The second step we do is linearizing the equation, retaining only the leading order terms. The idea is to replace n with $n_0 + n_1(r,t)$, and the u with $u_0 + u_1(r,t)$. Now again, in the case of static equilibrium, we can put $u_0$ equal to zero and that equation we have just written becomes $\partial n_0/\partial t$ by expliciting this time derivative, $\partial n_1/\partial t$, and then we will have -taking $u_0 = 0$, $\nabla\cdot(n_0\, u_1)$..., and... $\nabla\cdot(n_1\, u_1) = 0$ Now $n_0$ does not depend on time nor on space, so this is equal to zero. Here we have a $n_1$ term, we have $n_0\, u_1$, and here we have a term that is the product of two quantities that are perturbed quantities, and therefore which is much smaller than the other two terms that we are retaining. It's quadratic in terms of the perturbed quantities while these terms are linear in the perturbed quantities. This is therefore, in other terms, a higher order term, and therefore can be neglected. What we come up with is an equation that relates $\partial n_1/\partial t$ -with taking $n_0$ constant out, ... $n_0\, \nabla\cdot u_1 = 0$. At this point we can apply our Fourier analysis. We write, in fact, that $n_1(r,t)$, which is a function of space and time, is given by the integral of the Fourier modes.

Notes

Summary

2m 33s

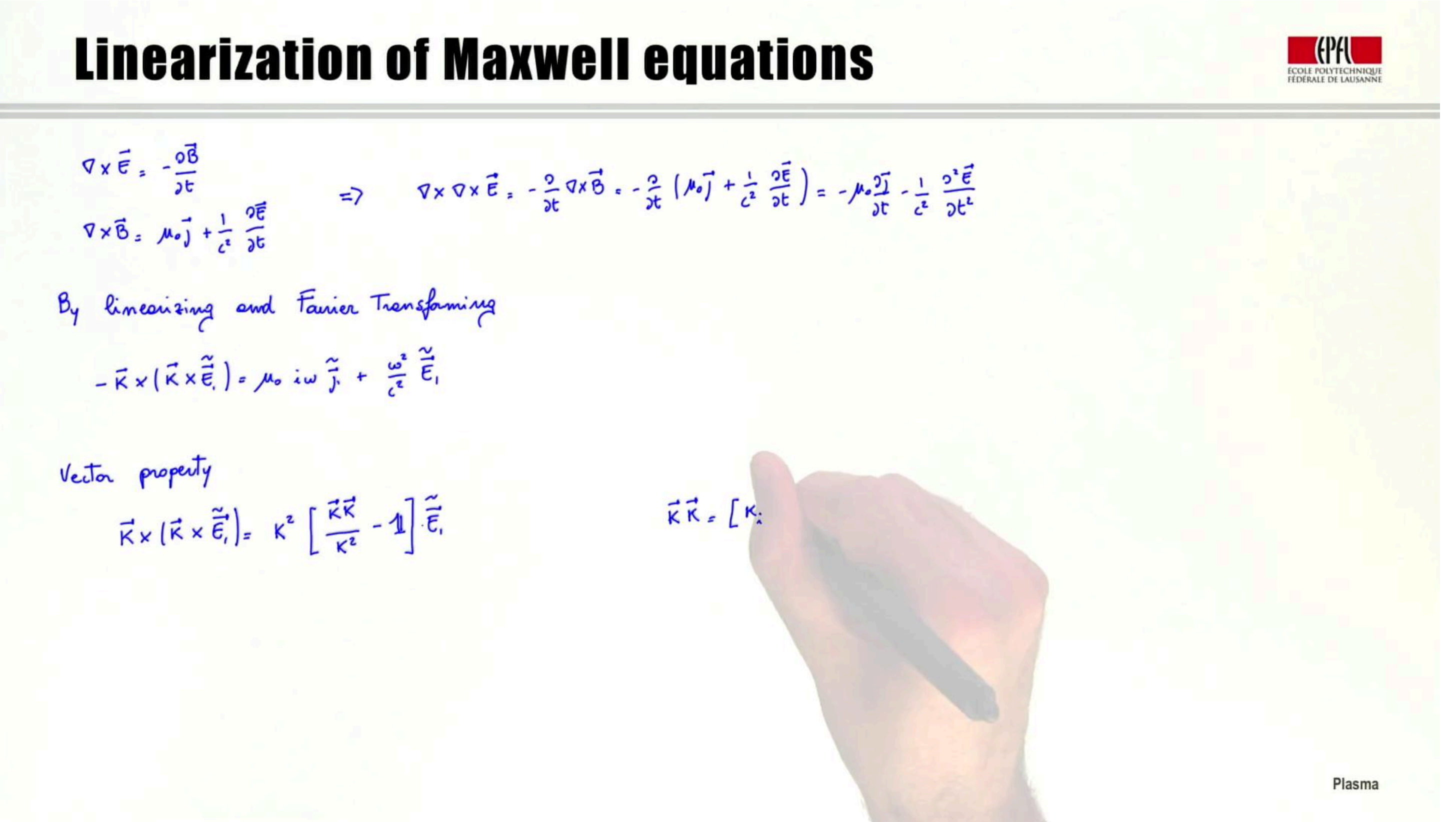



Therefore it will be given by the double integral over all wavenumbers, all frequencies of the Fourier transform, which I will denote with a *tilde*, of $\tilde{n}_1(K,\omega)$ exp {i (K · r - ωt) } The advantage of doing that is that by Fourier transforming, operators like the nabla (∇) operator will become very easy in Fourier space. For example, ∇ becomes the operator i K· and ∂/∂t becomes the -iω operator, it implies that by Fourier transforming this equation and looking at the evolution of each Fourier mode, as each Fourier mode is independent from all the others, we get that $-i\omega\, \tilde{n}_1 + n_0\, i\, K\cdot\tilde{u}_1 = 0$ which is the equation that relates the Fourier modes $\tilde{n}_1(K,\omega)$ and $\tilde{u}_1(K,\omega)$.

Notes

Summary

4m 48s




