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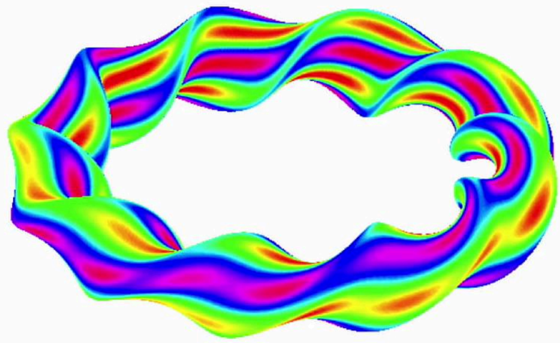
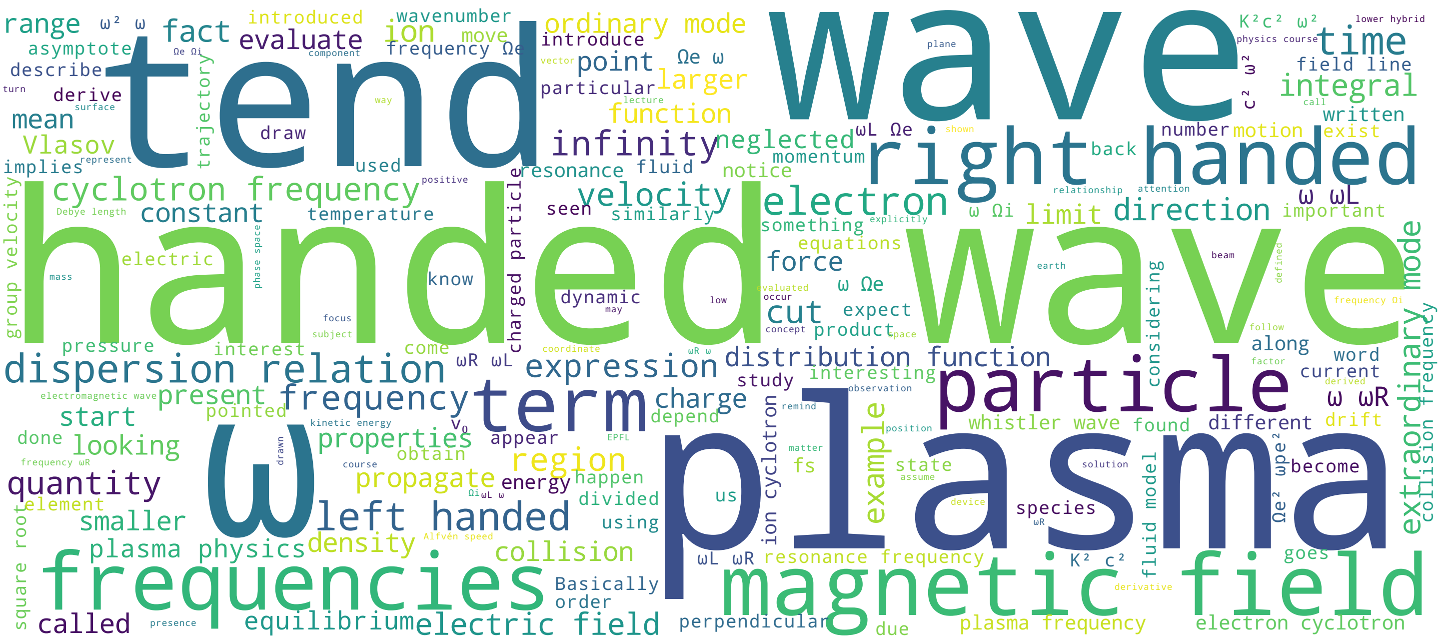


# Properties of the two-fluid waves

**Plasma Physics and Application to Fusion Energy, Astrophysics and Industry**

**Lecture 3d**

Paolo Ricci

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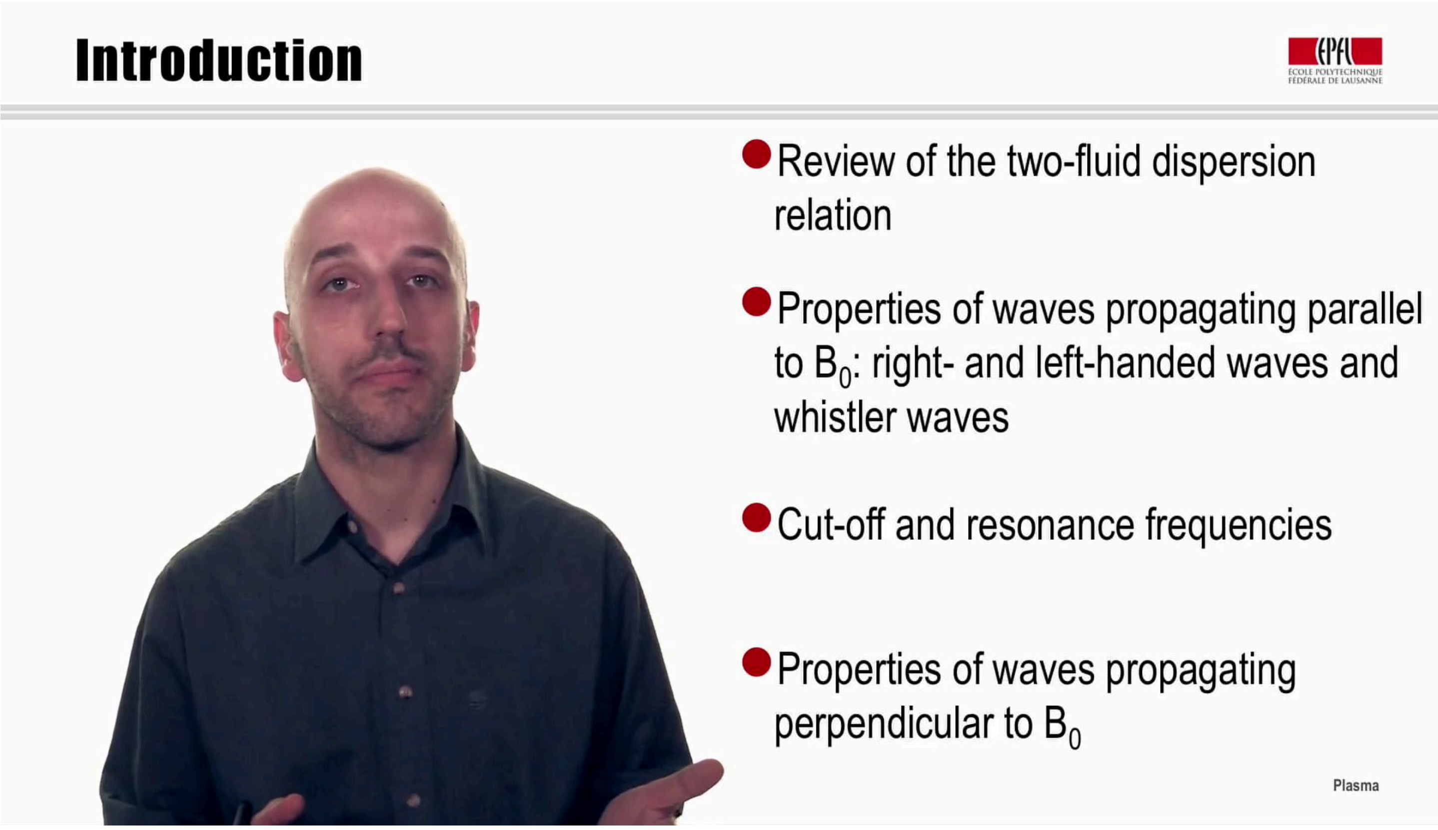



Welcome, welcome to the plasma physics course of EPFL. In the past module, we have derived the intrinsic modes, the waves that are present in a plasma within the two-fluid model. In the present one, we will actually show what are the properties of these waves. We will start by looking at the properties of the waves propagating in the direction parallel to the magnetic field that is right-handed and left-handed waves. We will see that in a certain range of frequencies, an interesting kind of wave appears: the *whistler wave*. The study of these waves, that are propagating parallel to magnetic field will give us the opportunity to introduce the concept of *cut-off* and *resonance*. Then we will turn our attention to the physical properties of the waves propagating perpendicular to the magnetic field.

Notes

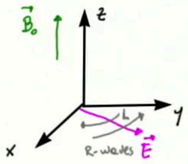
Summary

0m 05s

# Properties of right- and left-handed waves

$$\frac{K^2c^2}{\omega^2} = \frac{(\omega-\omega_R)(\omega+\omega_L)}{(\omega-|\Omega_e|)(\omega+\Omega_i)} \quad (R\text{- waves})$$

$$\frac{K^2c^2}{\omega^2} = \frac{(\omega+\omega_R)(\omega-\omega_L)}{(\omega+|\Omega_e|)(\omega-\Omega_i)} \quad (L\text{- waves})$$




Plasma

So let's start to give a look at the properties of the left-handed and right-handed waves. We will do that by first recalling the dispersion relation and then by looking at what is the range of frequency and wavenumber in which these waves can propagate. The dispersion relation for the right-handed waves reads as the index of refraction equal to $K^2c^2/\omega^2 = (\omega-\omega R)(\omega+\omega L)/((\omega-|\Omega e|)(\omega+\Omega i))$ and these are the right-handed waves. The left-handed waves, on the other hand are such that $K^2c^2/\omega^2 = (\omega+\omega R)(\omega-\omega L)/((\omega+|\Omega e|)(\omega-\Omega i))$ These are the left-handed waves. Just let's recall for a second why these waves are called this way. So why are these waves called right- and left-handed waves? Let's draw our coordinates system. x,y,z, With the equilibrium magnetic field along the z direction Now for these waves the electric field lies on the x,y plane and we've seen in the last module that this electric field rotates according to the right hand rule with respect to $B_0$ if we are dealing with the right-handed waves while it moves in opposite direction for the left-handed waves.

