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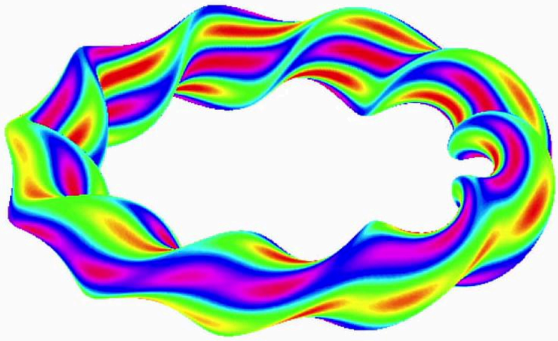
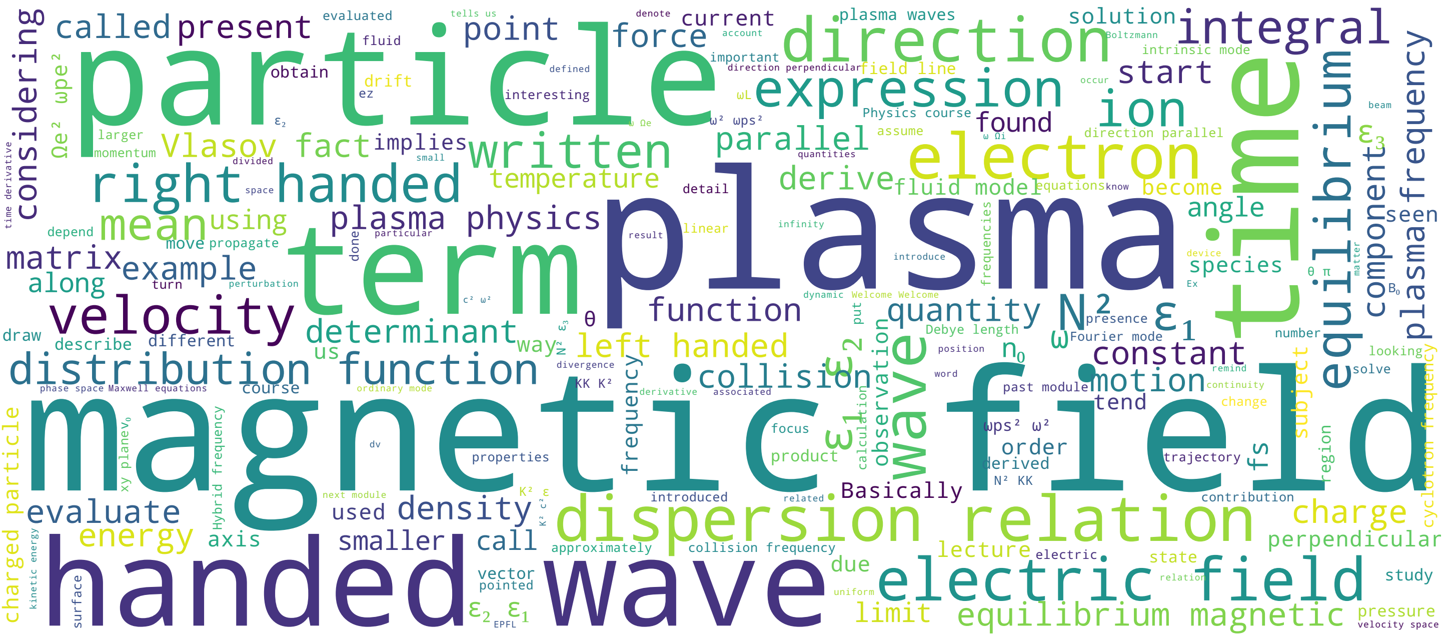


The two-fluid waves

Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

