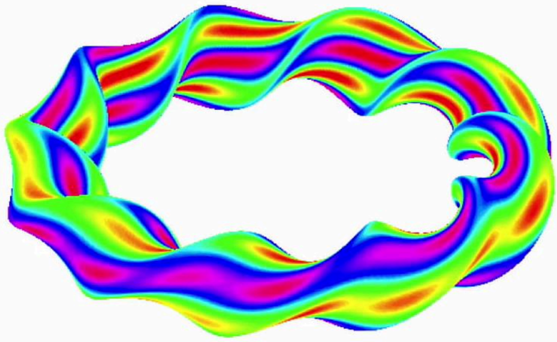
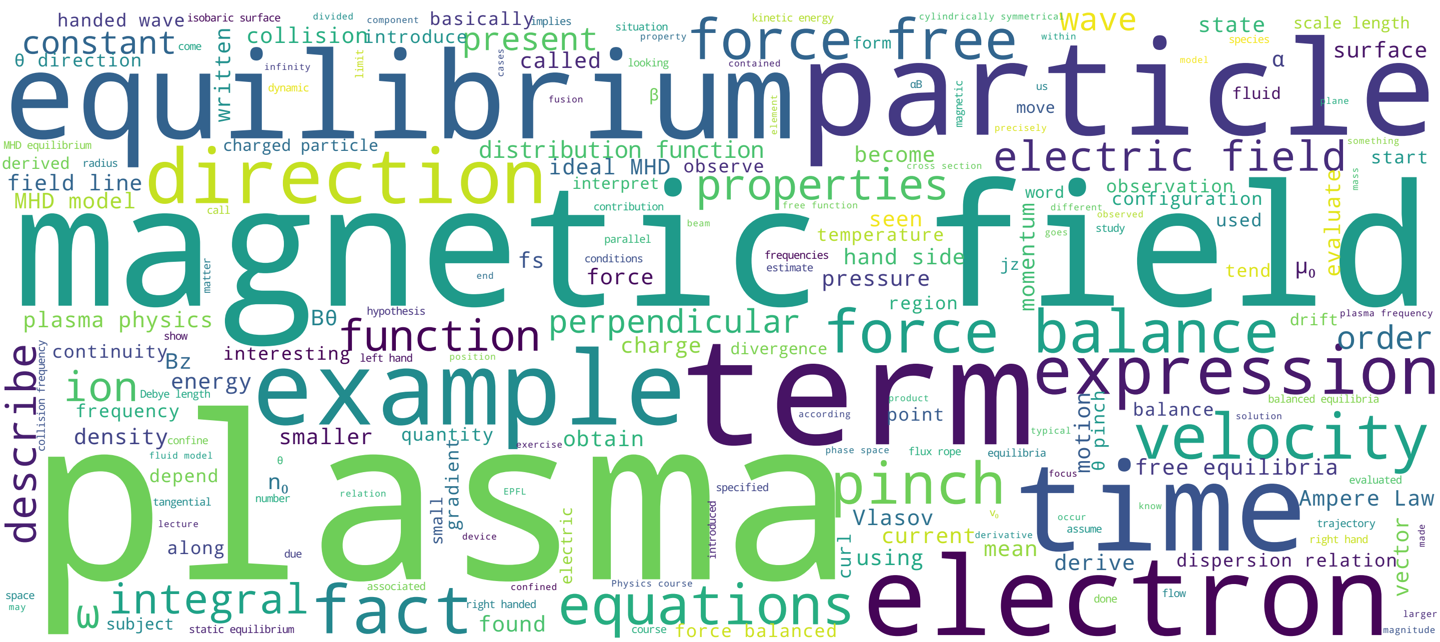


MHD equilibrium

Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

Lecture 3f

Paolo Ricci



Search MOOC

Video

Introduction

- Equations for static MHD equilibria
- Analysis of the force balance equations
- Force-free and force-balanced equilibria
- The Z-pinch and the ϑ-pinch

Plasma

Welcome to the Plasma Physics course of EPFL. Under which conditions is a standard fluid in equilibrium? That is, under which conditions do the properties of a fluid do not vary in time? This typically happens when there is a balance between the gravitational force and the forces due to pressure. In plasmas, things are much more complicated because plasmas are subject to electromagnetic fields that they have themselves generated. The goal of the present lecture is to study equilibria in plasmas. This is an extremely important subject because if you want to confine a plasma, this plasma has to be in equilibrium. We will approach this issue within the ideal MHD model. The first thing that we will do in this module will be to derive the equations that describe the equilibrium in plasma within the ideal MHD model. Then we will analyze in detail the most important equation that describes an equilibrium, that is the force balance equation. We will look at two different kinds of equilibria that can be present in plasma — the force-free and the force-balanced equilibria. Finally, I will talk about two examples of MHD equilibria — the Z-pinch and the θ-pinch.

Notes

Summary

0m 05s

The equations for the static ideal MHD equilibrium

Continuity equation: $\frac{\partial \rho_M}{\partial t} + \nabla \cdot (\rho_M \vec{V}) = 0$

Momentum equation: $\rho_M \frac{d\vec{V}}{dt} = \vec{j} \times \vec{B} - \nabla P$

Ohm's law: $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} = 0$

Closure equation: $\frac{d}{dt}\left(P \rho^{-\gamma}\right) = 0$

Maxwell equations: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \times \vec{B} = \mu_0 \vec{j}$