Notes

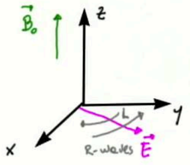
Summary

1m 09s

# Properties of right- and left-handed waves

$$\frac{K^2c^2}{\omega^2} = \frac{(\omega-\omega_R)(\omega+\omega_L)}{(\omega-|\Omega_e|)(\omega+\Omega_i)} \qquad \text{(R-waves)}$$

$$\frac{K^2c^2}{\omega^2} = \frac{(\omega+\omega_R)(\omega-\omega_L)}{(\omega+|\Omega_e|)(\omega-\Omega_i)} \qquad \text{(L-waves)}$$




$$\omega_R = \frac{1}{2}\left(|\Omega_e| + \sqrt{\Omega_e^2 + \omega_{pe}^2}\right) > |\Omega_e|$$

$$\omega_L = \frac{1}{2}\left(-|\Omega_e| + \sqrt{\Omega_e^2 + \omega_{pe}^2}\right) > 0$$

Properties

- $|\Omega_e| < \omega < \omega_R$ : no R-waves

  $\Omega_i < \omega < \omega_L$ : no L-waves

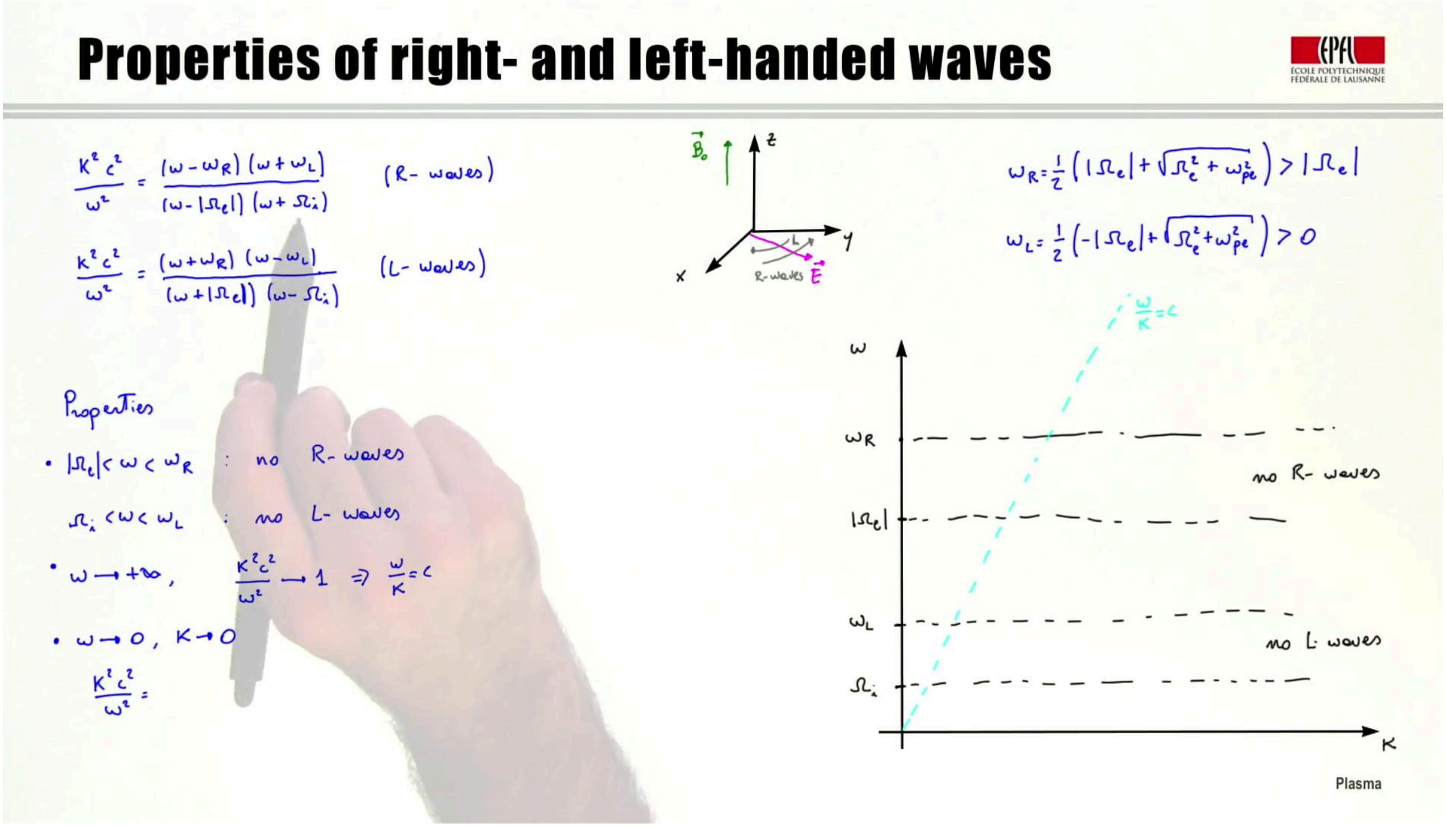
Plasma

The frequencies that I have introduced here, ωR and ωL are defined such that ωR = 1/2 { |Ωe| + sqrt( Ωe² + 4 ωpe² ) } and let me point out that this is a quantity that is larger than the electron cyclotron frequency, while ωL = 1/2 {-|Ωe| + sqrt( Ωe² + 4 ωpe² ) } that is a quantity that is larger than zero. What are the properties of these waves? First of all, in order to propagate, these waves should have a frequency such that this quantity here or this quantity here is positive. As K² c² / ω² is positive and if ω is found to be in between the electron cyclotron frequency |Ωe| and ωR... I remind you that this inequality is valid then this quantity here, - as these other two terms are always positive, will be negative. Therefore we will not have right-handed waves and similarly for the left-handed waves this term and this term here are always positive, so the only negative contribution can come from this factor and this factor here and in particular we see that for... - as typically the ion cyclotron frequency is much smaller than ωL, ... that for frequencies between the ion cyclotron frequency Ωi and ωL, we have no left-handed waves.