Lecture 3c

Paolo Ricci



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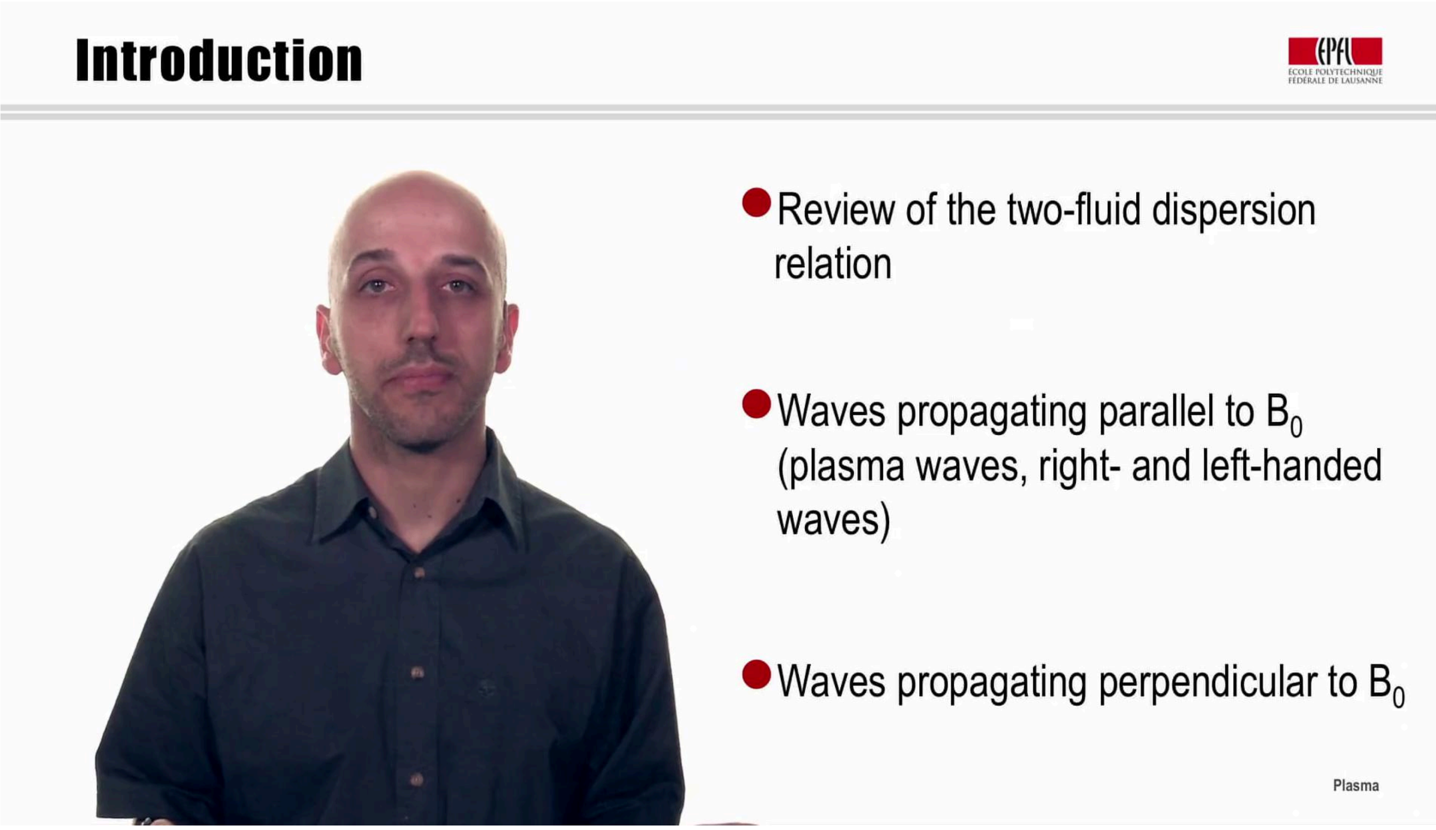
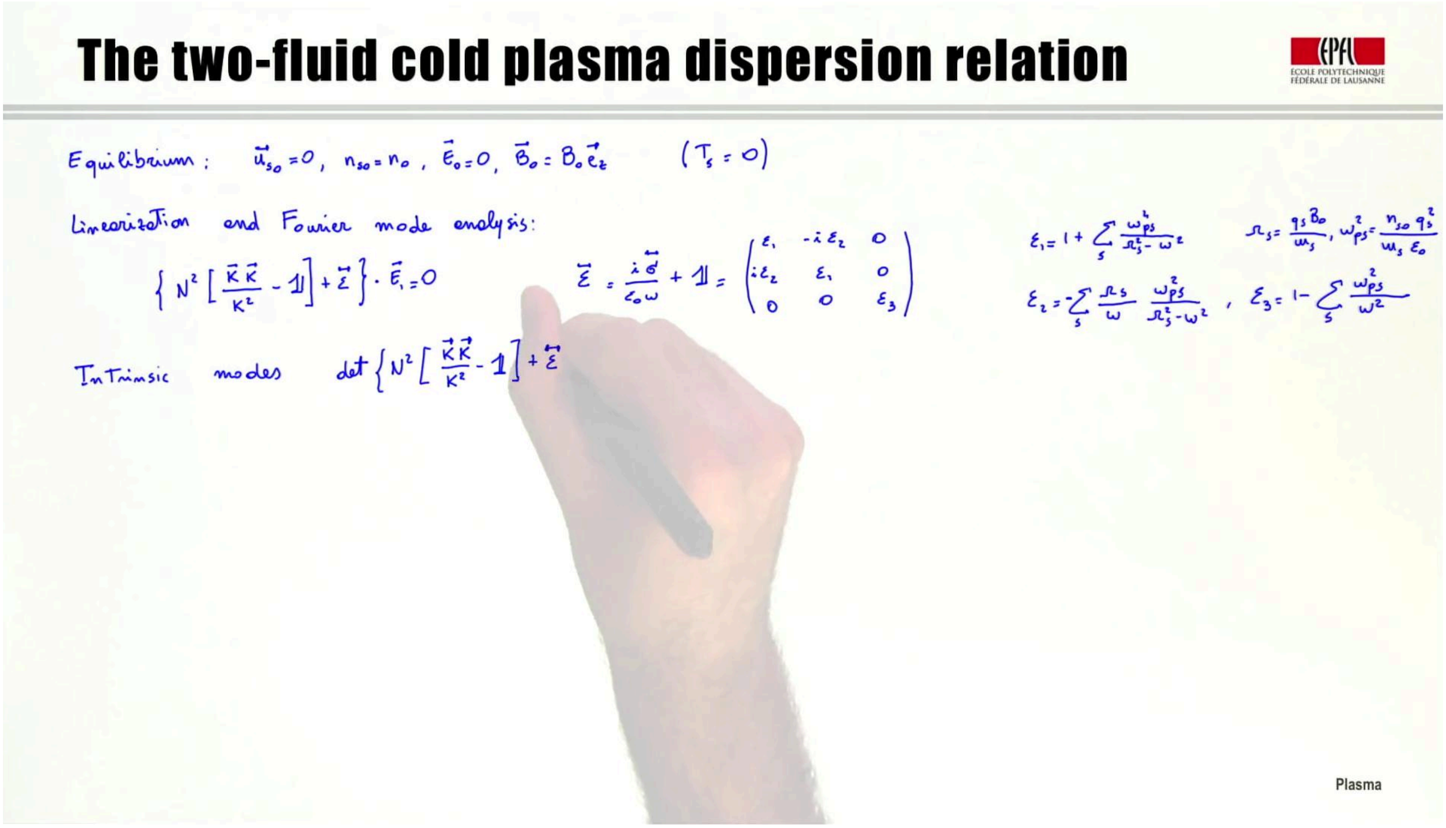