We are now ready to linearize and Fourier transform both the Maxwell equations and the two-fluid equations. Let's start with the Maxwell equations. The equation that are interesting for us are $\nabla \times E = -\partial B/\partial t$ and $\nabla \times B = \mu_0 j + 1/c^2\ \partial E/\partial t$. From these two equations, taking the curl of the first equation we have that $\nabla\times(\nabla\times E) = -\partial/\partial t(\nabla\times B)$ which is equal to $-\partial/\partial t\ (\mu_0 j + 1/c^2\ \partial E/\partial t)$ Bringing the time derivative in, this is equal to $-\mu_0\partial j/\partial t - 1/c^2\ \partial^2 E/\partial t^2$. This is the equation we want to linearize. The first thing we notice is that equilibrium terms in this equation do not play any role. In fact, here we are dealing with spatial or temporal derivative, and therefore the constant uniform value of the equilibrium do not matter. Therefore, only the perturbed terms enter in this equation and if we Fourier transform this equation and we linearize it, we find $-K\times(K\times\tilde{E}_1) = \mu_0\ i\omega\ \tilde{j}_1 + \omega^2/c^2\ \tilde{E}_1$ Here we have an ugly beast, the $K\times(K\times\tilde{E}_1)$. Actually this expression can be simplified by using vector properties that I will remind in a few seconds. $K\times(K\times\tilde{E}_1) = K^2$ - the modulus of $K^2$, and then the tensor $KK/K^2$ minus the unity dyad, times $\tilde{E}_1$. What do I mean with the KK tensor?

Notes

Summary

6m 14s

## Linearization of Maxwell equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \quad \nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \times \vec{B} = -\frac{\partial}{\partial t}\left(\mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\right) = -\mu_0 \frac{\partial \vec{j}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

By linearizing and Fourier Transforming

$$-\vec{k} \times (\vec{k} \times \tilde{\vec{E}}_1) = \mu_0 \, i\omega \, \tilde{\vec{j}}_1 + \frac{\omega^2}{c^2}\tilde{\vec{E}}_1$$

Vector property

$$\vec{k} \times (\vec{k} \times \tilde{\vec{E}}_1) = k^2 \left[\frac{\vec{k}\vec{k}}{k^2} - \mathbb{1}\right] \cdot \tilde{\vec{E}}_1$$

$$\vec{k}\vec{k} = [k_i k_j] = \begin{bmatrix} k_x^2 & k_x k_y & k_x k_z \\ k_x k_y & k_y^2 & k_y k_z \\ k_x k_z & k_y k_z & k_z^2 \end{bmatrix} \qquad \mathbb{1} = [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-\frac{c^2 k^2}{\omega^2}\left[\frac{\vec{k}\vec{k}}{k^2} - \mathbb{1}\right] \cdot \tilde{\vec{E}}_1 = \frac{i}{\varepsilon_0 \omega}\tilde{\vec{j}}_1 + \tilde{\vec{E}}_1$$

$N^2$ (index of refraction)