Notes

Summary

3m 41s

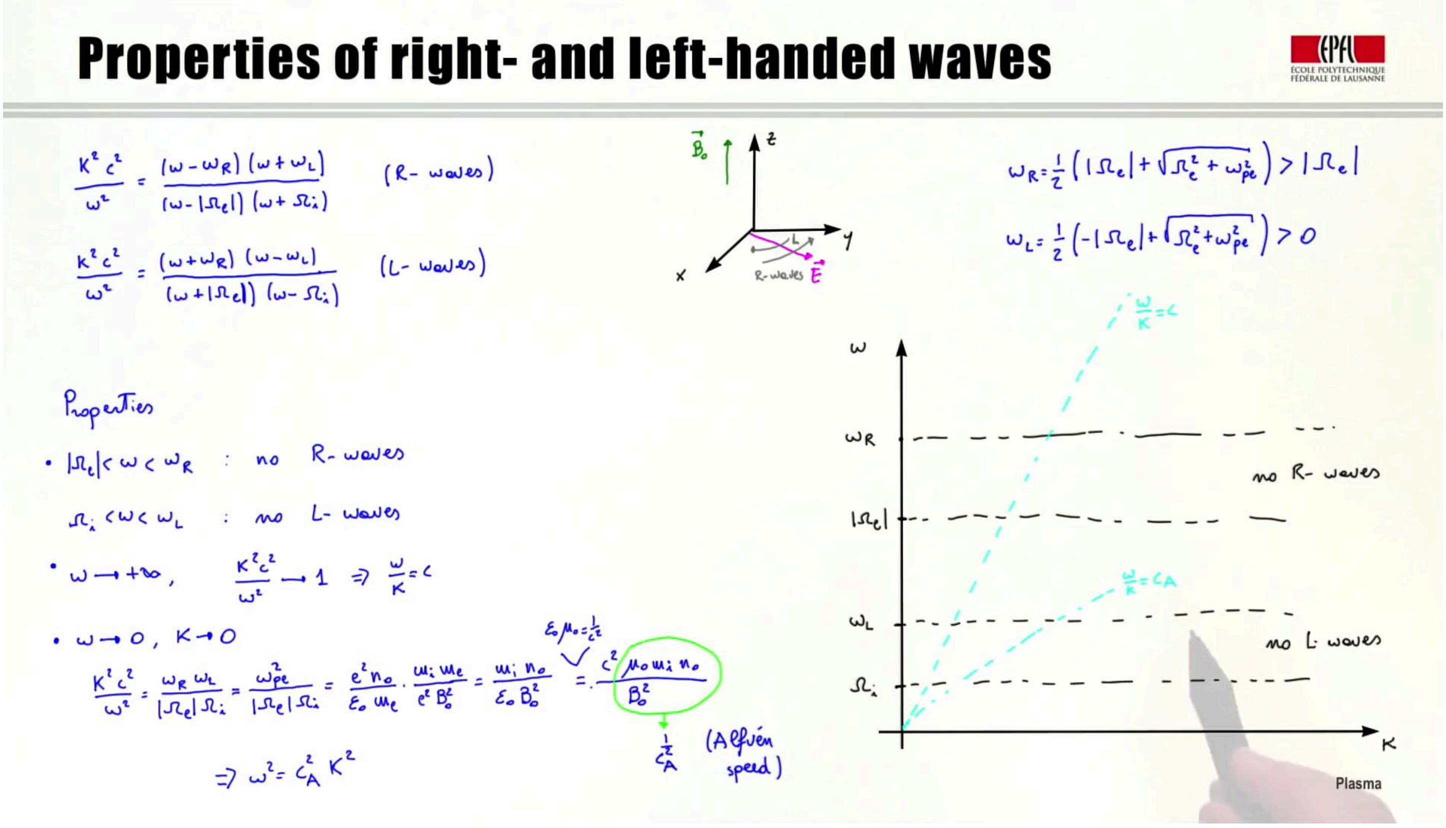



If we draw a diagram that represents the frequency as a function of the wavenumber, the frequencies of interest will be along this scale located in the following order: Ωi, then ωL, then the electron cyclotron frequency |Ωe|, and ωR and in this region we won't have left-handed waves and similarly in this region we won't have any right-handed waves. A second thing that may be interesting to look at is how these waves behave when ω tends to infinity. In this case, we will have that $K^2 c^2 / \omega^2$ will tend to -- ω will tend to infinity both this fraction and this fraction here will tend to 1. In other words, we will have that ω/K will be equal to c. What will happen? Basically we will have waves that are propagating at the speed of light. There are electromagnetic waves, the plasma has no time to respond at frequencies that are so high. There will be therefore an asypmtote here that is ω/K = c where these waves will tend to. And then how about the limit of ω going to zero and K going to zero? We will have that $K^2 c^2 / \omega^2$ will be equal to... -- we will be able to neglect ω with respect to ωR, ωL, |Ωe|, and Ωi. ...

Notes

Summary

6m 01s

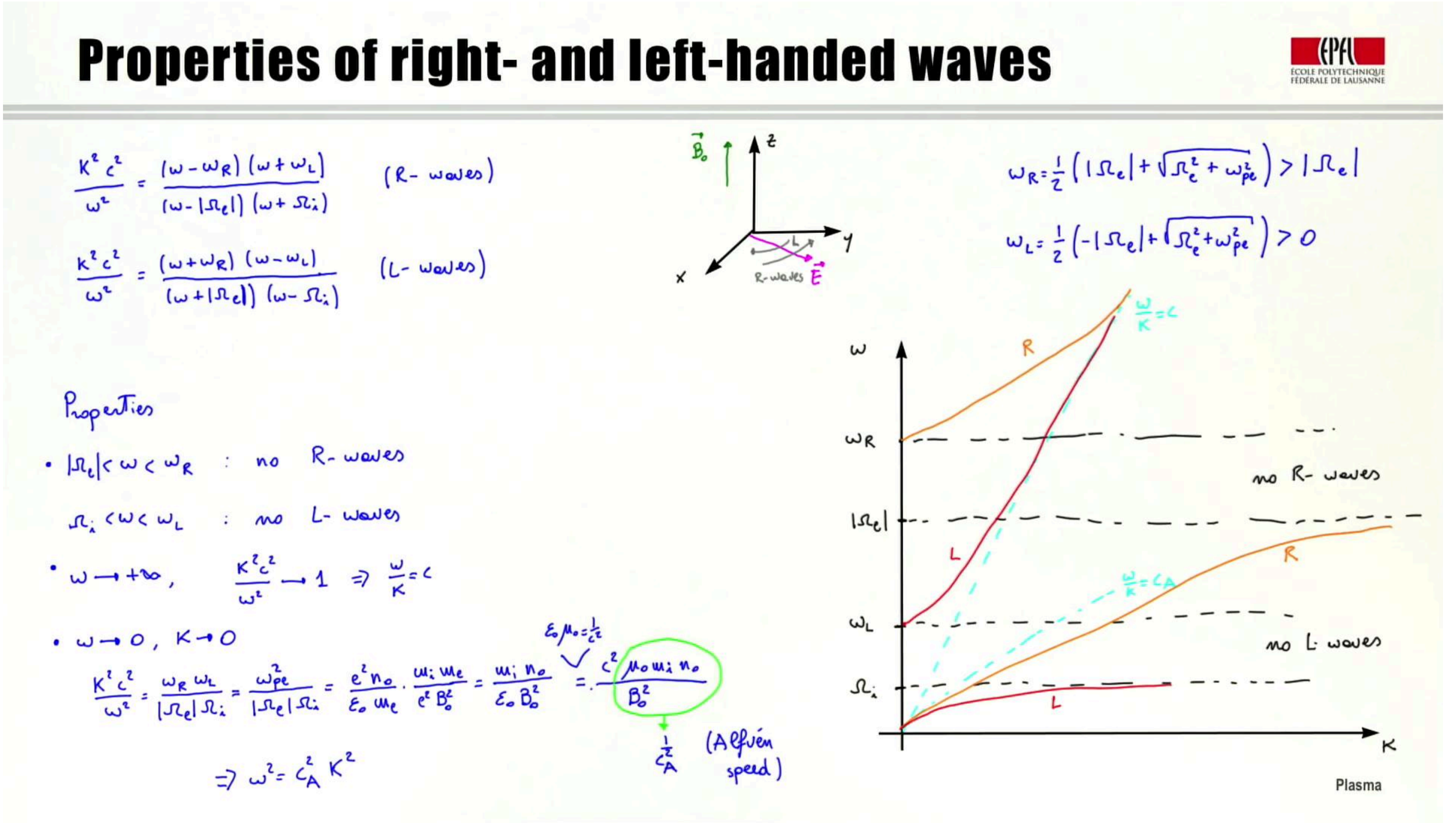



therefore, this will be equal to ωR ωL / (|Ωe| Ωi) and let me point out that we will have the same expression for both the right- and left-handed waves. Now if we explicit the product of ωR and ωL, we find ωpe² / (|Ωe| Ωi) Now we can explicitly write that the expression for the plasma frequency that is ωpe² = e² $n_0$ / ( $\varepsilon_0$ me ) and then we can write explicitly the 1/ (|Ωe| Ωi ) term as mi me / (e² $B_0^2$) We can simplify e². We can simplify me and me and we get mi $n_0$ / ( $\varepsilon_0$ $B_0^2$ ), which is equal --if we notice that $\varepsilon_0$ $\mu_0$ = 1/c², ...to c² $\mu_0$ mi $n_0$ / $B_0^2$ The quantity that I have just drawn here is actually the inverse of a square speed, 1/cA². The speed cA that is called *Alfvén speed*. If we look at this we notice that c² and c² simplify and therefore that ω² = cA² K². This implies that there is another asymptote that is given by ω/K = cA. We have now all the elements to derive the relationship between ω and K for the right-handed waves and the left-handed waves. Let's start by considering the right-handed waves. First of all, what we see is that for ω going to zero, K tends to zero, therefore the wave will start from this point, then will follow these aymptotically this expression and then the right-handed waves will exist up until |Ωe|.