Welcome! Welcome to the Plasma Physics course of EPFL. In the past module, we have derived the two-fluid dispersion relation. If you want, the relation between the frequency and the wavenumber of the modes that are intrinsically present in a plasma, according to the two-fluid description. Now it's time to look more into details at what are these modes, what are the waves that are actually described by the two-fluid model. For this purpose I will very briefly recall the two-fluid dispersion relation, and then I will show you that there are waves that are propagating both in the parallel and the perpendicular direction to the equilibrium magnetic field.

Notes

Summary

0m 06s

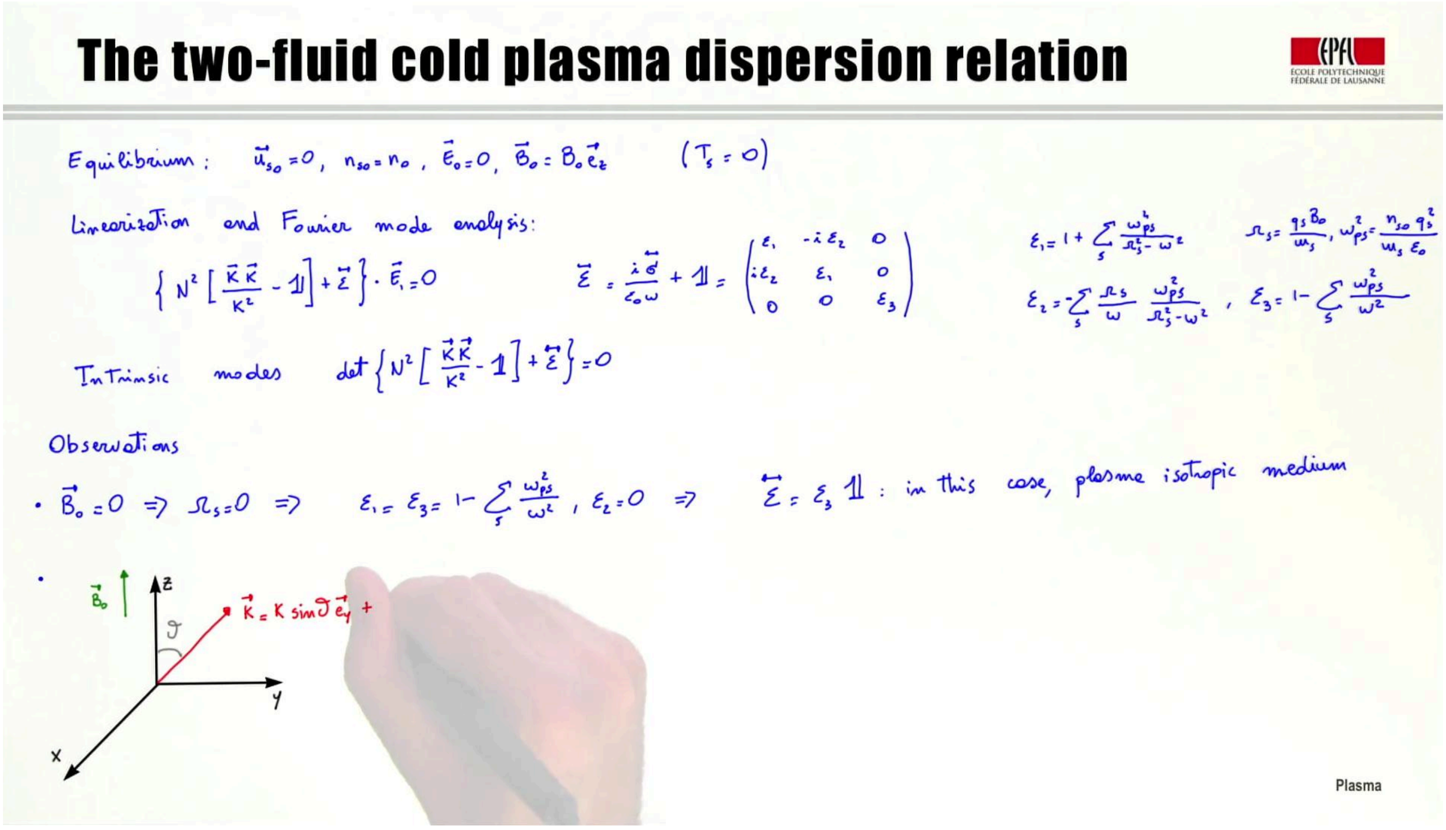


So to start, what is the equilibrium we are considering? We are considering a static equilibrium, therefore, the equilibrium velocity of the plasma species equal to zero. We are considering a plasma that is neutral, of course, in the equilibrium state, and therefore where the equilibrium electric field is equal to zero and we will also consider an equilibrium magnetic field that is uniform and static and we will turn our coordinates system in such a way that the equilibrium magnetic field becomes parallel to the ez vector. As we are considering the cold plasma dispersion relation, I just remind you that we will assume that both the equilibrium and perturbation of the temperature are equal to zero. The linearization and Fourier mode analysis of the two-fluid equations and the Maxwell equations show us that $(N^2 [KK/K^2 - \mathbb{1}] + \varepsilon) \cdot E_1 = 0$ where ε is given by $\varepsilon = i\sigma/(\varepsilon_0\, \omega) + \mathbb{1}$ and as we have seen in the past module, this can be written as... introducing 3 functions ε_1, ε_2 and ε_3, where $\varepsilon_1 = 1 + \sum \omega ps^2/(\Omega s^2 - \omega^2)$ Just to remind you 2 definitions, $\Omega s = qs\, B_0/ms$: cyclotron frequency and the plasma frequency: $\omega ps^2 = ns_0\, qs^2/(\varepsilon_0\, ms)$, and ε_2 is given by $\varepsilon_2 = - \sum \Omega s/\omega\ \omega ps^2/(\Omega s^2 - \omega^2)$ and $\varepsilon_3 = 1 - \sum \omega ps^2/\omega^2$ The intrinsic modes present in the system are the ones for which the determinant of this matrix is equal to zero.

Notes

Summary

0m 56s

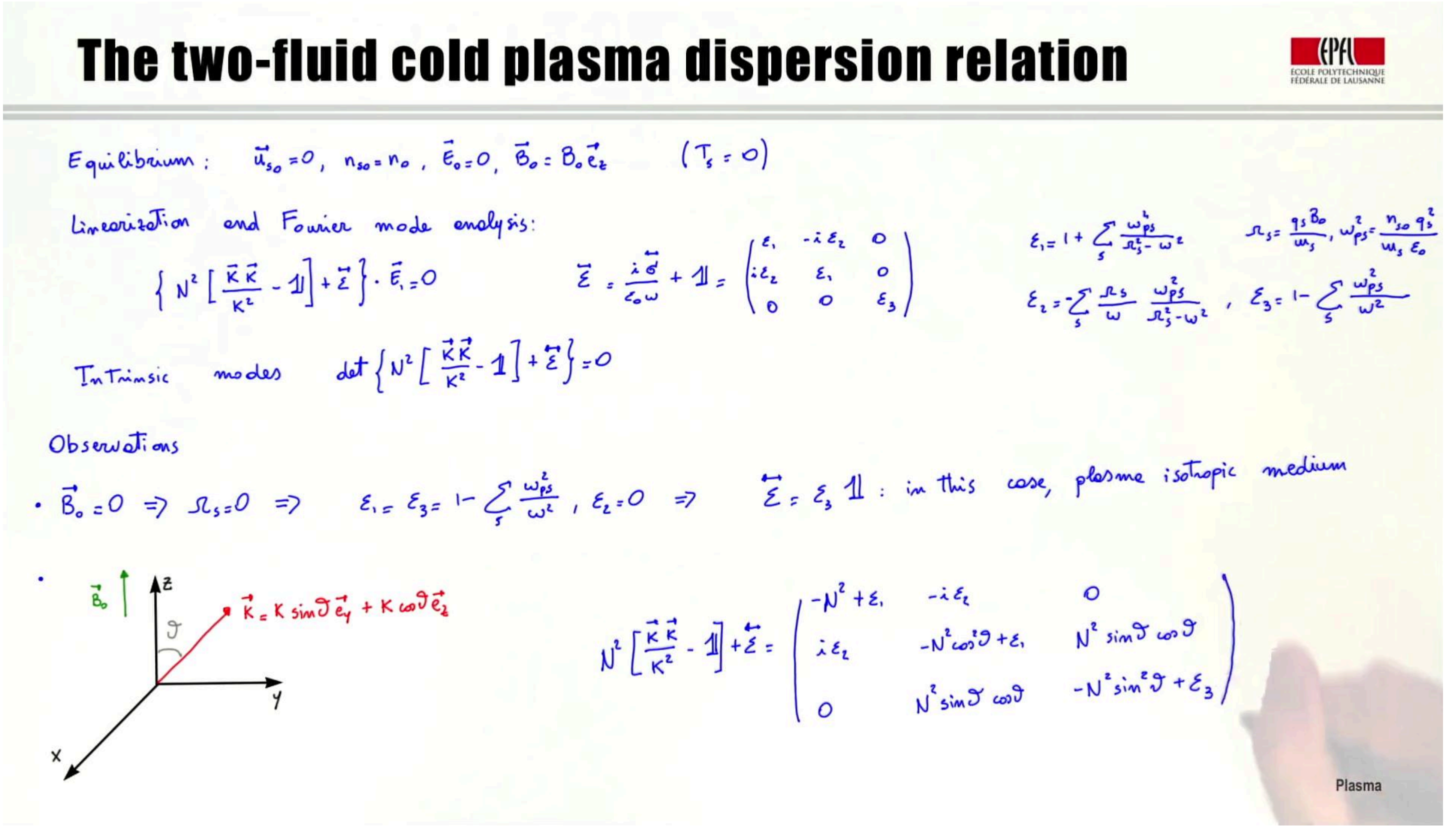


A few observations. First of all, by looking at this dispersion relation we can observe that the plasma seems to be an anisotropic medium. The direction, the third direction, the z-direction along the magnetic field is actually very different than what happens in the plane perpendicular to it. Well let's give a look. Is this due to the presence of the equilibrium magnetic field? Let's take the limit of equilibrium magnetic field equal to zero which implies that the cyclotron frequency is equal to zero and this implies that ε_1, if $\Omega s^2 = 0$, ε_1 becomes equal to ε_3 i.e. equal to $1 - \sum \omega ps^2/\omega^2$ while ε_2, being proportional to Ωs, becomes equal to zero. Well, the off-diagonal terms of ε, therefore, are equal to zero and ε_1 becomes equal to ε_3 It means that $\varepsilon = \varepsilon_3\,\mathbb{1}$ In this case, the plasma is an isotropic medium. Second observation that I would like to do: well let's look at our space, xyz. We have that the equilibrium magnetic field is in the direction of the z-axis. We can turn the x and y axes in such a way that the K vector lies on the yz plane, and denoting θ as the angle between the z-axis and K. We can write K as the K component along the y direction, plus the component in the z direction.