$\nabla \cdot \vec{B} = 0$

$$\xrightarrow[\vec{V} = 0]{\frac{\partial}{\partial t} = 0}$$

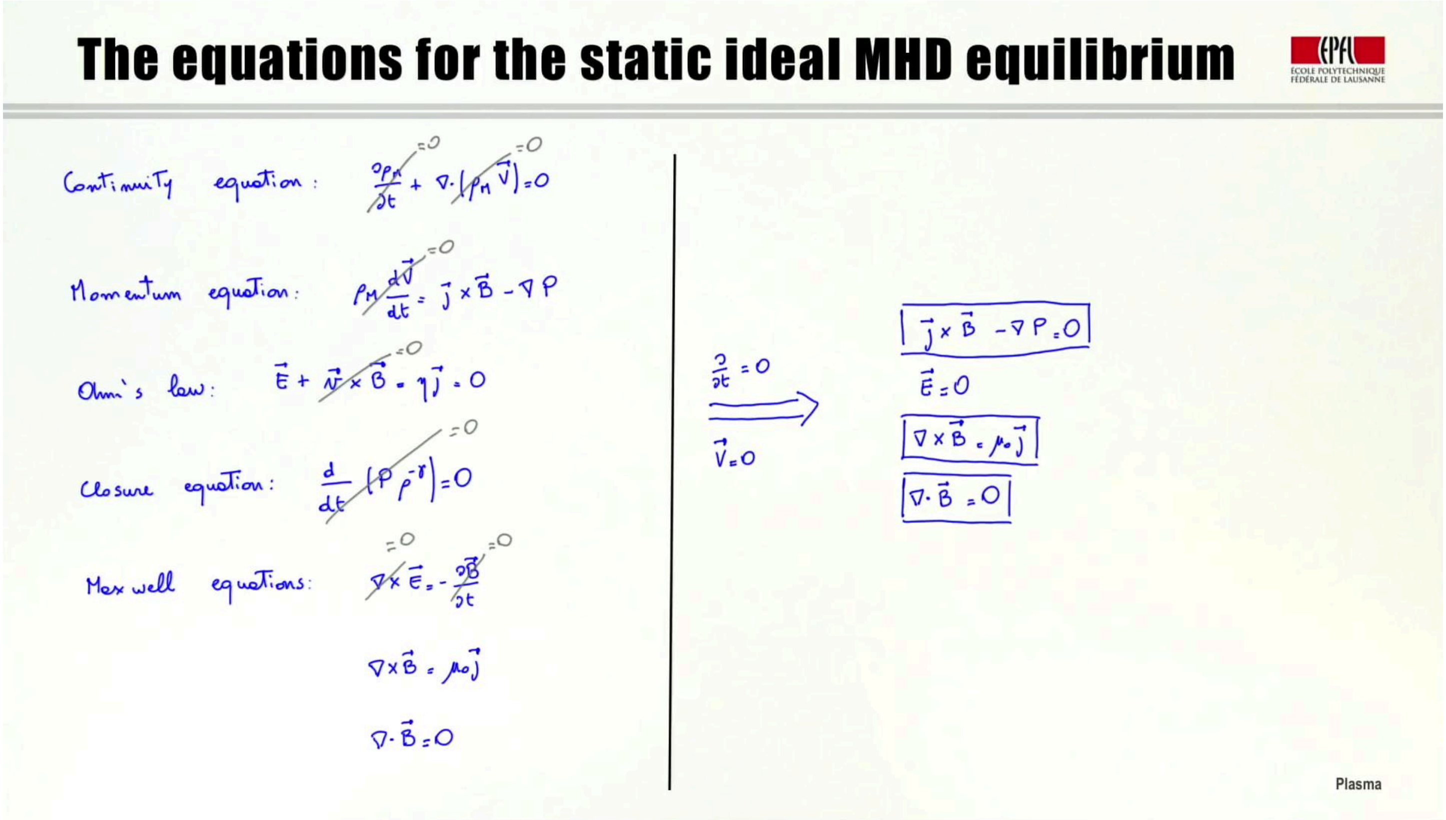
Plasma

So let's derive together the equations that describe the plasma equilibrium within the MHD model. I will start by recalling the MHD equations and then we will simplify those to look at the equilibrium situation. The equations for the MHD model are the continuity equation, which states that the mass density varies because of the plasma flow; the momentum equation, which states that the fluid moves because of the j × B and the ∇P forces; Ohm's Law, which states that the sum of the electric field plus the V × B terms is equal to ηj which, in the ideal limit, where the plasma resistivity is equal to zero, is equal to zero; then a closure equation and then the equations for the field — Maxwell's equations. ∇ × E = -∂B / ∂t, Ampere's Law, and ∇ · B = 0. From this equations we want to derive the equations that describe the static equilibrium. That is, we want to reduce this equation to the limit where the properties of the plasmas do not vary in time. That is ∂ / ∂t = 0, equilibrium, in other words. We will look at the case of static equilibrium that is the case where the velocity of the plasma is equal to zero. Let me say that equilibria that are not static — where an equilibrium flow is present are very interesting.