Notes

Summary

8m 13s

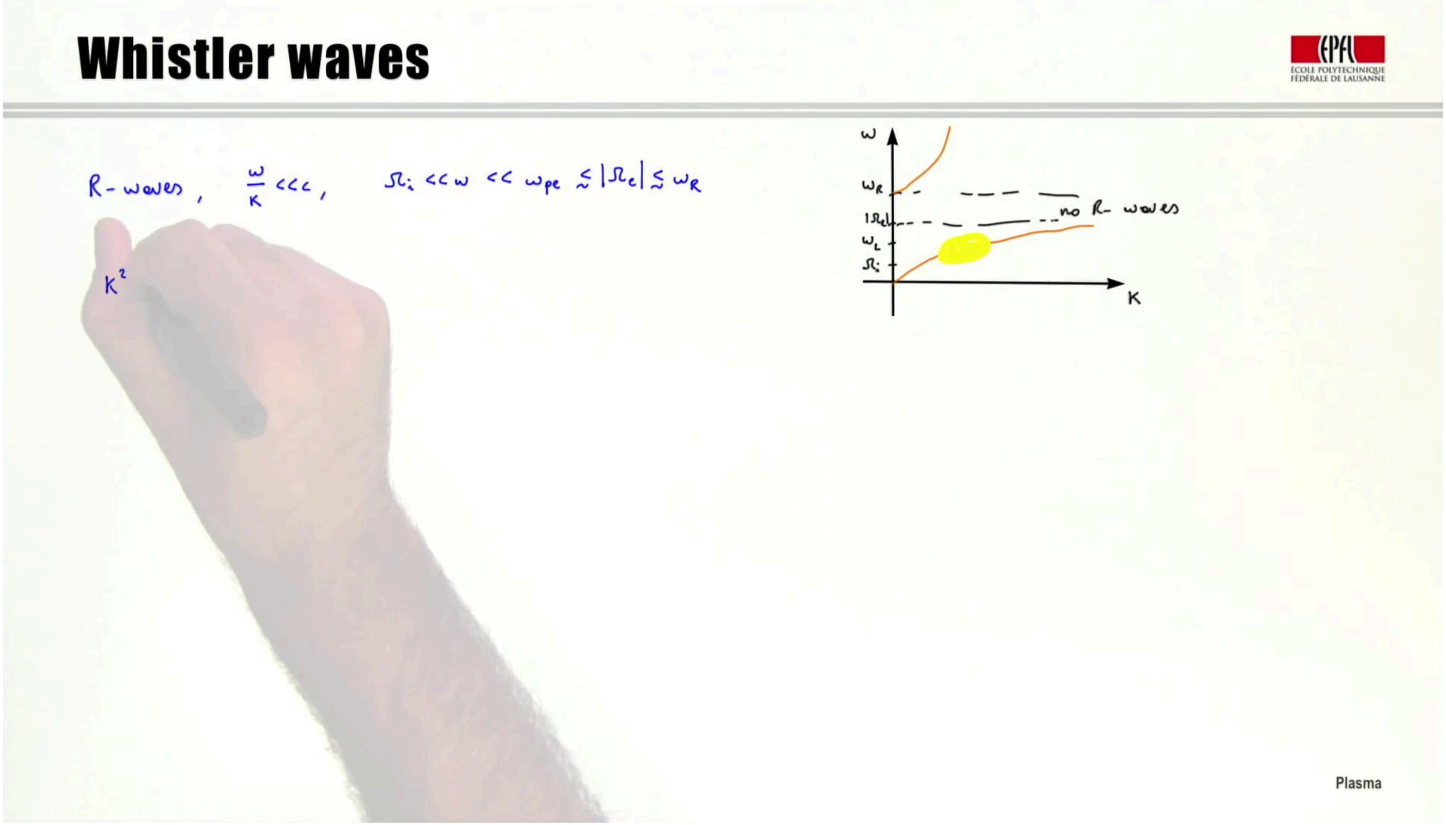



When ω will tend to |Ωe|, this term will tend to 0, and therefore this will blow up and we expect that K will tend to infinity. We expect, in other words, something like that. With K going to infinity and the dispersion relation that follows ω/K = cA at low K and low ω. The wave then will appear again at ωR and when ω is equal to ωR, we will have K = 0. And then when ω will tend to infinity, this asymptote will be followed. We expect, in other words, something like that. These are right-handed waves. What about the left-handed waves? Well they do not exist for ω that is in between Ωi and ωL and they have the same asymptotic behavior therefore at the low value of ω and K they will follow again, same asymptote then K will tend to infinity and then at the left-handed wave will appear again at ωL and will tend to infinity following this asymptote, ω/K = c These are the relationships between frequency and the wavenumber for both left-handed and right-handed waves.

Notes

Summary

11m 15s

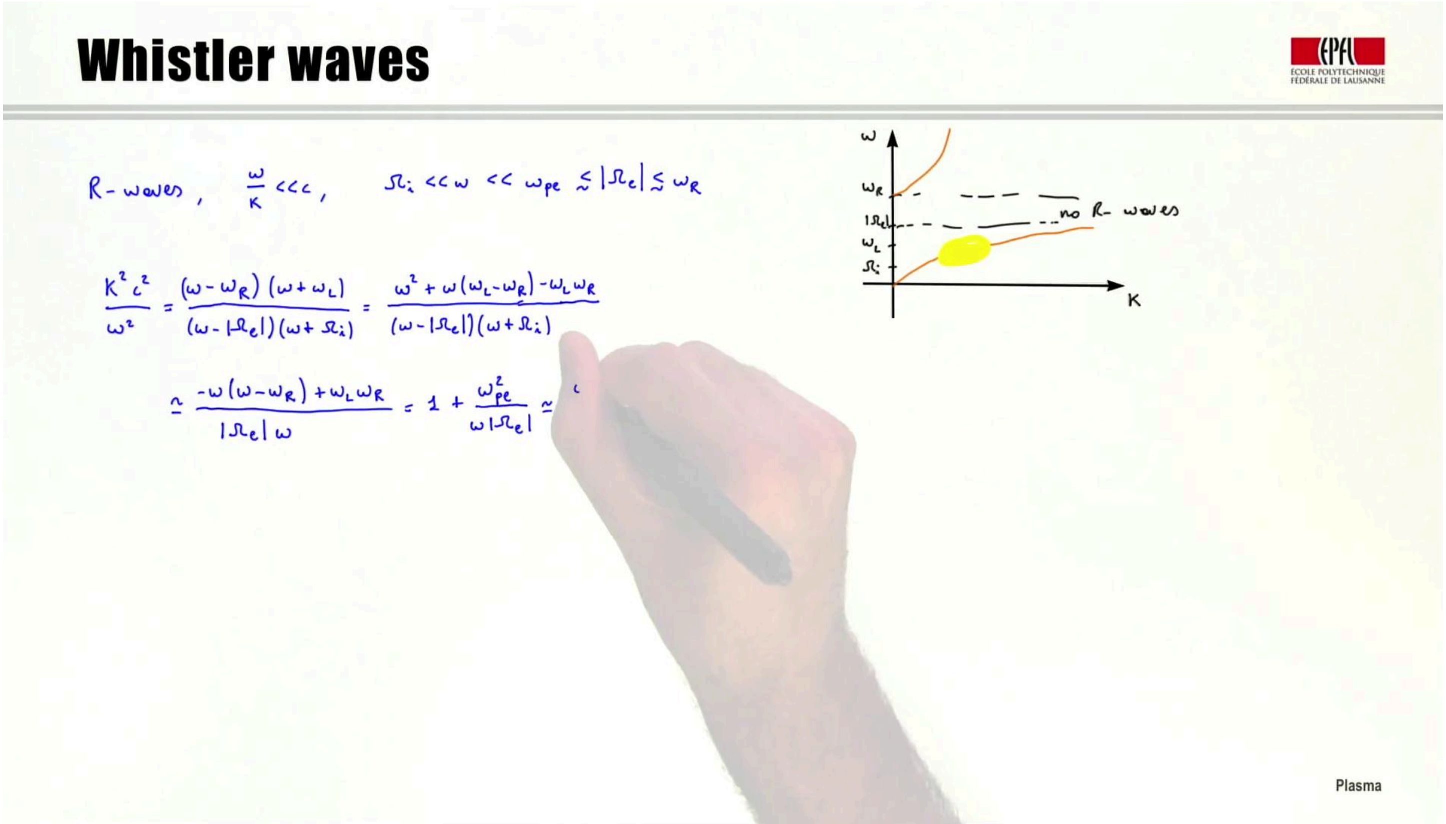



There is a range of frequencies where the right-handed waves display very interesting properties and in this range of frequency they are called *whistler waves*. Let's look at this property. So we will consider the right-handed waves in the limit of $\omega/K \ll c$ and in a range of frequencies that are much larger than the ion cyclotron frequency $\Omega i$ and much smaller than the plasma frequency $\omega pe$, that is typically smaller or comparable to the electron cyclotron frequency $|\Omega e|$ and the frequency $\omega R$. Where are we in the dispersion relation diagram that we have just drawn? We had pointed out the presence of the ion cyclotron frequency $\Omega i$, $\omega L$, $\Omega e$ and $\omega R$ frequencies. We've shown that there are no right-handed waves in this region and that the relationship between $\omega$ and K is given by something like that. In this range of frequency that we are considering here, we are looking at frequencies that are much larger than the ion cyclotron frequency but much smaller than the electron one. We are looking at waves that are propagating in this region here. What does the dispersion relation look like in this range of frequencies ?