Notes

Summary

3m 50s

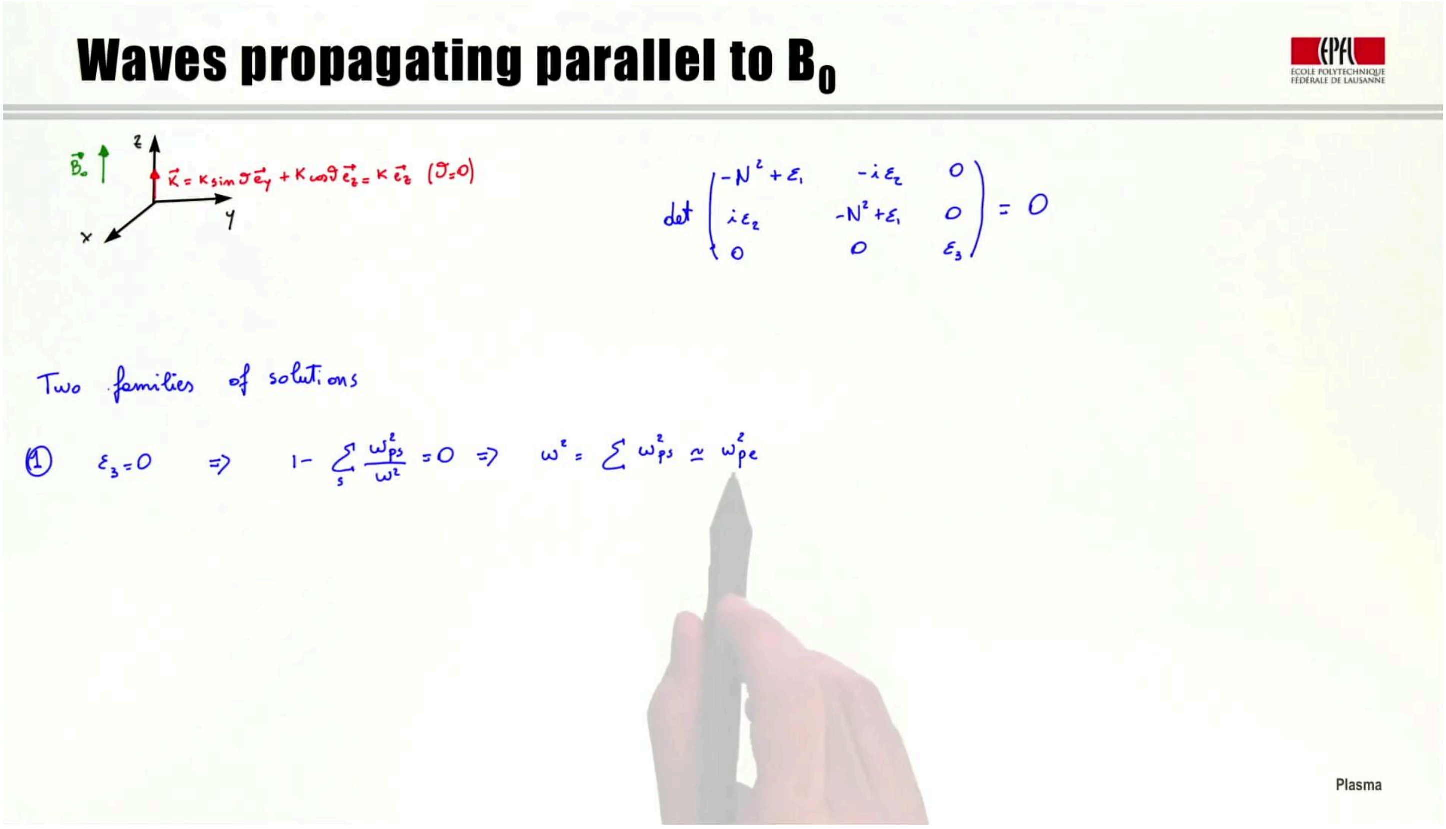


With this expression for K, we have that this term here, (N^2 [KK/K^2 -𝟙] + ε) becomes equal to -N^2 +ε_1, i ε_2, 0, 0, N^2 sinθ cosθ, N^2 sinθ cosθ, and for the last term, - N^2 $\sin^2$ θ + ε_3 What are the intrinsic modes present in our plasma?

Notes

Summary

6m 23s



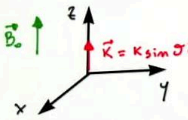
We will show that there are modes that are propagating parallel to the magnetic field. That means that the angle between the magnetic field and the K vector, the angle θ is equal to zero and modes that propagate in the direction perpendicular to the magnetic field where $\theta = \pi/2$. We can start by analyzing the modes propagating parallel to the magnetic field. In the three-dimensional space, a magnetic field that is in the z direction and the same thing for K, which we plot equal to K sinθ ey + K cosθ ez okay, we will write equal to K ez. Basically, we will consider the case θ=0. In this case, the dispersion relation, the determinant of this matrix equal to zero. What we can immediately notice is that there are two families of solutions. In fact, the determinant of this matrix can be equal to zero, because ε_3 is equal to zero, or because the determinant of this 2-by-2 two matrix is equal to zero. The first case is such that $\varepsilon_3 = 0$, which means $1 - \sum \omega_{ps}^2/\omega^2 = 0$ which implies that $\omega^2 = \sum \omega_{ps}^2$ and now as the electron plasma frequency is much higher than the ion one, a good approximation can be written as $\omega^2 = \sum \omega_{ps}^2 \simeq \omega_{pe}^2$ What have we found?

Notes

Summary

7m 06s

Waves propagating parallel to B_0



$\vec{K} = K \sin\vartheta\, \vec{e}_y + K \cos\vartheta\, \vec{e}_z = K \vec{e}_z \quad (\vartheta = 0)$

$$\det \begin{pmatrix} -N^2+\varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & -N^2+\varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} = 0$$

Two families of solutions

(1) $\varepsilon_3 = 0 \Rightarrow 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} = 0 \Rightarrow \omega^2 = \sum \omega_{ps}^2 \simeq \omega_{pe}^2$: plasma waves! $(\vec{B}_0 \,/\!/\, \vec{K} \,/\!/\, \vec{E}_1)$

(2) $(-N^2+\varepsilon_1)^2 - \varepsilon_2^2 = 0$

$-N^2 + \varepsilon_1 = \pm \varepsilon_2$

$N^2 = \varepsilon_1 + \varepsilon_2 = \varepsilon_R$ (right-handed waves)

$N^2 = \varepsilon_1 - \varepsilon_2 = \varepsilon_L$ (left-handed waves)