Notes

Summary

1m 43s



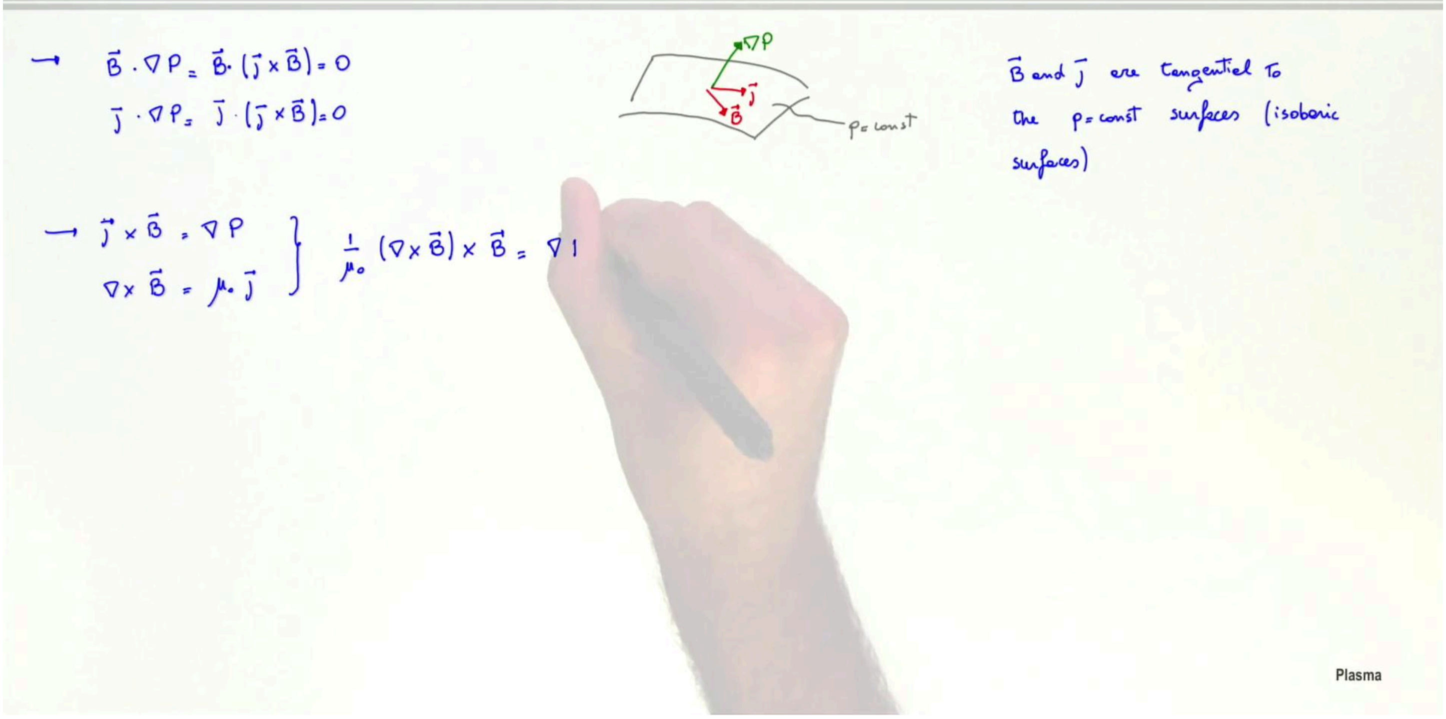
However, for the sake of simplicity, for this course we will focus our attention only on the simpler static equilibria. Within this hypothesis, we observe for the continuity equation that because of $\partial / \partial t = 0$, this goes to zero and as $V = 0$, this term goes to zero. So the continuity equation basically reduces, in static equilibrium condition, to $0 = 0$: it is always satisfied. For the momentum equation, we have that this term as $V= 0$ drops and therefore the momentum equation becomes a force-balance equation, that is $j \times B - \nabla P = 0$. Since the plasma velocity is equal to zero, we also have that this term is equal to zero. Therefore, for a static ideal MHD equilibrium, we have that the electric field has to be equal to zero. In the hypothesis of $\partial / \partial t = 0$ and $V = 0$, this term goes to zero. So it becomes $0 = 0$. Then as the electric field is equal to zero, this is equal to zero and on the other end $\partial / \partial t = 0$ implies that this is equal to zero. This equation has to be kept: $\nabla \times B = \mu_0 j$ as well as $\nabla \cdot B = 0$. These are the equations that give the condition for a static ideal MHD equilibrium. Taking out the fact that E has to be equal to zero, the three most important equations to describe the equilibrium are the first one, the force-balance equation, $j \times B - \nabla P = 0$ Ampere's Law, $\nabla \times B = \mu_0 j$ and $\nabla \cdot B = 0$.

Notes

Summary

4m 18s

Observations on force balance equation, $\mathrm{j} \times \mathrm{B} = \nabla P$



Let's now make a couple of observations on probably the most important equation to describe plasma equilibrium — the force-balance equation, j × B = ∇P. The first observation that I would like to make is the following one. If we take the product, the scalar product between B and the gradient of the pressure, which according to the force-balance equation is given by B · (j × B), you find that as (j × B) is perpendicular to B, this is equal to zero. At the same time, if one tries to evaluate j · ∇P, one finds that this is also equal to zero. Therefore, if we have a surface of constant pressure — an isobaric surface — along which P is constant, and therefore with respect to which the ∇P vector is perpendicular, we find that B and j have to be perpendicular to ∇P and therefore they have to be tangent to the surface. B and j are tangential to the P = constant surfaces, the so-called isobaric surfaces. The second observation that I would like to make comes from putting together the force-balance equation and Ampere's Law. By putting these two together, what we find is — more precisely, by replacing the value of j in the force-balance equation with the curl of B we find that 1 /μ_0 (∇ × B) that 1 /μ_0 (∇ × B) × B = ∇P.

Notes

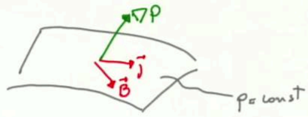
Summary

6m 35s

Observations on force balance equation, $\mathbf{j} \times \mathbf{B} = \nabla P$

→ $\vec{B} \cdot \nabla P = \vec{B} \cdot (\vec{j} \times \vec{B}) = 0$

$\vec{j} \cdot \nabla P = \vec{j} \cdot (\vec{j} \times \vec{B}) = 0$



$\vec{B}$ and $\vec{j}$ are tangential to the p = const surfaces (isobaric surfaces)

$$\left.\begin{array}{l} \rightarrow \vec{j} \times \vec{B} = \nabla P \\ \nabla \times \vec{B} = \mu_0 \vec{j} \end{array}\right\} \quad \underbrace{\frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B}}_{\text{terms of order } \sim \frac{B^2}{\mu_0 \ell}} = \underbrace{\nabla P}_{\sim \frac{p}{\ell}}$$