Notes

Summary

13m 10s

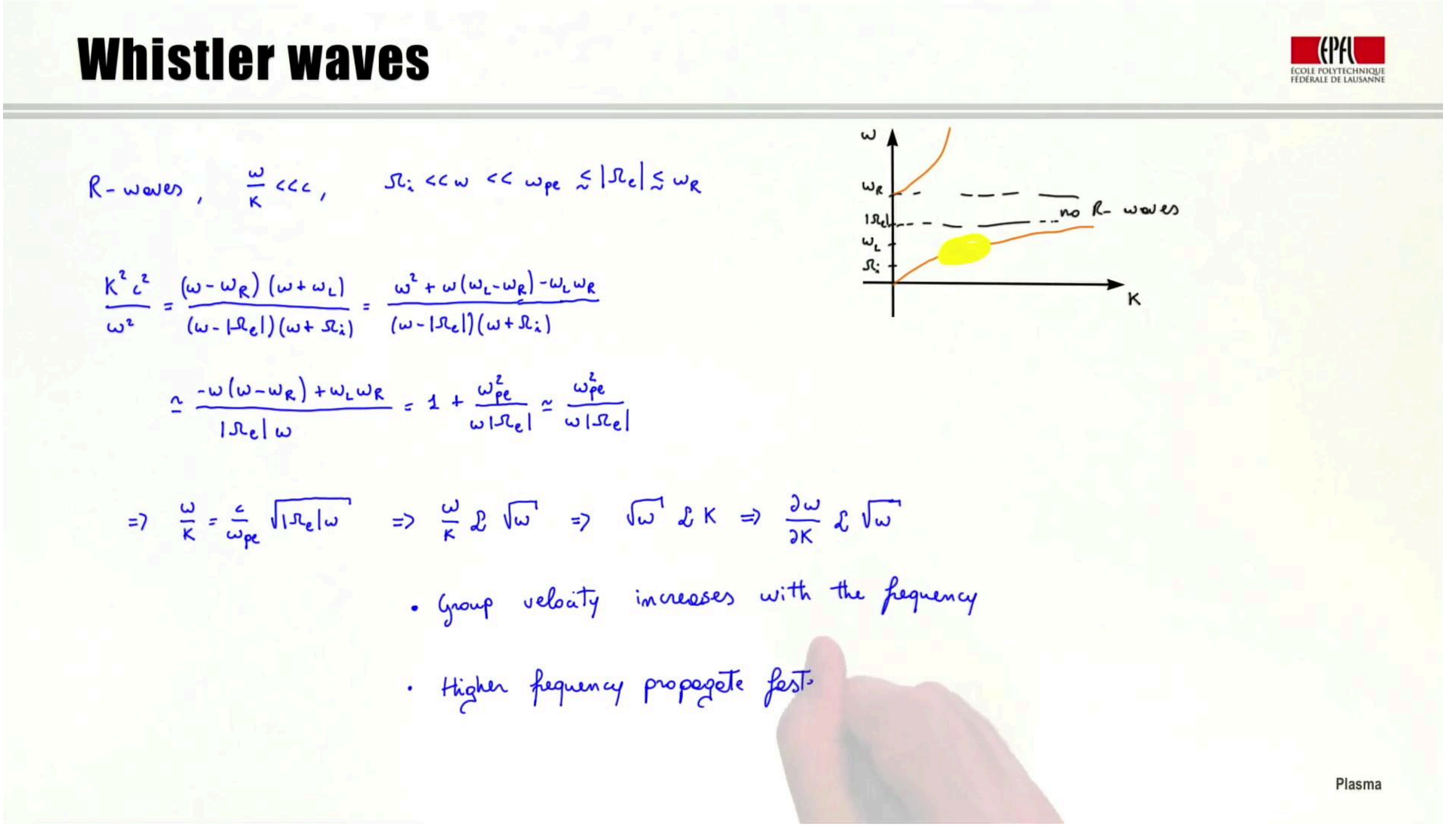



We have K²c²/ω² = (ω-ωR)(ω+ωL)/((ω-|Ωe|)(ω+Ωi)) We can leave the denominator of this expression as it is, and for the numerator we can develop this product and we have ω² + ω(ωL-ωR) - ωL ωR Now this, in the regime of interest can be actually simplified -up to now we haven't done any simplification, and in particular ω can be neglected with respect to |Ωe| and Ωi can be neglected with respect to ω. We've also that ωL is typically smaller than ωR and therefore it can be neglected in the term ( ωL- ωR), and therefore this can be written as {-ω(ω-ωR) + ωL ωR}/(|Ωe| ω) Now ω and ω here can be simplified, ω can be neglected with respect to ωR and in the limit of ωpe smaller than |Ωe|, ωR becomes comparable to |Ωe|. What we have therefore here is 1 and then the product of ωL and ωR gives the plasma frequency squared ωpe², that is then divided by ω |Ωe|. Now this, as we've already said is larger than |Ωe| and ω and therefore this 1 can be neglected with respect to this term.

Notes

Summary

14m 48s




This is therefore equal to ωpe² / ( ω |Ωe|) Here we've got a relatively simple result because that we can write it as as ω/K = c/ωpe sqrt(|Ωe| ω) Now these terms are constant: c, ωpe, |Ωe| and therefore ω/K is proportional essentially to square root of ω which implies that the square root of ω is proportional to K which implies that -if we want to evaluate the group of velocity of the wave by deriving ω with respect to K, we find that the group velocity is proportional to the square root of ω. This is an interesting result because it tells us, first of all, that the group velocity increases with the frequency as the group velocity goes like the square root of ω, and therefore that higher frequencies will propagate faster.

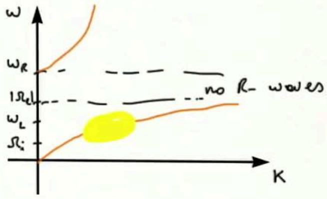
Notes

Summary

16m 57s

# Whistler waves

R- waves, $\frac{\omega}{K} <<< c$, $\Omega_i << \omega << \omega_{pe} \lesssim |\Omega_e| \lesssim \omega_R$

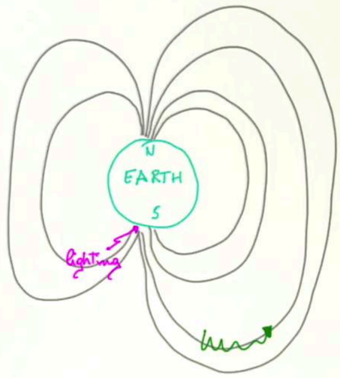



$$\frac{K^2c^2}{\omega^2} = \frac{(\omega-\omega_R)(\omega+\omega_L)}{(\omega-|\Omega_e|)(\omega+\Omega_i)} = \frac{\omega^2+\omega(\omega_L-\omega_R)-\omega_L\omega_R}{(\omega-|\Omega_e|)(\omega+\Omega_i)}$$

$$\simeq \frac{-\omega(\omega-\omega_R)+\omega_L\omega_R}{|\Omega_e|\,\omega} = 1+\frac{\omega_{pe}^2}{\omega|\Omega_e|} \simeq \frac{\omega_{pe}^2}{\omega|\Omega_e|}$$

$$\Rightarrow \frac{\omega}{K} = \frac{c}{\omega_{pe}}\sqrt{|\Omega_e|\omega} \Rightarrow \frac{\omega}{K}\propto\sqrt{\omega} \Rightarrow \sqrt{\omega}\propto K \Rightarrow \frac{\partial\omega}{\partial K}\propto\sqrt{\omega}$$

- Group velocity increases with the fequency
- Higher fequency propagate faster




Plasma

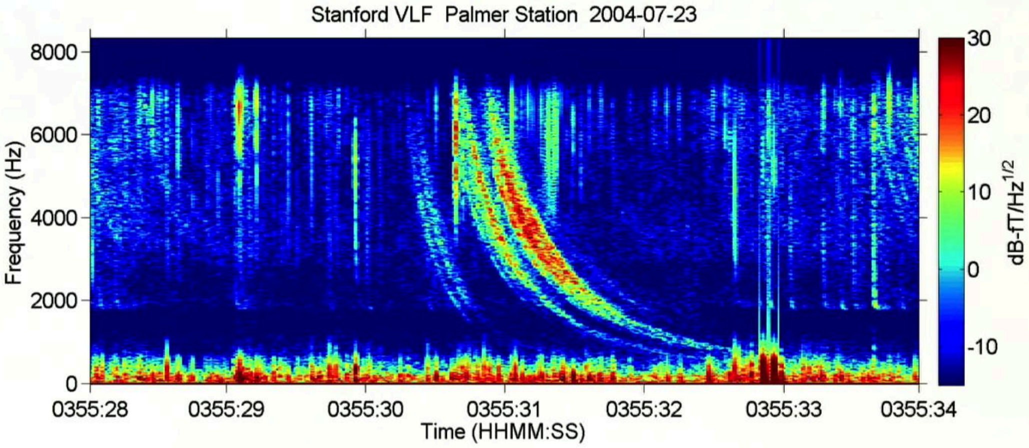
That fact that the whistler waves propagate with a group velocity that varies with the frequency is not only interesting academically it has actually consequences that are important. For example, for the propagation of waves in the ionosphere something that is of practical interest for radio transmissions, for example. In fact, let's consider our planet. There is the North Pole and the South Pole and as we know there is a magnetic field, a polar magnetic field that goes around the earth. Now let's suppose that a lightening strikes this region here, a number of waves with all frequencies will be produced and then will travel along the magnetic field surrounding the earth. The velocity at which these wave propagate along the magnetic field will depend on their frequency.