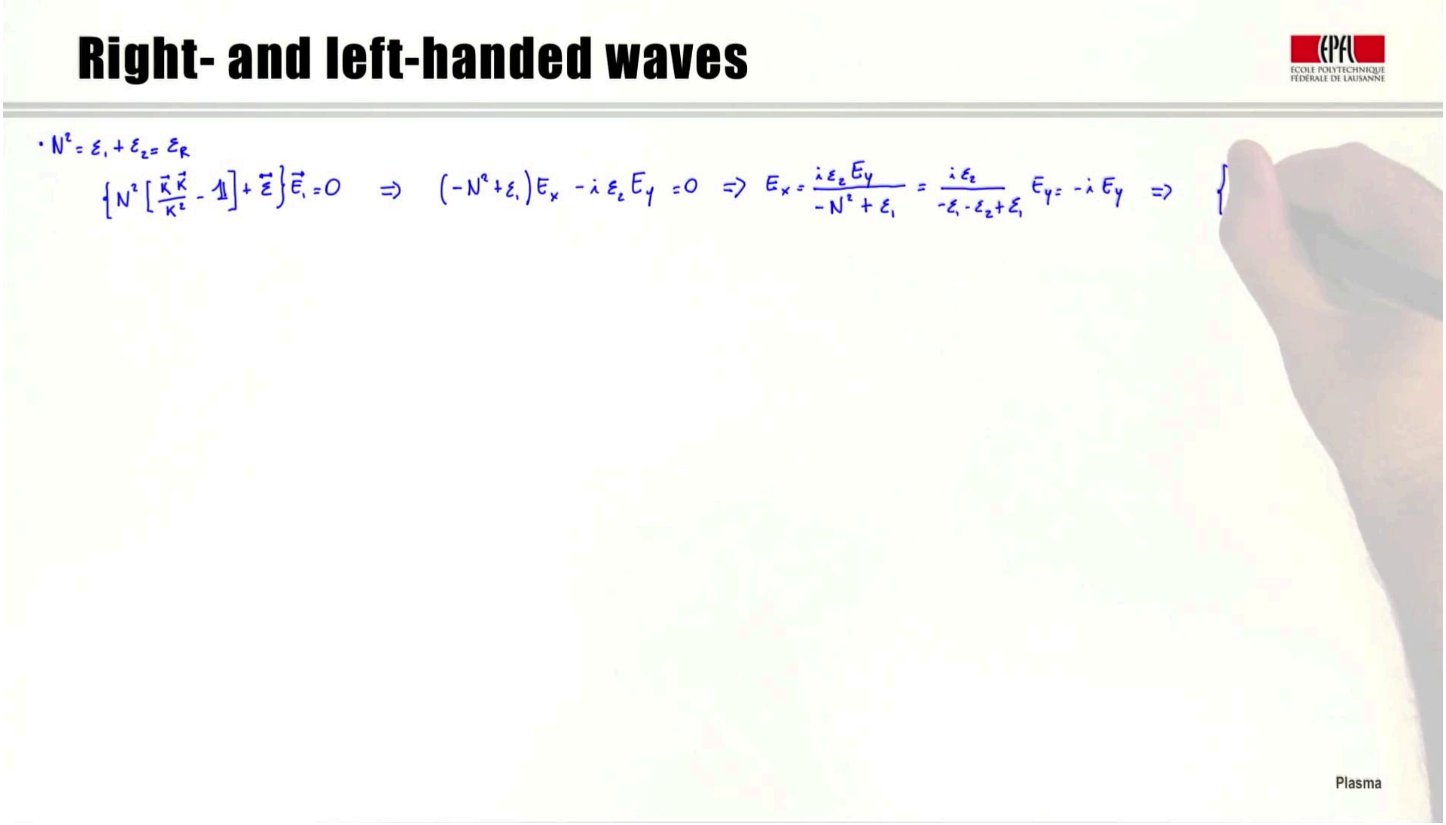
Plasma

$\omega^2 \simeq \omega_{pe}^2$ These are the *plasma waves*, that were the subject of our first lecture of the Plasma Physics course. It's a wave that takes place only in the direction parallel to the magnetic field for which B_0 is parallel to K which is also parallel to the perturbed electric field E_1. These are the waves that are present in a plasma because the plasma wants to restore the quasi-neutrality. They are not affected by the presence of the magnetic field. It was nice to retrieve the dispersion relation of the plasma waves. Now, let's look at something new that we see here. The determinant of this 2-by-2 two matrix. This determinant can be written as $(-N^2 + \varepsilon_1)^2 - \varepsilon_2^2 = 0$ and this is $-N^2 + \varepsilon_1 = \pm \varepsilon_2$ from which we derive that $N^2 = \varepsilon_1 + \varepsilon_2 = \varepsilon R$ and this will give, as we will see in a moment, the dispersion relation of the *right-handed waves* and $N^2 = \varepsilon_1 - \varepsilon_2 = \varepsilon L$ and this gives the dispersion relation of the *left-handed waves*.

Notes

Summary

9m 21s



We have therefore found that there are three families of waves that propagate in the direction parallel to the magnetic field: plasma waves, right handed waves, and left handed waves. Let's focus on the last two. First of all, let's look at why they are called in this way, and then on what is their dispersion relation. So let's look at the right-handed waves, for which $N^2 = \varepsilon_1 + \varepsilon_2$ and for which we have defined this quantity as εR. Now, the equation that we had found for wave dynamics says that (N^2 [KK/K^2 -𝟙] + ε)·E_1 has to be equal to zero. Let's see what the first equation of this linear system tells us about the relationship between the electric field in the different directions. If we look component by component, this linear system tells us for the first equation (with θ = 0) that ($-N^2$+ ε_1) Ex - i ε_2 Ey has to be equal to 0, as there is no contribution, no terms are related to Ez. This means that Ex will be equal to i ε_2 Ey / ($-N^2$ + ε_1) Now we can replace the expression for N^2 that we have from here and this will be equal to i ε_2 / (- ε_1 - ε_2 + ε_1) Ey which is therefore equal to, simplifying, -ε_1 with + ε_1 and dividing ε_2 by ε_2 -i Ey. What does it mean?

Notes

Summary

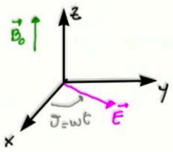
11m 24s

Right- and left-handed waves

• $N^2 = \varepsilon_1 + \varepsilon_2 = \varepsilon_R$

$$\left\{N^2\left[\frac{\vec K \vec K}{K^2} - \mathbb{1}\right] + \vec{\vec\varepsilon}\right\}\vec E_1 = 0 \Rightarrow (-N^2+\varepsilon_1)E_x - i\varepsilon_2 E_y = 0 \Rightarrow E_x = \frac{i\varepsilon_2 E_y}{-N^2+\varepsilon_1} = \frac{i\varepsilon_2}{-\varepsilon_1-\varepsilon_2+\varepsilon_1} E_y = -iE_y \Rightarrow \begin{cases} E_x = E \\ E_y = iE \end{cases}$$

$$\vec E = \mathrm{Re}\left\{\left[E\vec e_x + iE\vec e_y\right]\exp\left[i(\vec K\cdot\vec r - \omega t)\right]\right\} \overset{\vec r = 0}{=} E\cos(\omega t)\vec e_x + E\sin(\omega t)\vec e_y$$