We introduce $\beta = \dfrac{p}{\left(\frac{B^2}{2\mu_0}\right)}$

$\beta << 1$ in typical laboratory plasmas

$\beta \sim 1$ in the Earth's magnetosphere

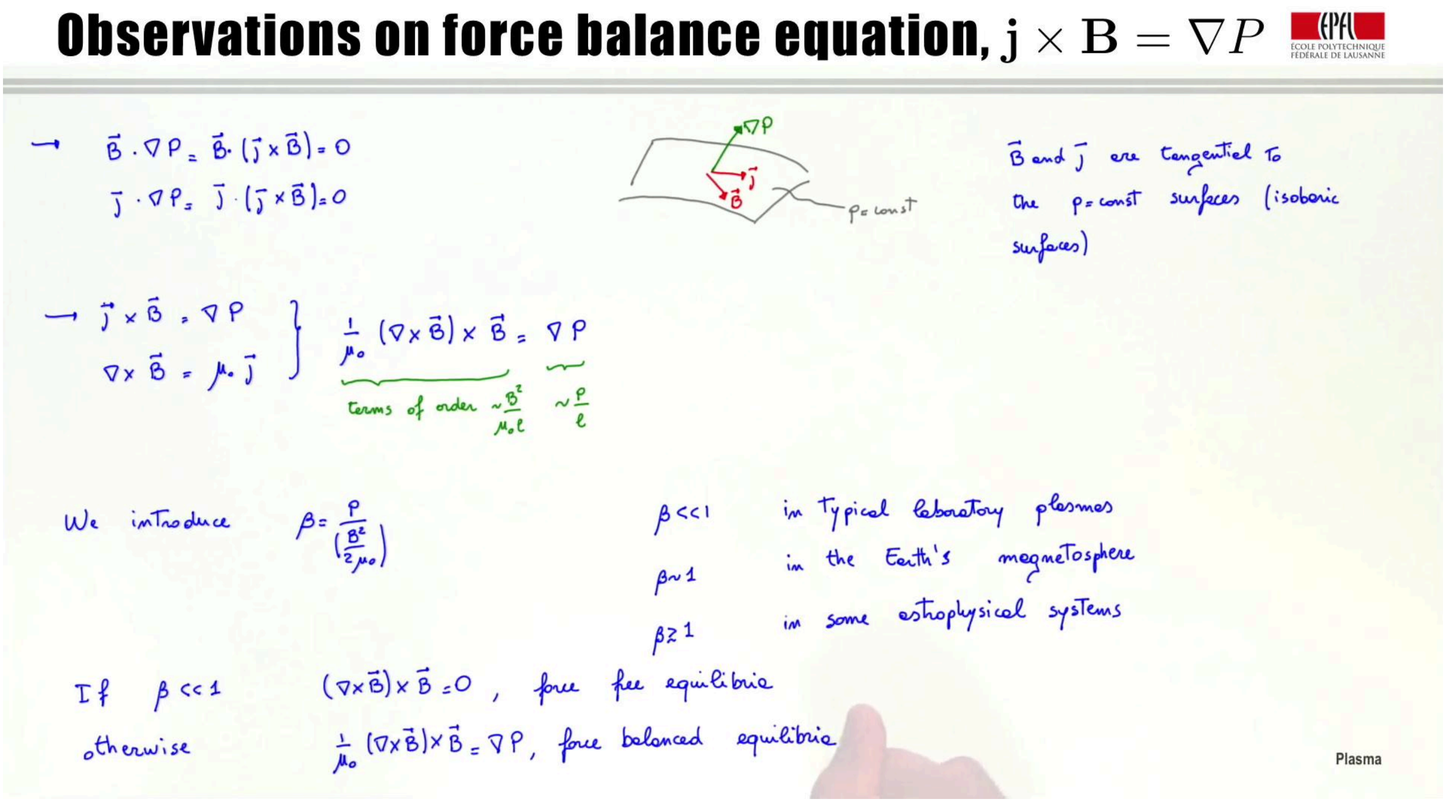
Plasma

Now if we try to estimate the order of magnitude of the terms that are contained in this left-hand side, we will find that there are terms of order B² / μ_0, and over a distance, ℓ. If we do the same for the right-hand side, we find that the term present on the right-hand side is of the order of pressure divided by scale length ℓ. What is this ℓ? It's the typical scale length of the system. We are estimating here that the spatial derivative of the curl operator can be estimated as 1 over the typical plasma scale length, that is, ℓ. In plasma physics we introduce an extremely important parameter, the so-called plasma β, which is the ratio between the estimate of the order of magnitude of the right-hand side of this equation and of the terms at the left-hand side. What is the typical value of this β, of this parameter? β is found to be much, much smaller than 1 in typical laboratory plasmas. In these cases, in fact, the plasma is very tenuous. If you want, most of the energy is contained in the magnetic field rather than in the plasma kinetic energy, but there are also less tenuous plasmas for which β is comparable to one, for example in the Earth's magnetosphere, where the magnetic field is relatively weak.

Notes

Summary

8m 51s



Then one can find plasmas in astrophysical systems, where β can be even larger than one. In this case, in fact, the magnetic field can be weak, and most of the energy can be in the form of thermal energy. If β is much, much smaller than one then this term here has to be equal to zero. The terms contained in (∇ × B) × B have to balance themselves. This is what we call force-free equilibria. Otherwise, we have that the equilibrium really comes from the balance of both these terms, the right- and left-hand sides, and therefore one has that 1 / μ_0 (∇ × B) × B = ∇p. These are the force-balanced equilibria.

Notes

Summary

10m 54s

Force-free equilibria

$$\vec{j} \times \vec{B} = \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B} = 0 \quad \Rightarrow \quad \nabla \times \vec{B} = \alpha \vec{B}$$

- Properties of α

$$\nabla \cdot (\nabla \times \vec{B}) = 0 = \nabla \cdot (\alpha \vec{B})$$

$$\nabla \cdot (\alpha \vec{B}) = \alpha \underbrace{\nabla \cdot \vec{B}}_{0} + \vec{B} \cdot \nabla \alpha = \vec{B} \cdot \nabla \alpha = 0$$

So let's take a look at the properties of force-free equilibria and let's take a look at an example. Actually, as you will see, such a beautiful structure that you can observe on the Sun's surface is an example of force-free equilibrium. Force-free equilibria, as we have just said, are equilibria for which j × B, which we can write in terms of the curl of B, thanks to Ampere's Law, is equal to zero. In other words, force-free equilibria appear when the curl of B is parallel to B. In this case, in fact, the vectoral product is equal to zero. Therefore, ∇ × B has to be equal to some function — α, scalar function — times B. It's interesting to take a look at what the properties are of this α. If we evaluate the divergence of the curl of B, this is equal to zero because of the properties of a vector. Because of this equation, this is equal to the ∇·(αB). If we try to evaluate the divergence of αB, we find that this is equal to α∇·B + B·∇α. This is equal to zero because of the property of the magnetic field and therefore we arrive to the fact that B · ∇α is equal to zero. If we consider surface of α equal to constant, we will find that the gradient of α, which is perpendicular to this surface, has to be perpendicular to B and therefore B has to be tangential to the α equal to constant plane.