Notes

Summary

18m 15s

## Whistler waves



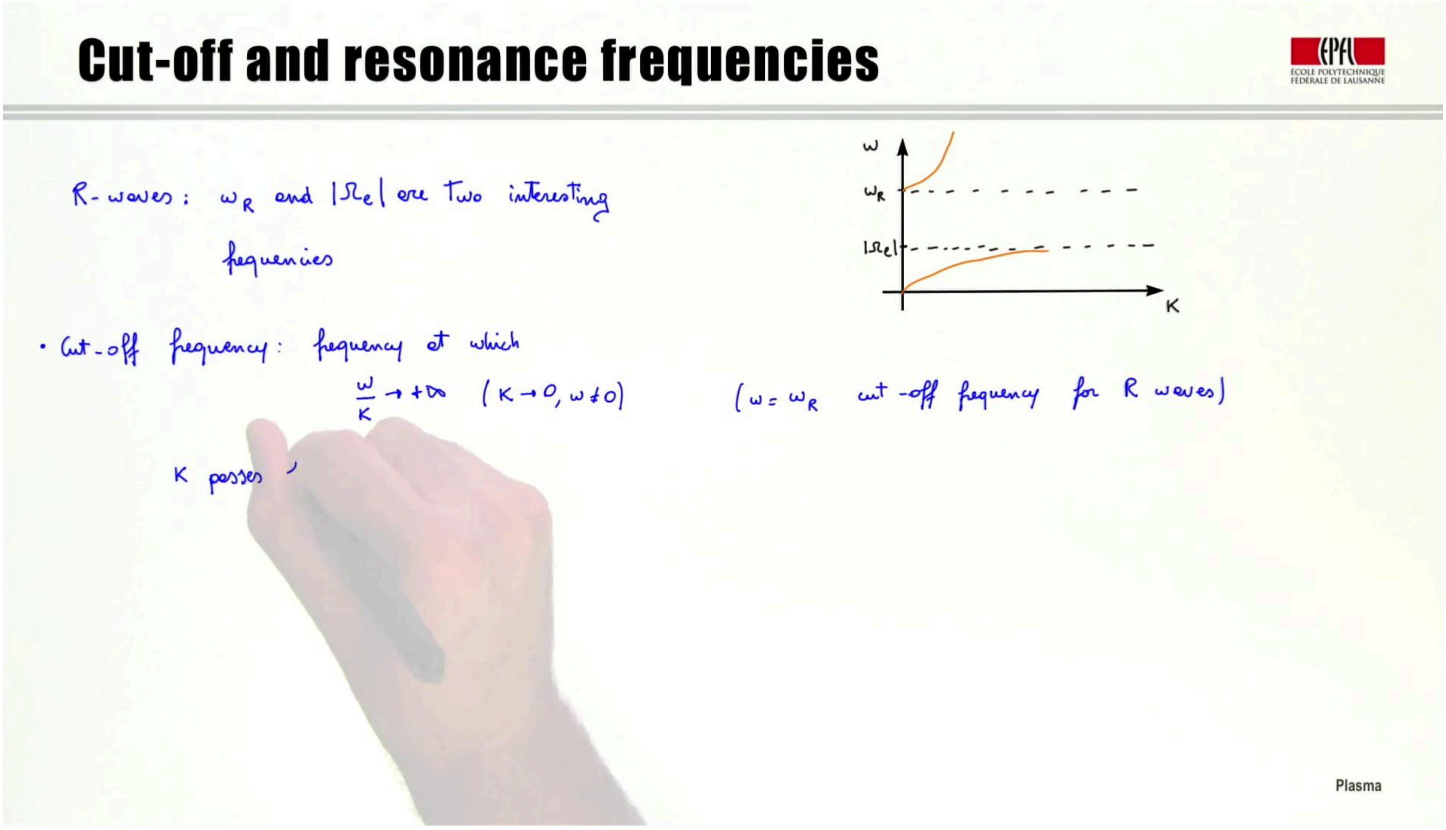

Plasma

And if we look at the spectrogram of a wave that has been traveling along the magnetic field, that is if we look at the frequency as a function of the time at which the wave has been recorded we observe that the higher frequency waves will arrive earlier than the low frequency waves. In radio transmissions, this causes a sound that is very similar to the one of whistler. This is also where the name *whistler waves* comes from, and here is the typical sound that can be heard, due to propagation of whistler waves. (static and whistling) Well you have noticed that high frequency waves arrive earlier than the low frequency ones.

Notes

Summary

19m 16s

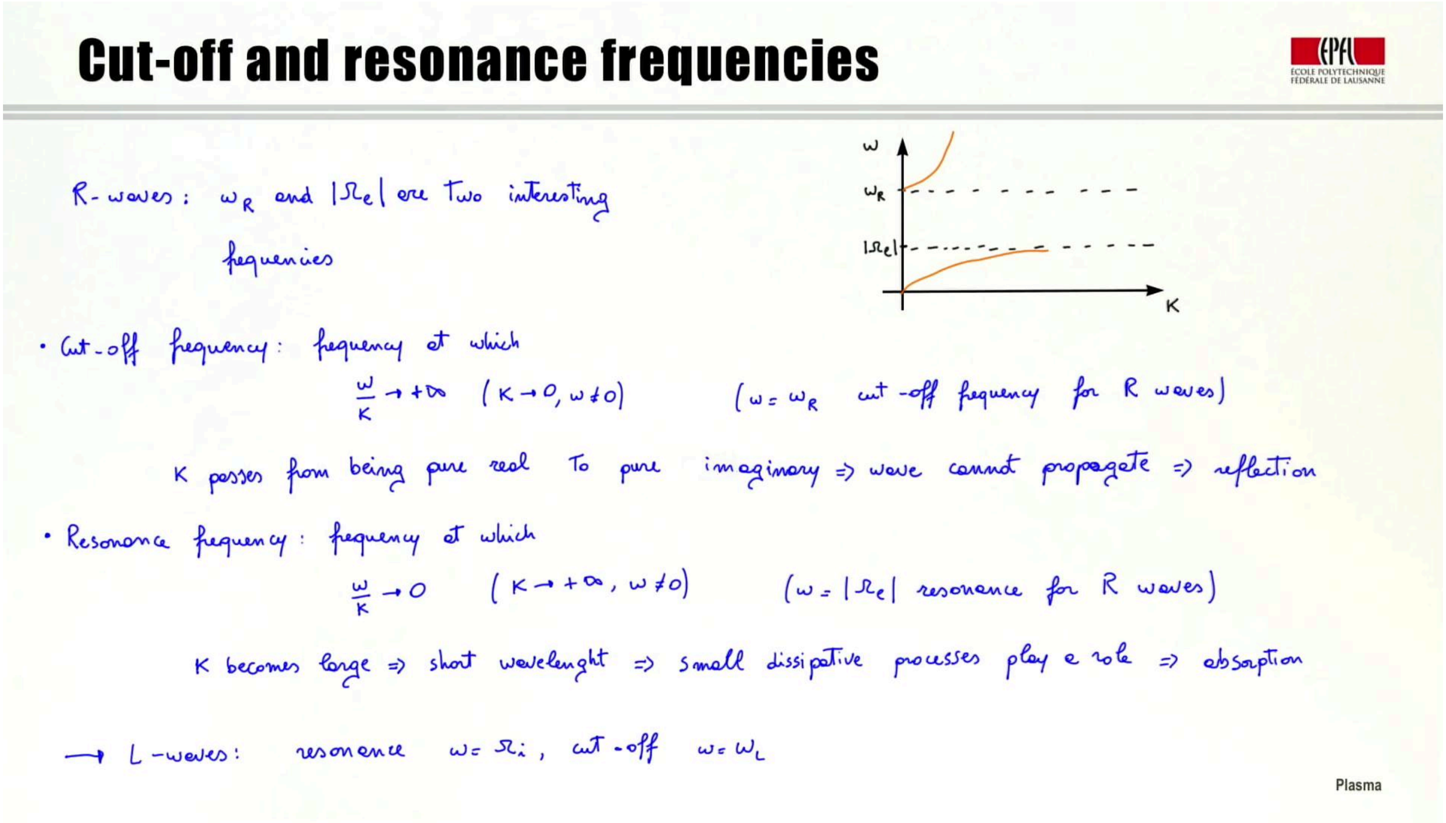



Now by looking at the dispersion relation of right and left-handed waves, we can notice that there are some frequencies that are quite interesting. The ones for which K goes to zero, or K goes to infinity. It's interesting to look at the physics of this frequencies, which are called *Cut-off and Resonant frequencies.* Okay, so let's go back to the dispersion relation that we were looking at. Let's focus on the dispersion relation of the right-handed waves. We notice that there was a region where the right-handed waves were not propagating, that is between |Ωe| and ωR frequencies, and that the dispersion relation shows an interesting behavior. As a matter of fact, for the right-handed waves it's clear that the electron cyclotron frequency |Ωe|, and the frequency ωR are two interesting frequencies. They are what we usually call cut-off and resonance frequencies. What do I mean by cut-off frequency? It's a frequency at which ω/K tends to infinity. In particular, this tends to infinity because K tends to zero, while ω is different from zero. In the present case, ω = ωR is a cut-off frequency. What happens in this region when we are close to the cut-off frequency?