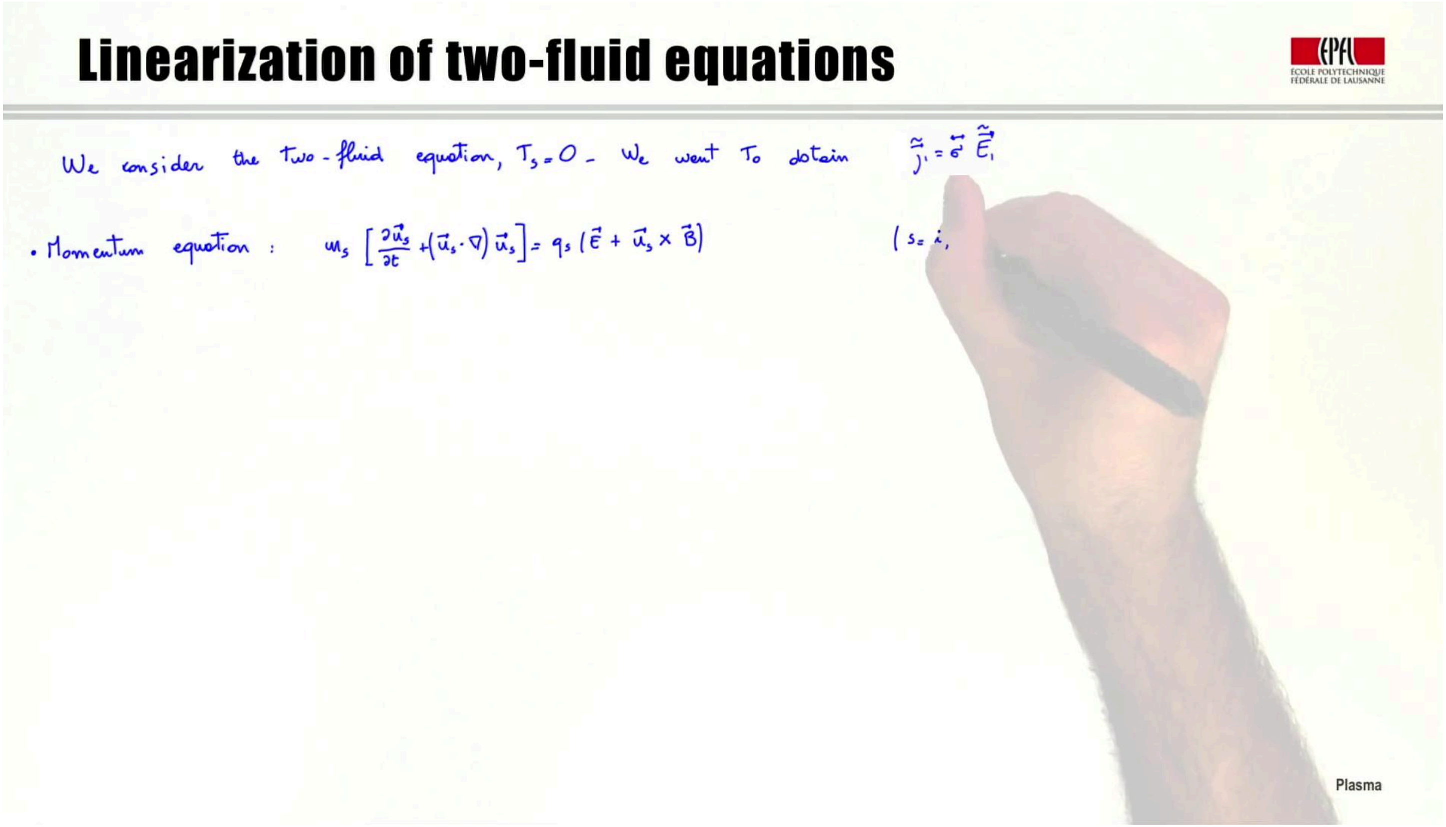
Plasma

Component by component, it's the matrix that has the i j element, the value K i K j, and which we can write using the XYZ coordinates as Kx, KxKy, KxKz, KxKy, Ky² and so on. And the unit dyad is the matrix that has the delta Kronecker as the i j element that is the identity matrix. And now, by using these vector properties we can expand this $K\times(K\times\tilde{E}_1)$, and by multiplying $(c^2/\omega^2)$, we find $-c^2 K^2/\omega^2$ and then this term here, times $E_1$ Fourier transformed [i.e. $\tilde{E}_1$]. Then here, recalling that $(\varepsilon_0\mu_0)^{-1} = c^2$, this can be written as $i/(\varepsilon_0\ \omega)\ \tilde{j}_1 + \tilde{E}_1$ This quantity here is an dimensionless number and we will denote with $N^2$, and this is actually the index of refraction. What have we found? We've found the relation between the electric field and the current. Somehow we have to work from the fluid side to try to get a relation between the current and the electric field, -a second relation, such that by using both equations, we will be able to derive the dispersion relation, the relation that will relate ω, the [angular] frequency of the waves, or the instability that we are analyzing, and their wavenumber.