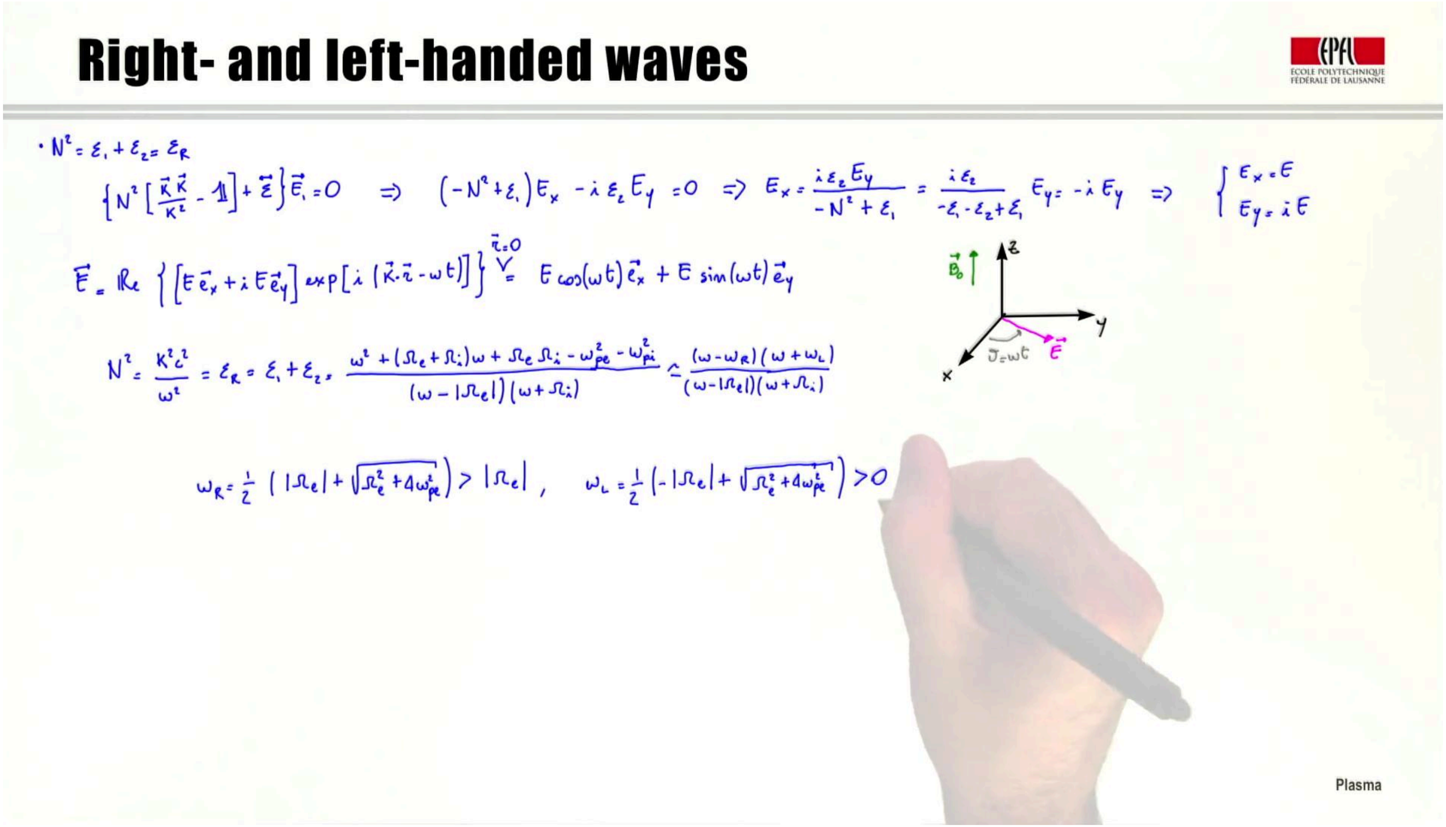
Plasma

If we call Ex E, we will have that Ey will be, for the right handed waves, equal to iE. Okay, this is a relation in Fourier space. How can it be translated in real space? We will have that the electric field will be given by the real part of the electric vector E ex + iE ey multiplied by the exponential factor that enters in the Fourier mode exp {i (K·r - ωt) } Now, let's suppose that we place ourselves for simplicity at the r = 0 location. We will have that the real part of this expression, for r = 0 will be given by, -for the x direction, by E cos(ωt)ex + E sin(ωt)ey. Let's try to draw what we have just derived mathematically. So we have our 3D coordinates, xyz the equilibrium magnetic field is in the z direction. The electric field will be on the xy plane and the direction of the electric field on the xy plane will evolve in time according to the law that we have just derived, In particular, if when we draw an angle between the electric field and the x axis and we call this θ, [not the angle between K and ez] we see that this θ will be equal to ωt. You see that the electric field rotates in the xy plane in a direction such that by using the right hand rule, the direction of rotation coincides with the B_0 field.

Notes

Summary

14m 00s



This is why this is called the right-handed wave. Now, let's give a look in more detail at what is the dispersion relation of the right handed waves. What we have is that $N^2 = K^2 c^2 / \omega^2$ which is equal to ε_R, that is $\varepsilon_1 + \varepsilon_2$. And then, if you substitute the expression for ε_1 and ε_2 that we have seen previously, you find that this will be given by ω^2 plus the electron and ion cyclotron frequencies [$(\Omega_e + \Omega_i)$] divided by-- let me just point in evidence that the sign of Ω_e -- Ω_e is a negative quantity, to put in evidence this negative sign and we will take the absolute value and put a minus here, [$(\omega - |\Omega_e|)$] and then multiply by $(\omega + \Omega_i)$. This expression here can actually be written as -- we won't make any change in the denominator, -- and we will decompose the numerator into the product of 2 terms which involve two frequencies: ω_R and ω_L which are given approximately -- this is why I am using an approximate equal here, by $\omega_R = 1/2 \{ |\Omega_e| + \sqrt{\Omega_e^2 + 4 \omega_{pe}^2} \}$ which is a general quantity that is greater than $|\Omega_e|$, and ω_L which is given by $\omega_L = 1/2 \{-|\Omega_e| + \sqrt{\Omega_e^2 + 4 \omega_{pe}^2} \}$ If you look at this, as this is a quantity that is always larger than $|\Omega_e|$, this will be always greater than zero.

Notes

Summary

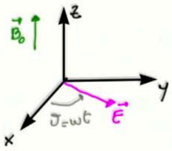
16m 33s

Right- and left-handed waves

- $N^2 = \varepsilon_1 + \varepsilon_2 = \varepsilon_R$

$$\left\{N^2\left[\frac{\vec K \vec K}{K^2} - \mathbb{1}\right] + \vec{\vec{\varepsilon}}\right\}\vec E_1 = 0 \Rightarrow (-N^2+\varepsilon_1)E_x - i\varepsilon_2 E_y = 0 \Rightarrow E_x = \frac{i\varepsilon_2 E_y}{-N^2+\varepsilon_1} = \frac{i\varepsilon_2}{-\varepsilon_1-\varepsilon_2+\varepsilon_1} E_y = -iE_y \Rightarrow \begin{cases} E_x = E \\ E_y = iE \end{cases}$$

$$\vec E = \mathrm{Re}\left\{\left[E\vec e_x + iE\vec e_y\right]\exp\left[i(\vec K\cdot\vec r - \omega t)\right]\right\} \overset{\vec r = 0}{=} E\cos(\omega t)\vec e_x + E\sin(\omega t)\vec e_y$$



$$N^2 = \frac{K^2c^2}{\omega^2} = \varepsilon_R = \varepsilon_1 + \varepsilon_2 = \frac{\omega^2 + (\Omega_e+\Omega_i)\omega + \Omega_e\Omega_i - \omega_{pe}^2 - \omega_{pi}^2}{(\omega-|\Omega_e|)(\omega+\Omega_i)} \simeq \frac{(\omega-\omega_R)(\omega+\omega_L)}{(\omega-|\Omega_e|)(\omega+\Omega_i)}$$

$$\omega_R = \frac{1}{2}\left(|\Omega_e| + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}\right) > |\Omega_e|, \quad \omega_L = \frac{1}{2}\left(-|\Omega_e| + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}\right) > 0$$

- $N^2 = \varepsilon_1 - \varepsilon_2 = \varepsilon_L$

$$N^2 = \frac{K^2c^2}{\omega^2} = \frac{(\omega+\omega_R)(\omega-\omega_L)}{(\omega+|\Omega_e|)(\omega-\Omega_i)}$$