Notes

Summary

20m 28s

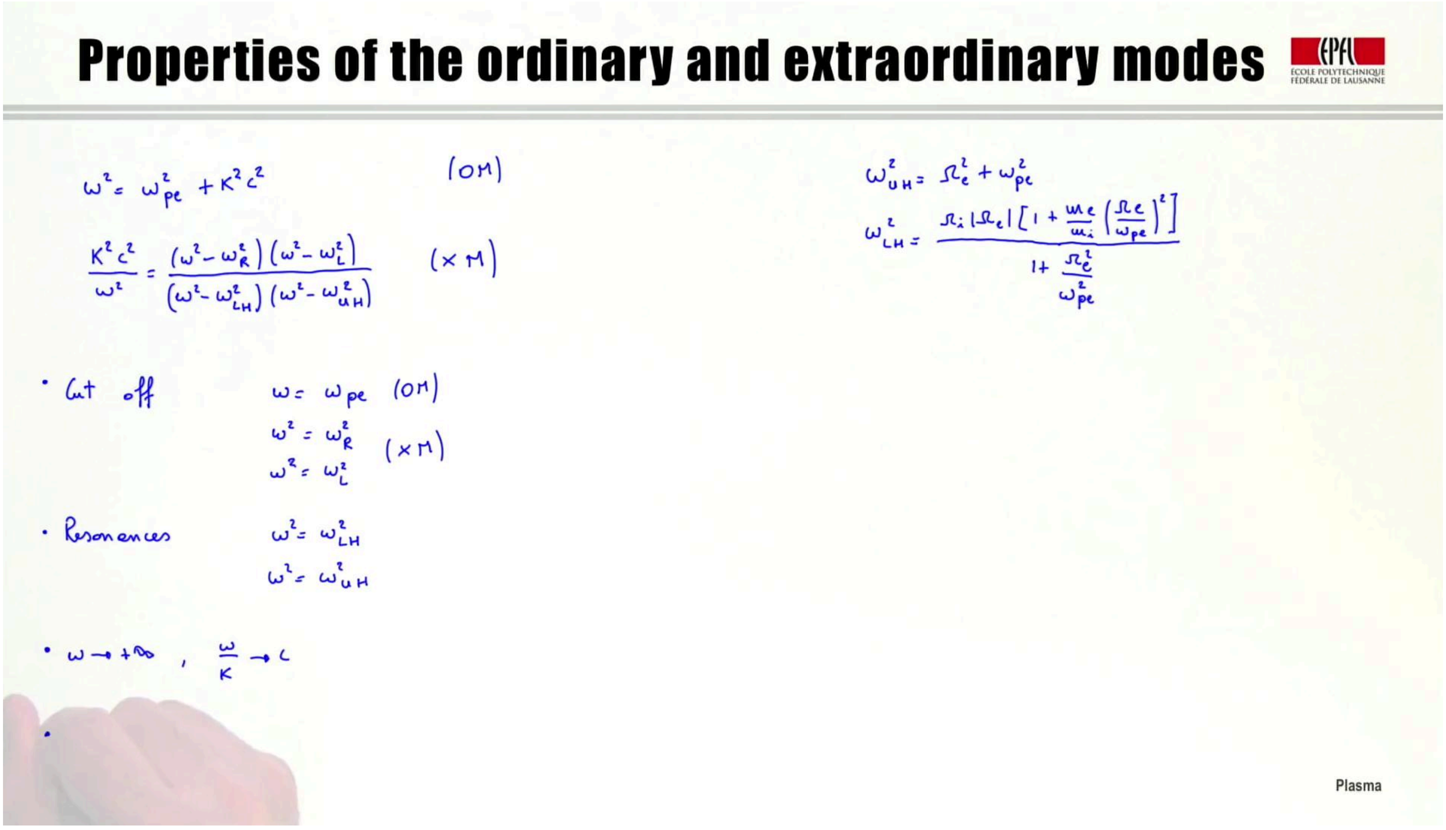



Well it happens that K passes from being purely real to being purely imaginary. That's the region where there is no propagation of the wave, therefore, the wave when it goes close to this frequency cannot propagate. And what happens to the wave? It's reflected. In other words, the cut-off frequencies, the frequency at which there is a reflection of the wave. Another interesting frequency here is the electron cyclotron frequency. This is what we call a resonance frequency. It's the frequency at which ω/K tends to zero, therefore, it's a frequency for which particularly this tends to zero because K tends to plus infinity and ω is different from zero. As I was saying for the right-handed waves ω = |Ωe| is a resonance. So what happens around this frequency? K becomes large. In this case the mode starts to have a shorter and shorter wavelength. Now, when a mode goes to very short wavelengths, small [scale] dissipative processes start to play a role and therefore, the wave is somehow dissipated because it's absorbed by the plasma through these small dissipative processes. Therefore, the resonance frequency, is a frequency at which there is absorption of the wave. This was for the right-handed waves. For the left-handed waves, we have a resonance for ω = Ωi and a cut-off for ω = ωL.

Notes

Summary

22m 06s

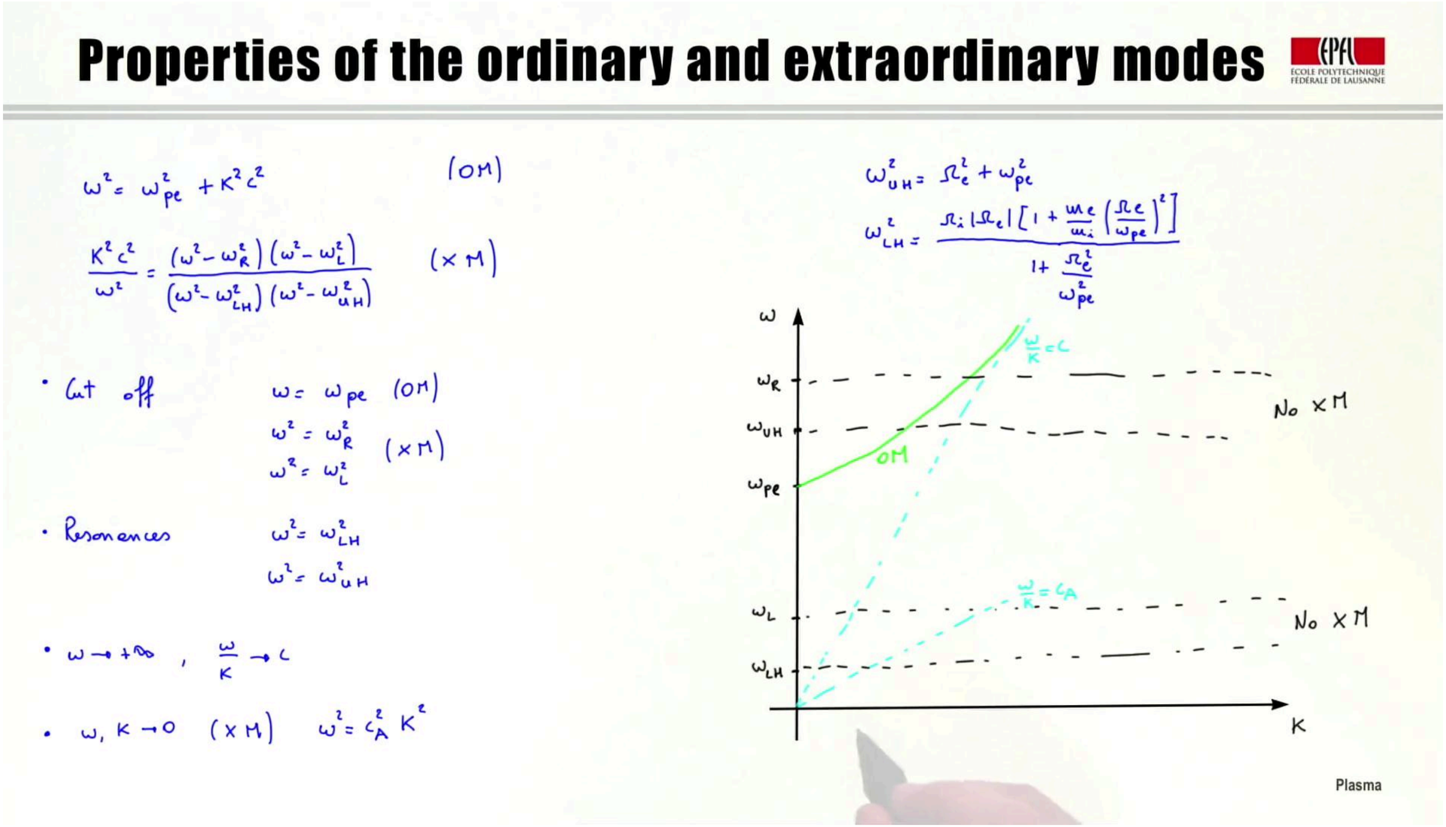



And now let's turn our attention to the modes propagating perpendicularly to the magnetic field, that is the ordinary and extraordinary modes. The dispersion relation of the ordinary mode says that ω² = ωpe² + K²c², while for the extraordinary modes we have that K² c² / ω² = (ω²-ωR²)(ω²-ωL²)/((ω²-ωLH²)(ω²-ωUH²)) This is the ordinary mode (OM) and this is the extraordinary mode (XM). I just remind you what the lower hybrid and upper hybrid frequencies are: ωUH² = Ωe² + ωpe² While the lower hybrid is given by: ωLH² = Ωi |Ωe| x... {1 + me/mi (Ωe/ωpe)²} / (1 + Ωe²/ωpe² ) First of all what are the cut-off frequencies of these modes? Those are the frequencies for which K tends to zero with ω different from zero and clearly, for the ordinary mode, it is given by ω = ωpe while for the extraordinary mode they are given by ω = ωR and ω = ωL. Resonances: there is no resonance for the ordinary mode, but there are 2 for the extraordinary one, given by the lower hybrid and upper hybrid frequencies, ω = ωLH and ω = ωUH. What is the behavior for ω that tends to infinity? For ω that tends to infinity then these 4 terms here tend to 1 and therefore K²c²/ω² tends to 1, which means that ω/K tends to c, and the same for the ordinary mode.