Notes

Summary

9m 10s

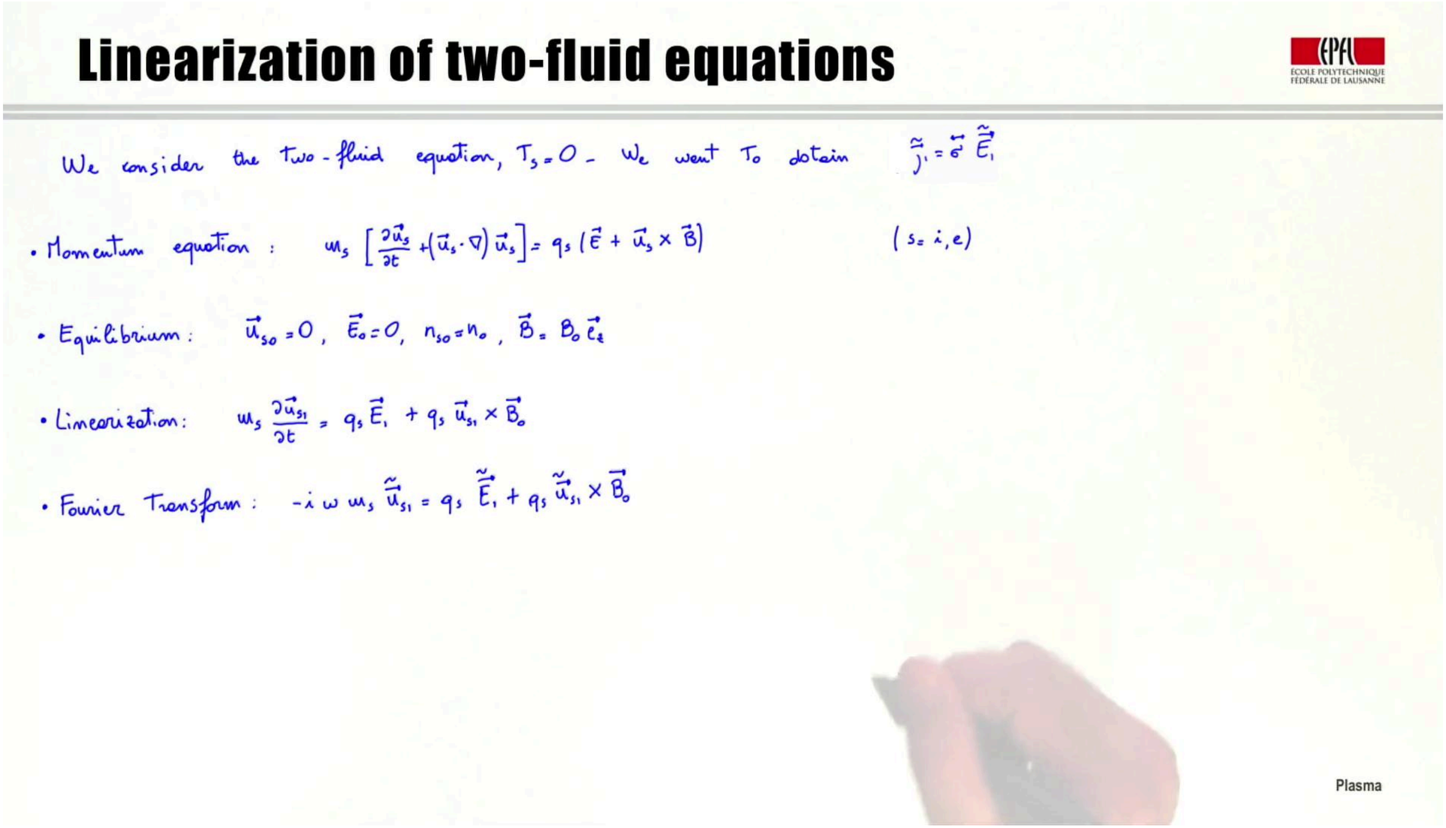



So now our goal is to derive a relation between the plasma current and the electric field by using the fluid equations. We will focus on a relatively simple case of the two-fluid model: the one of the cold plasma. We will therefore neglect all the terms related to the finite ion temperature. In other words, we will put the temperature of both electrons and ions equal to zero. This will considerably simplify our study without losing much of the physics that is behind waves and the instability described by the two-fluid model. So we consider the two-fluid equations in the limit of temperature equal to zero. Our goal is to obtain a relation between $\tilde{j}_1$, the plasma current, and the electric field $\tilde{E}_1$. We expect that this relation will be a linear relation that will pass through σ, that is the conductivity, that in general can be a tensor. That is, the mass of the species s times the convective derivative of us, this has to be equal to the electric force plus the Lorentz force. We will neglect the pressure term according to the hypothesis of having the plasma temperature equal to zero. And this equation will be written for the species that are present in our plasma that we will assume to be the ions and electrons.