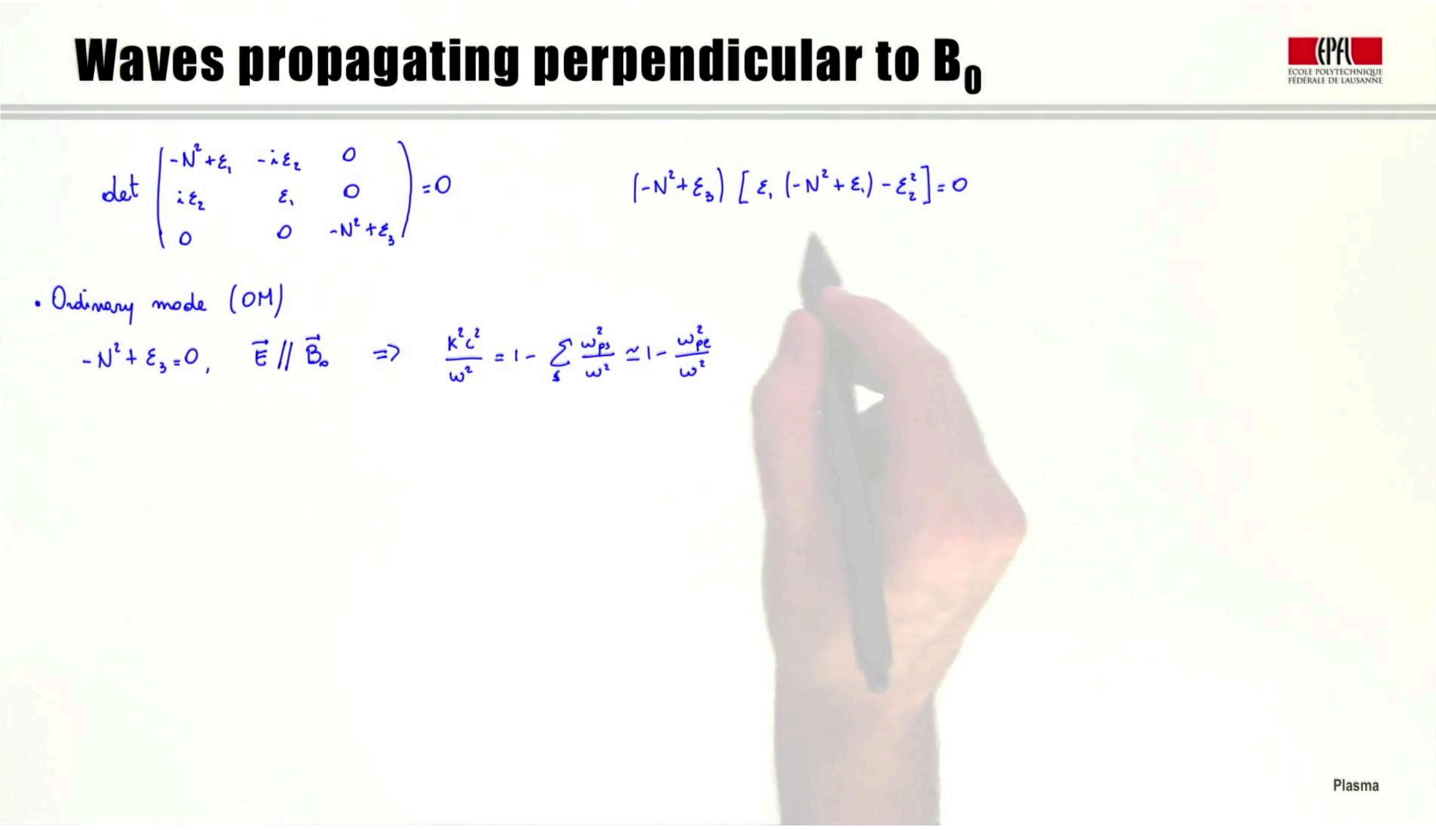
Plasma

So these ωR and ωL are, in practice, the solutions of the quadratic equation that we have here at the numerator and which are evaluated in the limit of the electron mass being much smaller than the ion mass. So this is the dispersion relation for the right-handed waves. Now we can derive a similar expression for the left handed waves by observing that $N^2 = \varepsilon_1 - \varepsilon_2$, that is what we have called εL. In this case we won't do the calculation as we did for the right-handed waves but let me say that the rotation of the electric field will be in the direction opposite to the magnetic field, B_0, and this is why this is called left-handed wave, and the dispersion relation turns out to be $N^2 = K^2c^2/\omega^2$ and this is equal to (ω+ωR) (ω-ωL) / ((ω+|Ωe|) (ω-Ωi)) We will study the physics of the left-handed waves and of the right-handed waves in the next module.

Notes

Summary

18m 59s

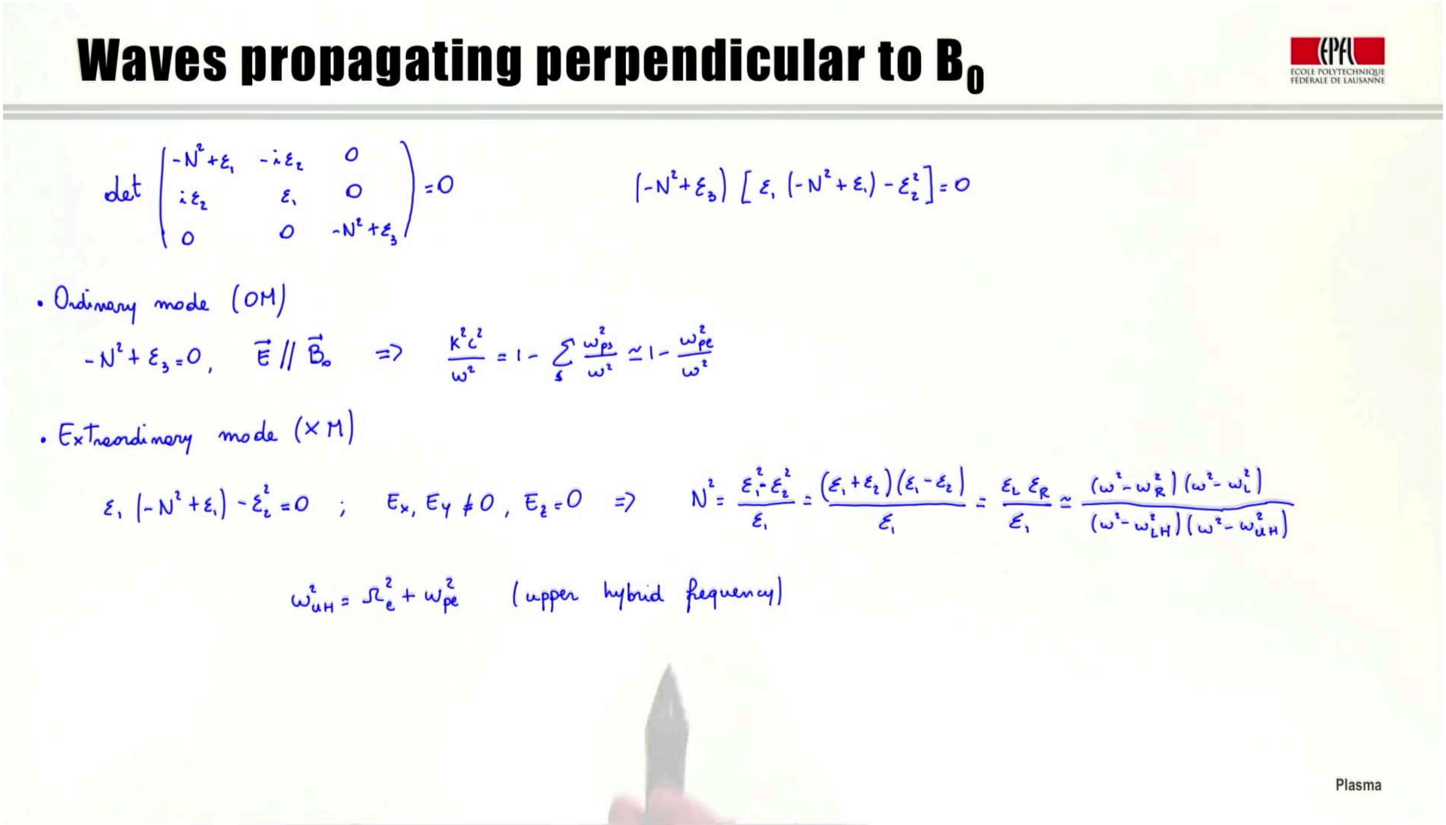


Now, let's focus our attention on the waves propagating in a direction perpendicular to the magnetic field the waves for which the K vector is in the y direction, and therefore $\theta = \pi/2$. Let's derive the dispersion relation for these waves. For waves propagating perpendicular to the magnetic field, $\theta = \pi/2$ and what we have to evaluate is the determinant of a matrix that is $-N^2+\varepsilon_1$, $i\varepsilon_2$, 0, $i\varepsilon_2$, ε_1, 0, 0, $-N^2+\varepsilon_3$, is equal to zero which can be written as the product of these terms here, times the determinant of the 2-by-2 matrix up here equal to zero. Here too we have two possibilities. The first one is that $-N^2+\varepsilon_3 = 0$. This is the ordinary mode we will denote with OM which therefore, $-N^2+\varepsilon_3 = 0$. In this case the electric field is again in the direction parallel to B, therefore E is parallel to B_0 and if you try to evaluate the dispersion relation you plug the correct expression for ε_3, you find that $K^2 c^2/\omega^2 = 1 - \sum \omega_{ps}^2/\omega^2$ which is approximately equal to, -- as the ion plasma frequency is much lower than the electron one, this is approximately equal to $1 - \omega_{pe}^2/\omega^2$. This is the so-called ordinary mode that satisfies this first term is equal to zero.