Notes

Summary

11m 54s

Force-free equilibria

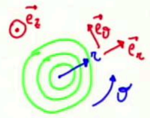
$$\vec{j}\times\vec{B} = \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} = 0 \quad \Rightarrow \quad \nabla\times\vec{B} = \alpha\vec{B}$$

- Properties of α

$$\nabla\cdot(\nabla\times\vec{B}) = 0 = \nabla\cdot(\alpha\vec{B})$$

$$\nabla\cdot(\alpha\vec{B}) = \alpha\underbrace{\nabla\cdot\vec{B}}_{0} + \vec{B}\cdot\nabla\alpha = \vec{B}\cdot\nabla\alpha = 0$$

- Example of force free equilibrium, cylindrical symmetry $\left(\frac{\partial}{\partial\theta}=0\right)$



$\nabla\times\vec{B} = \alpha\vec{B}$ is satisfied by $\vec{B} = B_\theta\vec{e}_\theta + B_z\vec{e}_z$

with $B_\theta = \dfrac{B_0 K r}{1+K^2r^2}$

A demonstration similar to the one I just did will be used in the exercises to show an interesting property of force-free equilibria, that is, that the current is proportional to the magnetic field. Can we give an example of a force-free equilibrium? We will give it for a relatively simple configuration that is the one where there is cylindrical symmetry. We will consider a situation where the properties of the plasmas and the magnetic field are cylindrically symmetric, therefore they will depend on the radius, but they won't depend on θ, therefore for which ∂ / ∂θ = 0. In order to better describe this configuration, we will introduce er, that is, the unitary vector in the r direction, eθ. We won't forget that the properties can still depend on z where ez is out of the plane. Now one can show that the equation ∇ × B = αB is satisfied by a magnetic field that has the following form: It has a component along the θ direction and a component along the z direction. with Bθ equal to (B_0 K r) / (1 + k²r²) where B_0 is a constant that has the units of a magnetic field, where k is also a constant that has the units of 1/length and sets the size, the scale length of this equilibrium in the radial direction.

Notes

Summary

14m 10s

Force-free equilibria

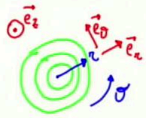
$$\vec{j}\times\vec{B} = \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} = 0 \quad\Rightarrow\quad \nabla\times\vec{B} = \alpha\vec{B}$$

- Properties of α

$$\nabla\cdot(\nabla\times\vec{B}) = 0 = \nabla\cdot(\alpha\vec{B})$$

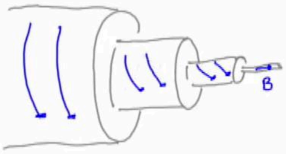
$$\nabla\cdot(\alpha\vec{B}) = \alpha\underbrace{\nabla\cdot\vec{B}}_{0} + \vec{B}\cdot\nabla\alpha = \vec{B}\cdot\nabla\alpha = 0$$

- Example of force free equilibrium, cylindrical symmetry $\left(\frac{\partial}{\partial\theta}=0\right)$

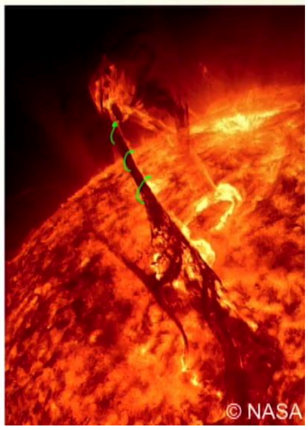


$\nabla\times\vec{B} = \alpha\vec{B}$ is satisfied by $\vec{B} = B_\theta\vec{e}_\theta + B_z\vec{e}_z$

with $B_\theta = \frac{B_0 K r}{1+K^2r^2}$ $\quad B_z = \frac{B_0}{1+K^2r^2}$ $\quad \left(\alpha = \frac{\mu_0 j_z}{B_z}, \quad j_z = \frac{2KB_0/\mu_0}{(1+K^2r^2)^2}\right)$



"Flux rope"