Notes

Summary

11m 32s

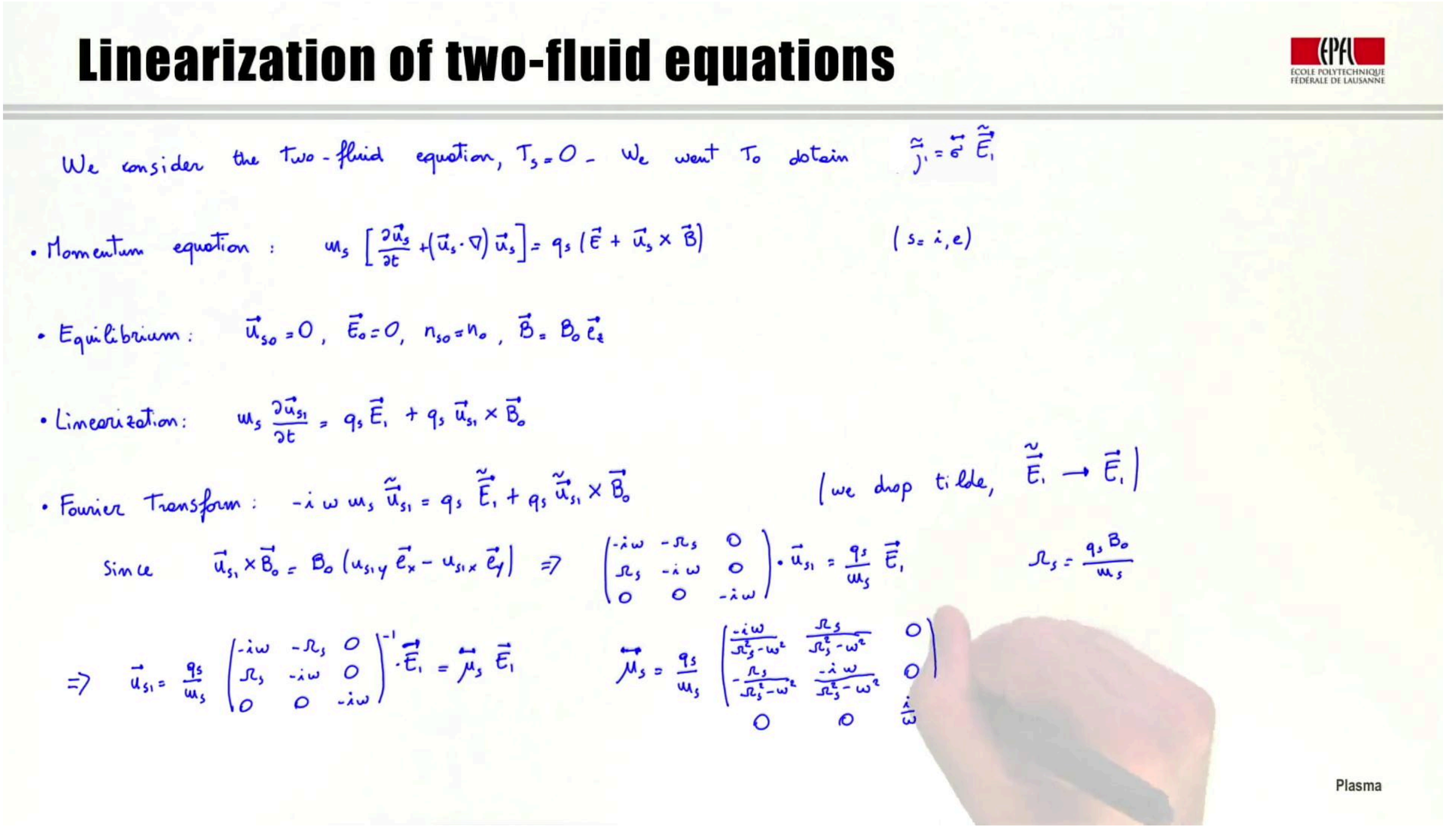



We will assume that we are considering a static equilibrium, therefore with $us_0 = 0$, that there is no electric field at equilibrium. So basically, the plasma will be neutral, in steady state, so the species will have the same density, and therefore the equilibrium electric field will be equal to zero. But we will assume there are at least the presence of an equilibrium magnetic field B that we assume to be uniform and we will choose our coordinate system in such a way that B becomes aligned to the ez direction. Now, we can linearize our equation. This term is higher order, a second order in perturbation only this one, the ∂/∂t remains. Therefore, we have ms $\partial \tilde{u}s_1/\partial t$ is equal to the perturbed part of the forces, which will be given by qs $\tilde{E}_1$ + qs... and then the Lorentz force. And it's now time to Fourier transform this equation. As usual, ∂/∂t will become a -iω term and therefore the Fourier component of $\tilde{u}s_1$ and $\tilde{E}_1$ we will have -iω ms $\tilde{u}s_1$ = qs $\tilde{E}_1$ + qs ($\tilde{u}s_1$ x $B_0$) From this moment on, as we will be only dealing with Fourier components $\tilde{u}s_1$ and $\tilde{E}_1$, **let's drop the tilde (~) notation for simplicity** .

Notes

Summary

13m 23s




Therefore, for the notation we will, for example, denote $\tilde{E}_1$, Fourier transformed simply as $E_1$, [$\tilde{E}_1(K,\omega) \rightarrow E_1(K,\omega)$] but we will keep in mind that this is the Fourier transform of $E_1$, and more particularly, the (K,ω) mode. Now since ($us_1$ x $B_0$) can be written component by component as $B_0$ ($us_1y$ ex - $us_1x$ ey), this equation can be written as -iω, (i.e. the ∂/∂t term), that will be a diagonal term, then we can bring the contribution of this term on the left-hand side and what we will find is a -Ωs [cyclotron frequency for species s] and Ωs here, 0, 0..... • $us_1$ = qs/ms $E_1$ where we leave on the right-hand side the $E_1$ term. Just as a reminder, the Ωs that we have introduced here, the cyclotron frequency, is equal to qs $B_0$ / ms. Therefore, we can explicitly write $us_1$ by inverting this matrix here. $us_1$ will be equal qs ms... times the inverse of this matrix. The inverse of the matrix times qs ms can be defined as a tensor we call μs, ...μs $E_1$...., where this tensor, μs is equal to qs ms.... And if you evaluate the inverse of this matrix -- thing that is a bit boring, but it's feasible, you get - iω/ ($\Omega s^2$-$\omega^2$), Ωs / ($\Omega s^2$-$\omega^2$), 0, and so on.