Notes

Summary

20m 43s



There is another kind of wave that is present in our system that is one for which the second term vanishes. These are the waves that are called the extraordinary mode, XM. So for these waves, $\varepsilon_1(-N^2+\varepsilon_1)-\varepsilon_2^2 = 0$ has to be equal to zero. For this wave we will have that Ex and Ey are in general different from zero but Ez will be constrained to be equal to zero, and then, if we try to solve the dispersion relation, we will find by expressing N^2 in terms of ε_1 and ε_2 that $N^2 = (\varepsilon_1^2 - \varepsilon_2^2) / \varepsilon_1$ which we can write as... -sorry, there is a minus here, it's the difference between ε_1^2 and ε_2^2. .. this can be decomposed as $(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 - \varepsilon_2) / \varepsilon_1$ These are two terms that are now familiar to us. They are called εL and εR, divided by ε_1 and now, this can be written by decomposing the product, εL εR as.... -it's a few lines of algebra that I spare you... $(\omega^2-\omega_R^2)(\omega^2-\omega_L^2)/((\omega^2-\omega_{LH}^2)(\omega^2-\omega_{UH}^2))$ and we will give the definition in a few seconds [same formula as before] $(\omega^2-\omega_R^2)(\omega^2-\omega_L^2)/((\omega^2-\omega_{LH}^2)(\omega^2-\omega_{UH}^2))$ What are the frequencies that we have just introduced? $\omega_{UH}^2 = \Omega_e^2 + \omega_{pe}^2$ plus the electron plasma frequency, and we have denoted that as *UH* as we call it *Upper Hybrid frequency.* It's a hybrid frequency as it involves both the cyclotron and the plasma frequency.

Notes

Summary

23m 28s

Waves propagating perpendicular to B_0

$$\det \begin{pmatrix} -N^2+\varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & -N^2+\varepsilon_3 \end{pmatrix} = 0 \qquad (-N^2+\varepsilon_3)\left[\varepsilon_1(-N^2+\varepsilon_1)-\varepsilon_2^2\right]=0$$

- Ordinary mode (OM)

$$-N^2+\varepsilon_3=0, \quad \vec{E} \parallel \vec{B}_0 \quad \Rightarrow \quad \frac{k^2c^2}{\omega^2} = 1-\sum_s \frac{\omega_{ps}^2}{\omega^2} \simeq 1-\frac{\omega_{pe}^2}{\omega^2}$$

- Extraordinary mode (XM)

$$\varepsilon_1(-N^2+\varepsilon_1)-\varepsilon_2^2=0 \;;\quad E_x, E_y \neq 0,\ E_z=0 \quad \Rightarrow \quad N^2 = \frac{\varepsilon_1^2-\varepsilon_2^2}{\varepsilon_1} = \frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1-\varepsilon_2)}{\varepsilon_1} = \frac{\varepsilon_L\varepsilon_R}{\varepsilon_1} \simeq \frac{(\omega^2-\omega_R^2)(\omega^2-\omega_L^2)}{(\omega^2-\omega_{LH}^2)(\omega^2-\omega_{UH}^2)}$$

$$\omega_{UH}^2 = \Omega_e^2+\omega_{pe}^2 \quad \text{(upper hybrid frequency)}$$

$$\omega_{LH}^2 = \frac{\Omega_i|\Omega_e|\left[1+\frac{m_e}{m_i}\left(\frac{\Omega_e}{\omega_{pe}}\right)^2\right]}{1+\frac{\Omega_e^2}{\omega_{pe}^2}} \quad \text{(lower hybrid frequency)}$$

Plasma

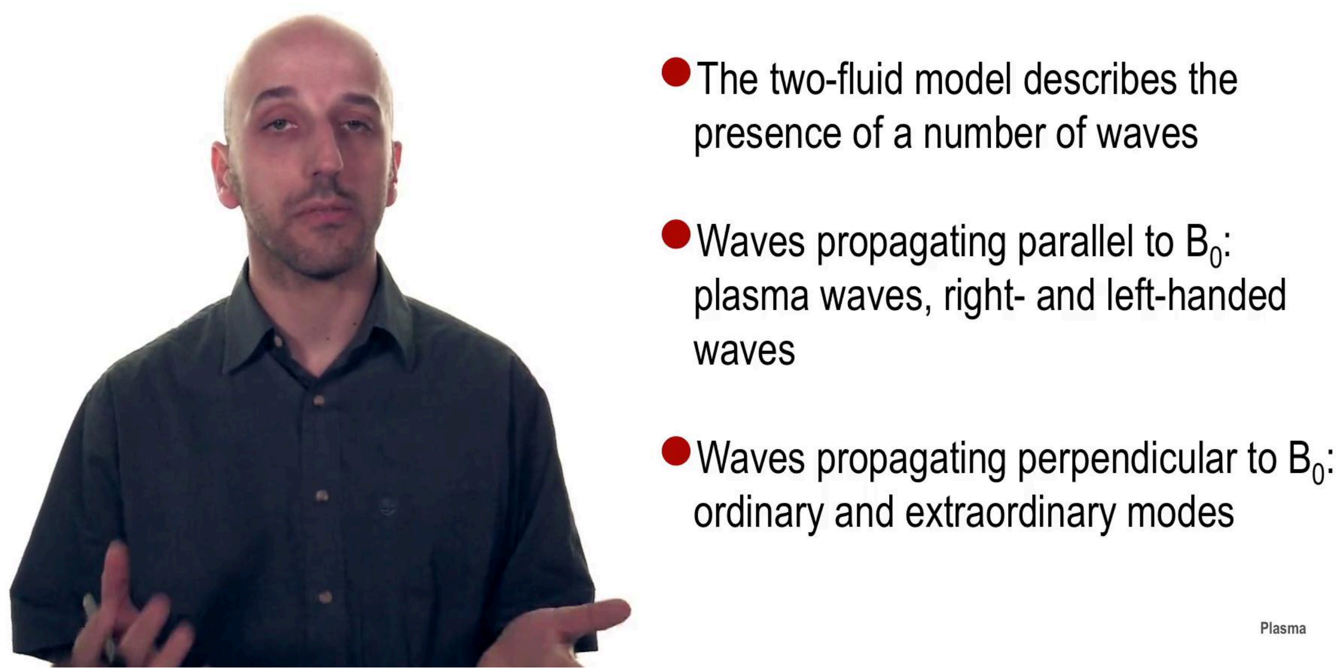
We've also introduced ω_{LH}^2 and this is equal to $\Omega_i\,|\Omega_e|$ and then times ($1 + m_e/m_i\,(\Omega_e/\omega_{pe})^2$) divided by ($1 + \Omega_e^2/\omega_{pe}^2$) This is what we have called *Lower Hybrid frequency*. This is the reason of having used the subscript LH.

Notes

Summary

26m 33s

Summary



To summarize the contents of the present module, let me first of all say that we have actually pointed out that the plasma, even within a relatively simple model such as the two-fluid model in the cold plasma limits, the plasma is actually a complex medium. It supports, in fact, a large number of intrinsic modes. We have pointed out that, propagating **parallel** to the equilibrium magnetic field there are: - plasma waves - right-handed waves, - left-handed waves. And propagating **perpendicular** to the magnetic field that we have extraordinary (XM) and ordinary (OM) modes. The physical properties of these modes, of these waves, will be the subject of the next module.

Notes

Summary

27m 13s