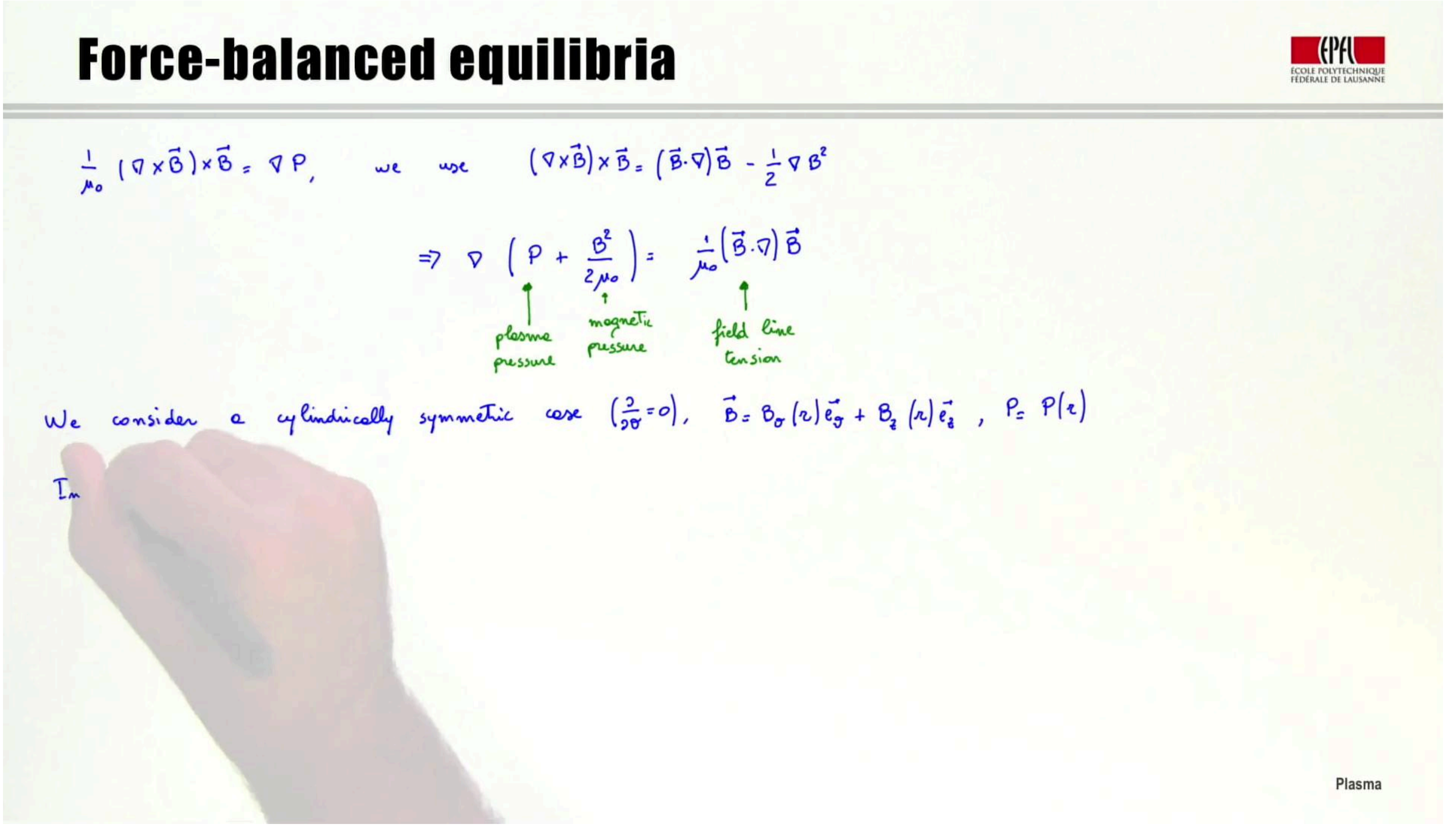


Bz is given by B_0 / (1 + k²r²). With a magnetic field of this form, one can readily evaluate the value of α, which is found to be equal to μ_0 jz over Bz, where jz is the current along the z direction that can be evaluated by considering Ampere's Law and is equal to r k Bz / (1 + k²r²)². How does such a configuration look? Let's imagine a cylinder and look at the different radii, the direction of the magnetic field, in other words we will — we will look at the magnetic field lines as they lie on these nested cylindrical surfaces and what we observe at small radii is that the θ component is very small and therefore most of the magnetic field is in the z direction. Therefore B will be in this direction. At larger radii, this term will grow and we will find something like that. At even larger radii, this will grow far more, up to the largest radii, where Bθ will dominate over Bz. Therefore the magnetic field will be mostly in the θ direction. What we have just drawn is what is called a flux rope, an MHD equilibrium that is a underlined, for example, by this nice beautiful structure that we can observe on the Sun's surface. In this region, in fact, the plasma β is very small and the equilibrium can be well described by the force-free equations. Here, for example, you will have a magnetic field that spirals around this flux rope and it's able to confine the plasma.

Notes

Summary

16m 28s



We have taken a quick look at properties of force-free equilibria. Now let's concentrate on force-balanced equilibria. In this case, the force-balance equation is given by $(1 / \mu_0)$ $(\nabla \times B) \times B = \nabla P$. To make some progress, we will use a property of vectors that says that this expression here is equal to $(B \cdot \nabla)B - 1/2\nabla B^2$. This implies that the expression, the force-balance equation that we have written here, can be written as the gradient of (P plus this term here, $B^2 / 2\mu_0$) equal to the term contained here. How can we interpret this equation here? This will give us a balance between this term here, that is the plasma pressure, a term here that we can interpret as — which still has the dimension of a pressure and that we can interpret as the pressure associated to the magnetic field, the *magnetic pressure*, and a term which is present if the B field is curved, something that we can interpret as the tension of the magnetic field lines. This is the general expression for an equilibrium. We may consider the simplest case, as we have seen in the previous slide, the one of a cylindrically symmetrical system with a magnetic field of the form... and a pressure given by a function of r.

Notes

Summary

19m 28s

Force-balanced equilibria

$\frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} = \nabla P$, we use $(\nabla\times\vec{B})\times\vec{B} = (\vec{B}\cdot\nabla)\vec{B} - \frac{1}{2}\nabla B^2$

$$\Rightarrow \nabla\Big(\underbrace{P}_{\text{plasma pressure}} + \underbrace{\frac{B^2}{2\mu_0}}_{\text{magnetic pressure}}\Big) = \underbrace{\frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B}}_{\text{field line tension}}$$

We consider a cylindrically symmetric case $\left(\frac{\partial}{\partial\theta}=0\right)$, $\vec{B} = B_\theta(r)\vec{e}_\theta + B_z(r)\vec{e}_z$, $P = P(r)$

In this case $\frac{d}{dr}\left(P + \frac{B_\theta^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}$

From $\vec{j} = \frac{1}{\mu_0}\nabla\times\vec{B}$, $\vec{j} = j_\theta\vec{e}_\theta + j_z\vec{e}_z$, with $j_\theta = -\frac{1}{\mu_0}\frac{dB_z}{dr}$ and $j_z = \frac{1}{\mu_0 r}\frac{d}{dr}(rB_\theta)$

Three functions: $B_\theta(r)$, $B_z(r)$, and $P(r)$, two can be specified arbitrarily, the third one can be determined, subject to appropriate boundary conditions.