Notes

Summary

15m 22s

# Linearization of two-fluid equations

We consider the two-fluid equation, $T_s = 0$. We want to obtain $\tilde{\vec{j}}_1 = \overleftrightarrow{\sigma}\,\tilde{\vec{E}}_1$

- Momentum equation: $m_s\left[\frac{\partial \vec{u}_s}{\partial t} + (\vec{u}_s\cdot\nabla)\vec{u}_s\right] = q_s(\vec{E} + \vec{u}_s\times\vec{B})$ $\quad (s = i, e)$
- Equilibrium: $\vec{u}_{s0} = 0$, $\vec{E}_0 = 0$, $n_{s0} = n_0$, $\vec{B} = B_0\vec{e}_z$
- Linearization: $m_s\frac{\partial \vec{u}_{s1}}{\partial t} = q_s\vec{E}_1 + q_s\vec{u}_{s1}\times\vec{B}_0$
- Fourier Transform: $-i\omega m_s\tilde{\vec{u}}_{s1} = q_s\tilde{\vec{E}}_1 + q_s\tilde{\vec{u}}_{s1}\times\vec{B}_0$ $\quad$ (we drop tilde, $\tilde{\vec{E}}_1 \to \vec{E}_1$)

Since $\vec{u}_{s1}\times\vec{B}_0 = B_0(u_{s1y}\vec{e}_x - u_{s1x}\vec{e}_y)$ $\Rightarrow$ $\begin{pmatrix} -i\omega & -\Omega_s & 0 \\ \Omega_s & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix}\cdot\vec{u}_{s1} = \frac{q_s}{m_s}\vec{E}_1$ $\qquad \Omega_s = \frac{q_s B_0}{m_s}$

$$\Rightarrow \vec{u}_{s1} = \frac{q_s}{m_s}\begin{pmatrix} -i\omega & -\Omega_s & 0 \\ \Omega_s & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix}^{-1}\cdot\vec{E}_1 = \overleftrightarrow{\mu}_s\vec{E}_1 \qquad \overleftrightarrow{\mu}_s = \frac{q_s}{m_s}\begin{pmatrix} \frac{-i\omega}{\Omega_s^2-\omega^2} & \frac{\Omega_s}{\Omega_s^2-\omega^2} & 0 \\ -\frac{\Omega_s}{\Omega_s^2-\omega^2} & \frac{-i\omega}{\Omega_s^2-\omega^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix}$$

mobility Tensor

$$\vec{j}_1 = \sum_s q_s n_{s0}\vec{u}_{s1} = \sum_s q_s n_{s0}\overleftrightarrow{\mu}_s\cdot\vec{E}_1 = \overleftrightarrow{\sigma}\,\vec{E}_1$$