Notes

Summary

24m 07s

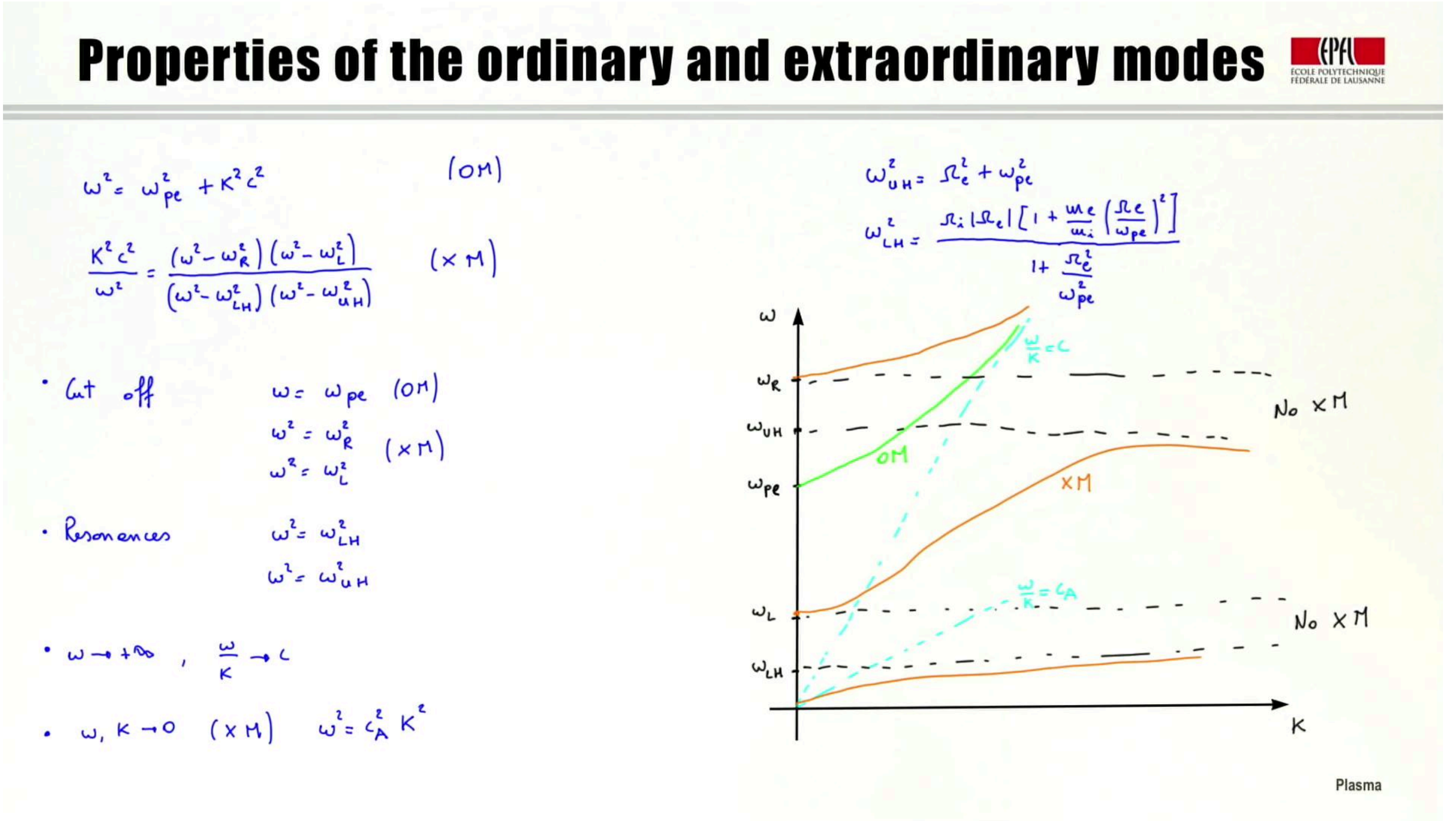



Now what happens in the limit of ω and K that tend to zero? First of all, this limit cannot be reached by the ordinary mode, so it has only relevance for the extraordinary mode. If you do the full calculation as we did for the left-handed and right-handed waves, you'll find that $\omega^2$ becomes equal to $cA^2$, the Alfvén speed squared times $K^2$. [ $\omega^2 = cA^2 K^2$ ] So we have now all the elements to draw the dispersion relation of the ordinary and extraordinary modes. First, let's point the frequencies of interest: these are: ωLH, ωL, ωpe, ωUH, and ωR. Then quite interesting there is an asymptote for ω/K = c and a second one that is for ω/K = cA. The ordinary mode cannot propagate for frequencies that are below the electron plasma frequency (ωpe) and will tend to ω/K = c - to the dispersion relation of the electromagnetic waves, for ω that grows, therefore, it will do something like that. This is the ordinary mode. And then if we look at the signs of this terms, we will see that the extraordinary mode does not propagate for frequencies such that ωLH < ω < ωL and same thing for frequencies ωUH < ω < ωR, and the dispersion relation we tend to ω/K = cA for small values of K and ω.

Notes

Summary

26m 36s

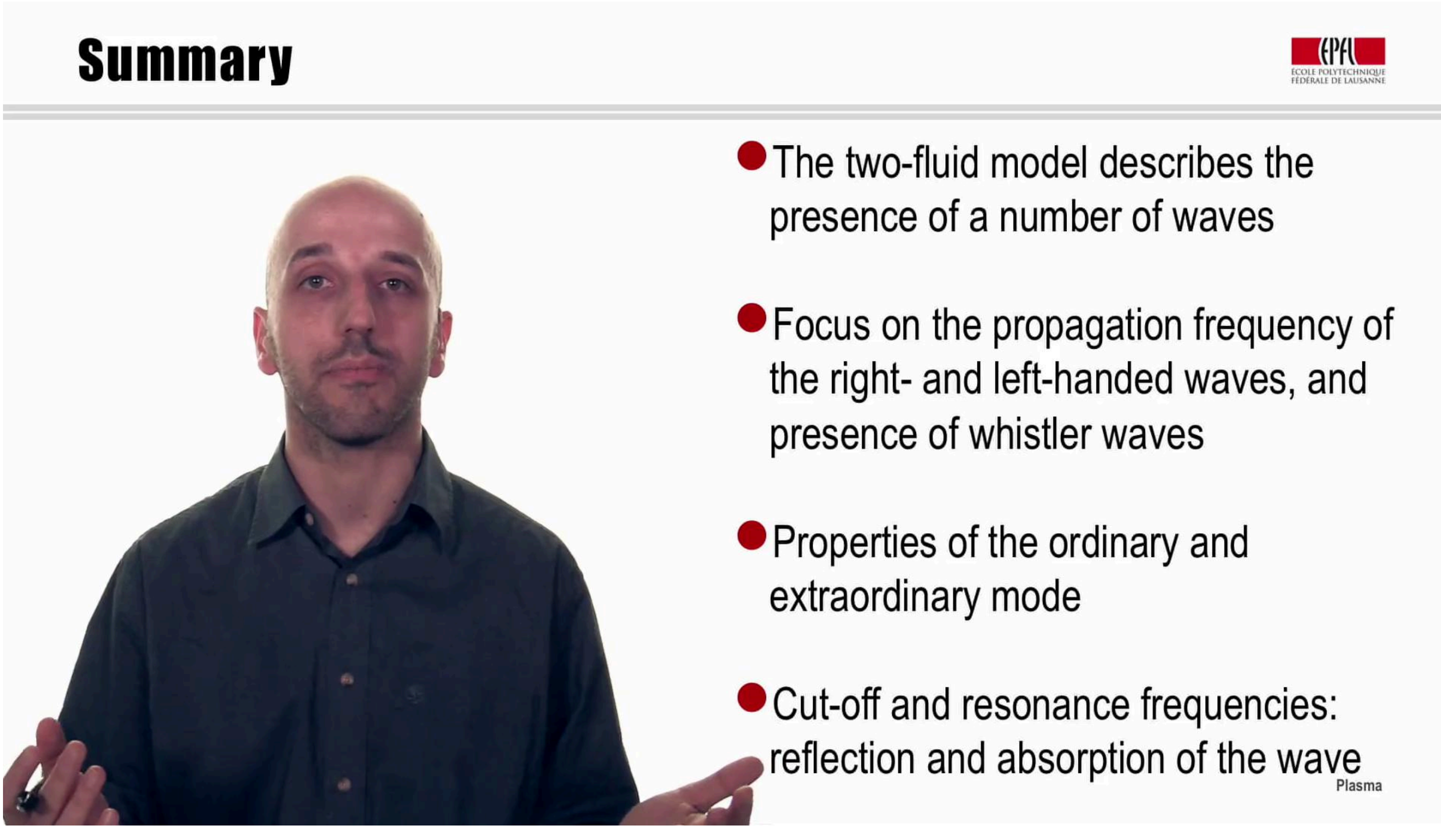



And ωLH will be a resonance frequency, therefore, the dispersion relation will do something like that. It will go back here with ωL as a cut-off frequency, and it will tend to ωUH which is the second resonance frequency, and then the mode will appear again at the cut-off frequency ωR and the dispersion relation will tend to that of the electromagnetic waves. This is the dispersion relation of the extraordinary mode.

Notes

Summary

28m 35s




In the present module we have studied some of the properties of the waves that propagate in a plasma according to the two-fluid model. First, we have pointed out the ranges of frequencies at which left-handed, right-handed waves and similiarly ordinary, extraordinary modes propagate. Then we have shown that in certain range of frequencies the right-handed waves have very interesting propagation properties, and in this range of frequencies, the right-handed waves are called whistler waves. And finally we've pointed out that there exist some frequencies, the so called *cut-off* and *resonance* frequencies that have a particular importance in the dynamics of the waves. A cut-off frequency corresponds to a wave reflection while the wave is absorbed at a resonance frequency.

Notes

Summary

29m 15s