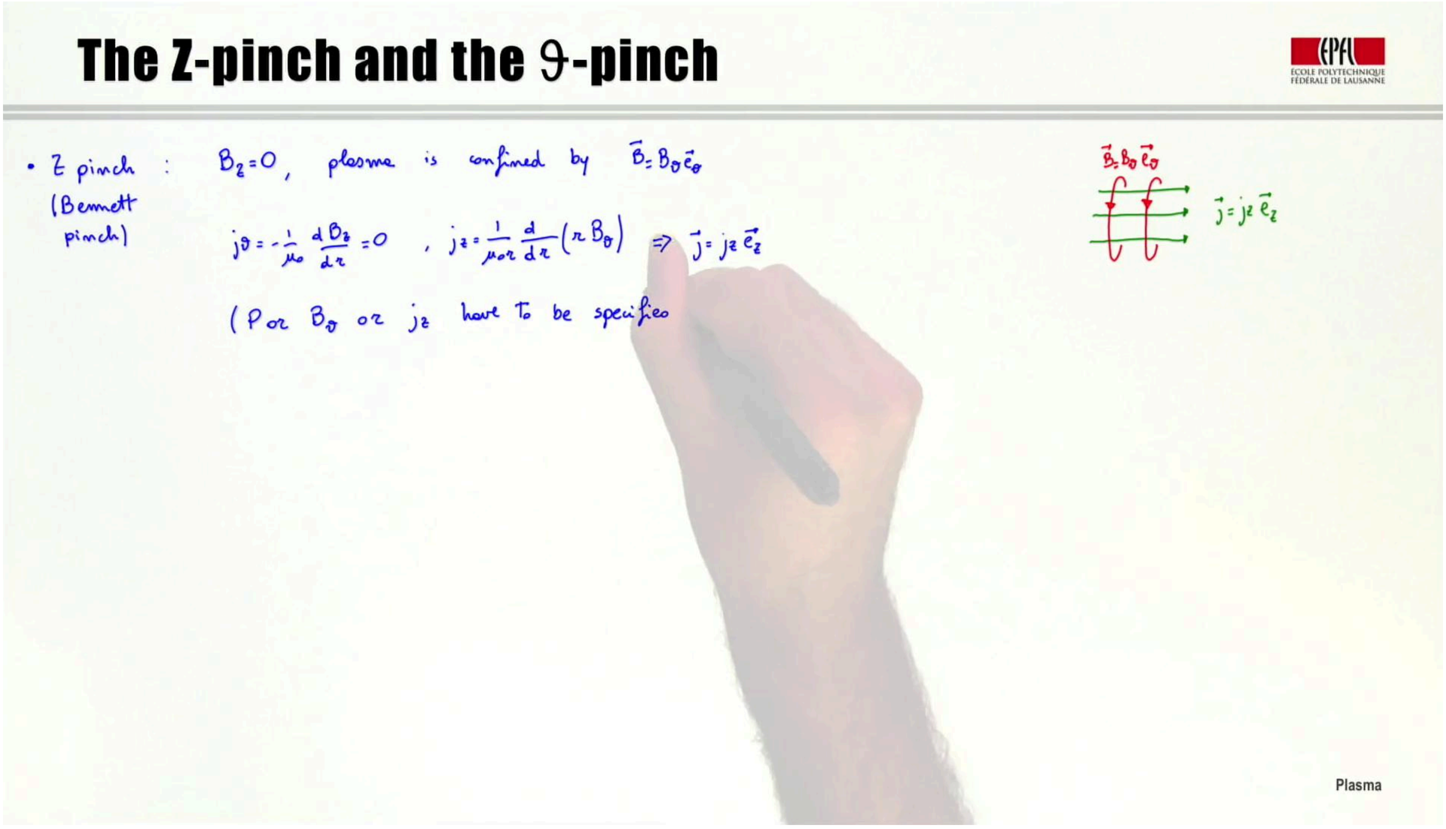
Plasma

In this case, the force-balance equation here can be written as d / dr, pressure, the sum of Bθ and Bz in evaluating to B^2 term, and this term, field line tension term, becomes $B\theta^2 / \mu_0 r$. Now by using Ampere's Law we can also find what is the expression of j associated with this magnetic field. We find, in fact, that j is equal to the sum of the components along θ and along z with jθ given by the derivative along the radius of Bz and jz by this expression. As a matter of fact, here we have three functions: P, Bθ and Bz. Two of these functions can be specified arbitrarily. Once you have specified two functions, the third one can be determined by using the force-balance equation, of course, subject to the appropriate boundary conditions.

Notes

Summary

21m 59s



We have just derived the conditions under which a cylindrically symmetric plasma is in equilibrium. Now let's take a look at two cases: the Z-pinch and the θ-pinch, where the plasma, cylindrically symmetrical, is in equilibrium. In the Z-pinch, which is also called Bennett's pinch, one assumes that Bz = 0. This specifies, actually, one of the free functions that we had found, and therefore that one specified that the plasma is confined by a magnetic field that is only in the θ direction. By using the relation between current and magnetic field that we have seen in the previous slide, one finds that jθ = 0 and therefore that j is only in the z direction. What we are therefore looking at is a configuration where there is a current that flows in the z direction and the plasma is confined by a magnetic field that is in the θ direction. Therefore, if you want, it pinches the plasma in the z direction. This configuration we have specified one of the two free functions and therefore we are left with one free function. We can therefore specify P or Bθ or jz.

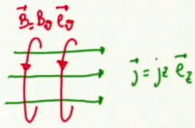
Notes

Summary

23m 42s

The Z-pinch and the ϑ-pinch

- Z pinch (Bennett pinch) : $B_z = 0$, plasma is confined by $\vec{B} = B_\theta \vec{e}_\theta$



$j_\theta = -\frac{1}{\mu_0}\frac{dB_z}{dr} = 0$, $j_z = \frac{1}{\mu_0 r}\frac{d}{dr}(r B_\theta) \Rightarrow \vec{j} = j_z \vec{e}_z$

(P or B_θ or j_z have to be specified)

Example :

$$j_z = \begin{cases} j_{z0} & (r \le a) \\ 0 & (r > a) \end{cases} \Rightarrow B_\theta = \begin{cases} \frac{\mu_0 j_{z0} r}{2} = \frac{\mu_0 I r}{2\pi a^2} & r \le a \\ \frac{\mu_0 I}{2\pi r} & r > a \end{cases}$$

$$I = \int_0^a j_z 2\pi r \, dr = \pi a^2 j_{z0}$$

From force balance equation $\frac{d}{dr}\left(P + \frac{B_\theta^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}$, with $P(r \ge a) = 0$, $P(r) = \mu_0 \left(\frac{I}{2\pi a}\right)^2 \left(1 - \frac{r^2}{a^2}\right)$, $r \le a$