$\overleftrightarrow{\sigma}$ : conductivity Tensor

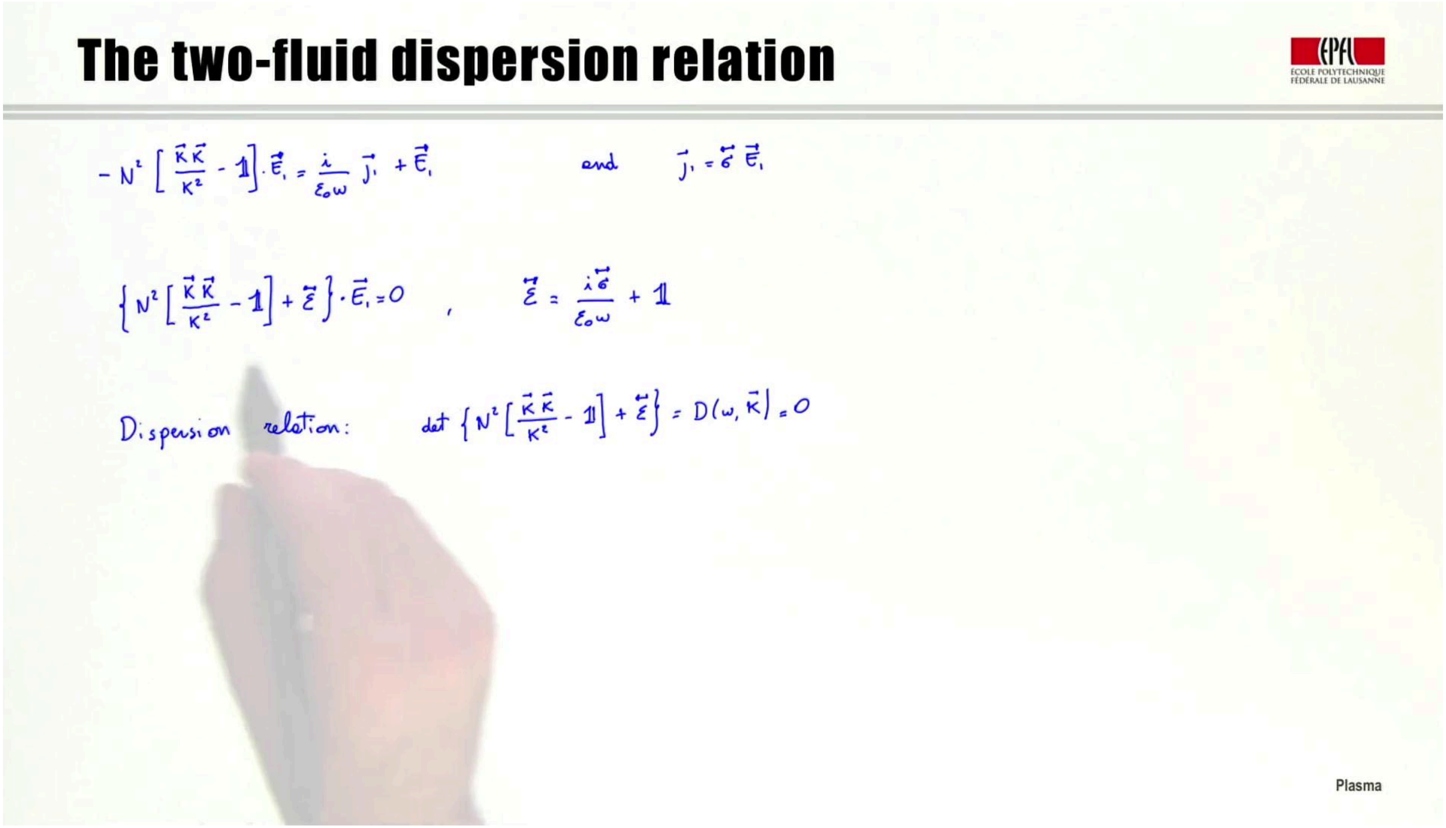
Plasma

We will call this tensor the *mobility tensor*, as it will tell us how a particle will start to move a subject to a certain electric field. Actually now it's possible to evaluate the perturbed current in the plasma because the perturbed current -as the equilibrium flow is equal to zero, will be given by the sum over all species of qs [times] the equilibrium density [ $ns_0$, times] $us_1$. This is the only terms that remains from the linearization of the current, and now, we can write $us_1$ in terms of the mobility tensor and the electric field,... times $E_1$. This quantity here is actually a tensor σ $E_1$. We have, in fact, obtained exactly the relation we were looking for. And σ, by the way, is the *conductivity tensor*.

Notes

Summary

17m 58s




We have now all the elements to derive the dispersion relation that is the function that gives us the relation between ω and K, the frequency and the wave number of the modes that are intrinsically present in the system. What we have to do is in fact to inject that the expression of the plasma current that we have derived within the two-fluid model into the Maxwell equations. In fact, by the linearization of the Maxwell equation, we have obtained that $-N^2$ (the index of refraction)... times this tensor here... $\cdot E_1$ is equal to the contribution of the plasma current and the electric field. And from the fluid equation we have a relation between $j_1$ and $E_1$. We can plug this expression here, and what we obtain is... where we have introduced the ε tensor which will take into account the contribution from the conductivity, and this term here. And now we have all the elements to derive-- to write down the equation for the dispersion relation of the two-fluid model. In fact, the modes that have a frequency and a wavenumber such that the determinant of this matrix here which we can call $D(\omega,K) = 0$, so the modes whose frequency and wavenumber are such that the determinant of this matrix is equal to zero, then these are intrinsically present modes, i.e.

Notes

Summary

19m 07s

# The two-fluid dispersion relation

$$-N^2\left[\frac{\vec{K}\vec{K}}{K^2}-\mathbb{1}\right]\cdot\vec{E}_1=\frac{i}{\varepsilon_0\omega}\vec{j}_1+\vec{E}_1 \quad \text{and} \quad \vec{j}_1=\overleftrightarrow{\sigma}\vec{E}_1$$

$$\left\{N^2\left[\frac{\vec{K}\vec{K}}{K^2}-\mathbb{1}\right]+\overleftrightarrow{\varepsilon}\right\}\cdot\vec{E}_1=0 \quad , \quad \overleftrightarrow{\varepsilon}=\frac{i\overleftrightarrow{\sigma}}{\varepsilon_0\omega}+\mathbb{1}$$

Dispersion relation: $\det\left\{N^2\left[\frac{\vec{K}\vec{K}}{K^2}-\mathbb{1}\right]+\overleftrightarrow{\varepsilon}\right\}=D(\omega,\vec{K})=0$

$$\overleftrightarrow{\varepsilon}=\begin{pmatrix}\varepsilon_1 & -i\varepsilon_2 & 0\\ i\varepsilon_2 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_3\end{pmatrix}$$

$$\varepsilon_1=1+\sum_s\frac{\omega_{ps}^2}{\Omega_s^2-\omega^2} \qquad \omega_{ps}^2=\frac{n_{s0}\,q_s^2}{m_s\,\varepsilon_0}$$

$$\varepsilon_2=-\sum_s\frac{\Omega_s}{\omega}\frac{\omega_{ps}^2}{\Omega_s^2-\omega^2}$$

$$\varepsilon_3=1-\sum_s\frac{\omega_{ps}^2}{\omega^2}$$

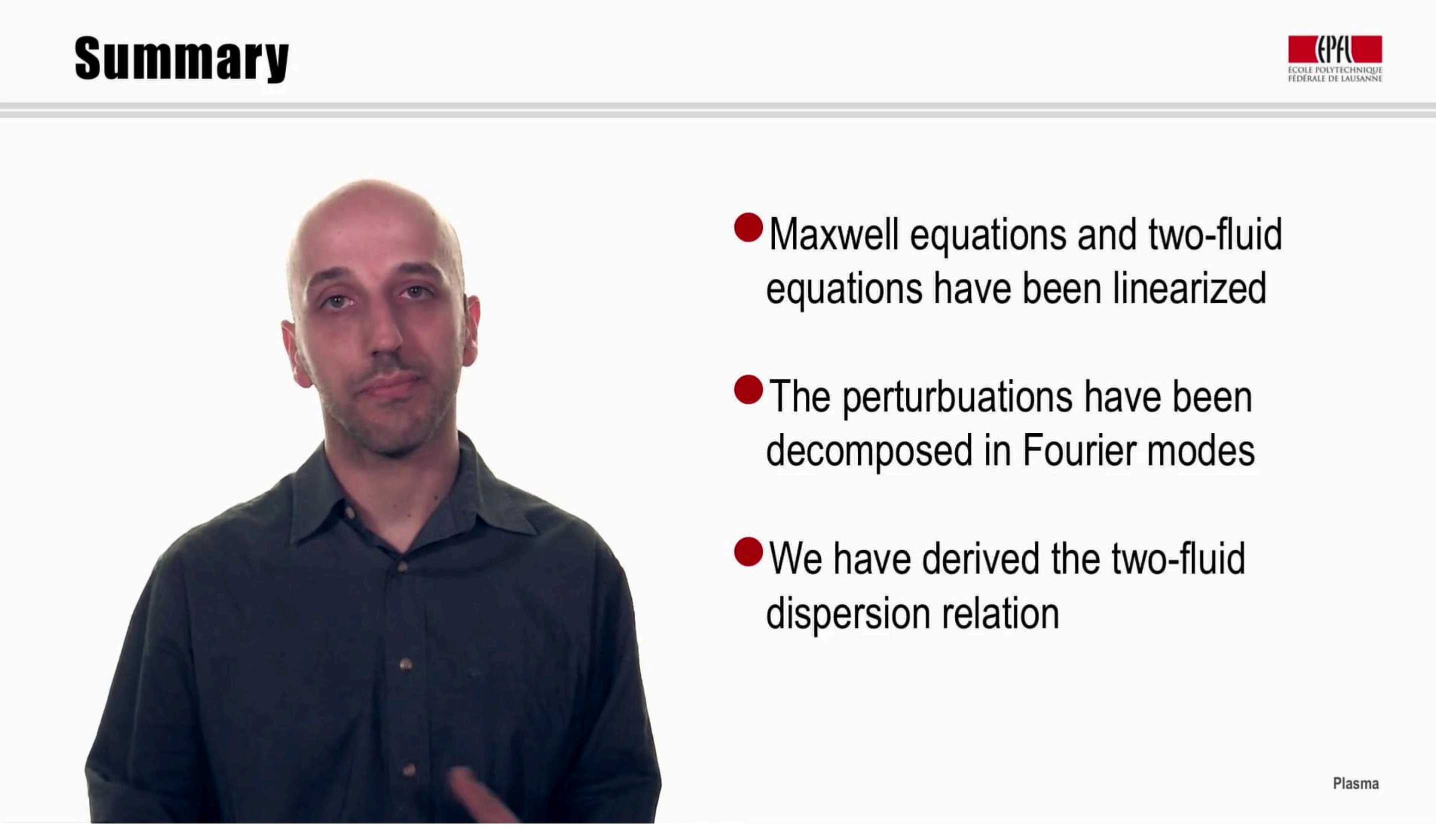
Plasma

modes which can be present in our system, and they can have an $E_1$ different from zero. Solving this equation will actually tell us what is the relation between the frequency and the wavenumber of these modes. Before concluding, let me just write explicitly what is the value of ε by introducing the conductivity that we have found. This can be written by introducing three parameters: $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$, where $\varepsilon_1$ is defined as one plus the sum over all species of the plasma frequency of species s, [squared], divided by the cyclotron frequency [squared] minus $\omega^2$. I just remind you that the plasma frequency squared of species *s* is equal to $ns_0$, the equilibrium density, [times] the charge squared, divided by the mass [ms] and $\varepsilon_0$. $\varepsilon_2$ is equal to minus the sum [over species] of the cyclotron frequency over the frequency of the wave, [times the] plasma frequency [squared] divided by the difference between the cyclotron frequency squared and the frequency of the wave squared. And $\varepsilon_3$, finally, will be given by 1 minus the [sim over species] of plasma frequencies squared divided by $\omega^2$.

Notes

Summary

21m 05s




To summarize, in the present module, what we have done is to linearize and Fourier transform the Maxwell equations and the two-fluid equations, in order to come up with a dispersion relation, a relation between the frequency of the wave number of the modes, of the waves and instability that are intrinsically present in our system. In the next module, we will consider exactly what are the waves and instabilities present in our system and we will look at their physical properties.

Notes

Summary

22m 46s