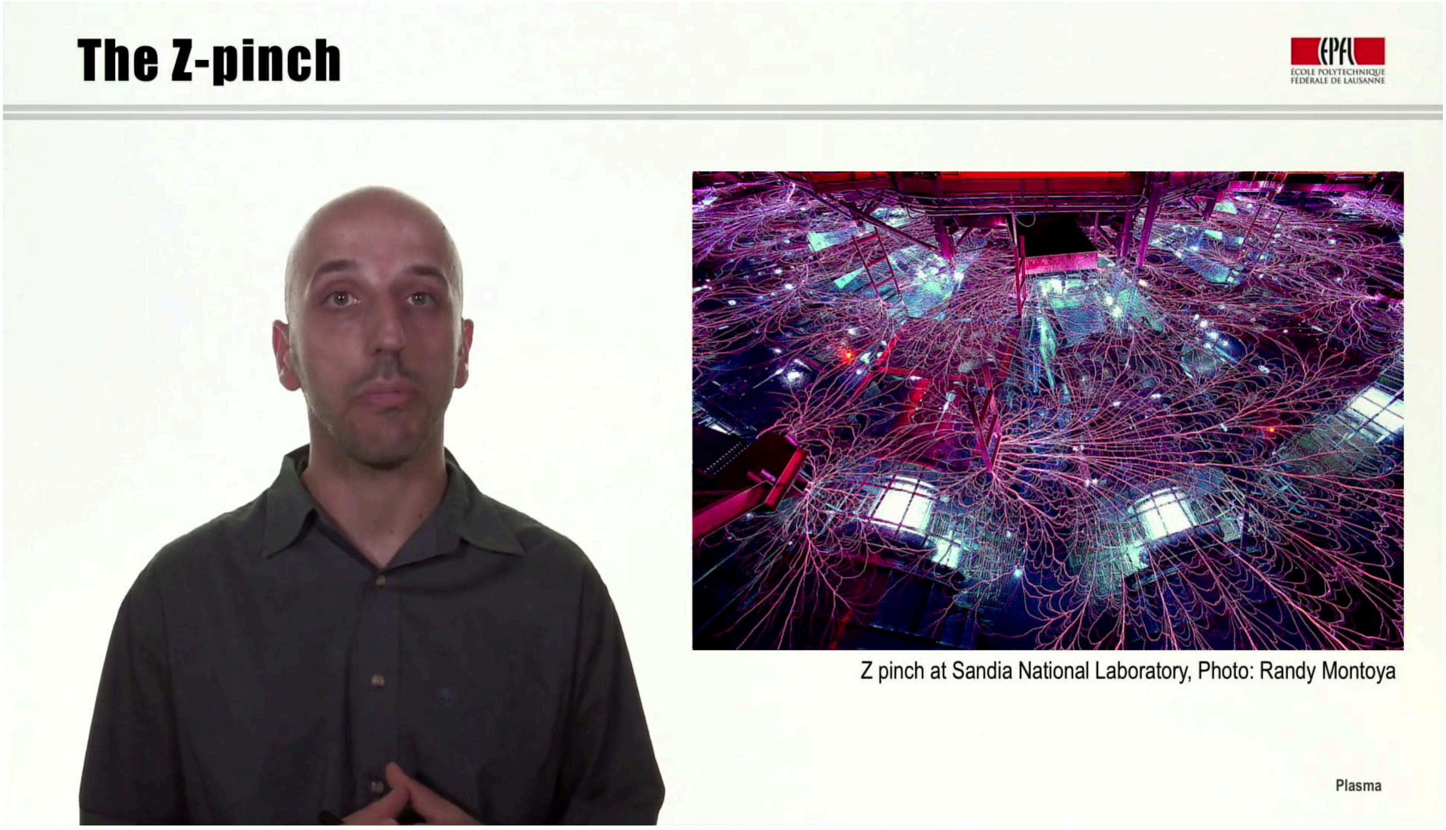
Plasma

Let's take a look at an example of Z-pinch equilibrium, a simple example, the one where a uniform current passes through the plasma and the current is equal to zero outside the plasma — therefore, a plasma where jz = jz0 inside the plasma and is equal to zero outside. The total current flow in the plasma is given by I, that is the integral over jz over the full plasma cross-section, which is equal to π a² jz0. Having this expression, one can derive Bθ. To do that, one can simply integrate this expression — if you do these steps, what you obtain is that Bθ, which is a function of r, is equal to this expression, which can also be written in terms of I, for r < a and r larger than a. From the force-balanced equation, then, one can derive the value of P. This has to be subject to the correct boundary condition, that is the plasma pressure has to be equal to zero for r ≥ a, from which one obtains P(r), which is valid for r < a.

Notes

Summary

25m 40s



Let me say that studying the Z-pinch is not a purely academic exercise. In fact, among the first machines that were used to make fusion, the Z-pinch had an important role. Nowadays, the Z-pinch still exists as a source of x-rays. The plasma, typically, is made by heavy atoms — is compressed by using the forces that act on a Z-pinch and a huge burst of x-rays is produced. An example is shown here with the Z-pinch at the Sandia National Laboratory in New Mexico.

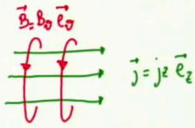
Notes

Summary

27m 34s

The Z-pinch and the ϑ-pinch

- Z pinch (Bennett pinch) : $B_z = 0$, plasma is confined by $\vec{B} = B_\theta \vec{e}_\theta$



$j_\theta = -\frac{1}{\mu_0}\frac{dB_z}{dr} = 0$, $j_z = \frac{1}{\mu_0 r}\frac{d}{dr}(r B_\theta)$ $\Rightarrow$ $\vec{j} = j_z \vec{e}_z$

(P or B_θ or j_z have to be specified)

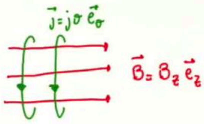
Example :

$$j_z = \begin{cases} j_{z0} & (r \le a) \\ 0 & (r > a) \end{cases} \Rightarrow B_\theta = \begin{cases} \frac{\mu_0 j_{z0} r}{2} = \frac{\mu_0 I r}{2\pi a^2} & r \le a \\ \frac{\mu_0 I}{2\pi r} & r > a \end{cases}$$

$$I = \int_0^a j_z 2\pi r \, dr = \pi a^2 j_{z0}$$

From force balance equation $\frac{d}{dr}\left(P + \frac{B_\theta^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}$, with $P(r \ge a) = 0$, $P(r) = \mu_0 \left(\frac{I}{2\pi a}\right)^2 \left(1 - \frac{r^2}{a^2}\right)$, $r \le a$

- ϑ pinch : $B_\theta = 0$, $\vec{j} = j_\theta \vec{e}_\theta$



Force balance equation $\frac{d}{dr}\left(P + \frac{B_z^2}{2\mu_0}\right) = 0$

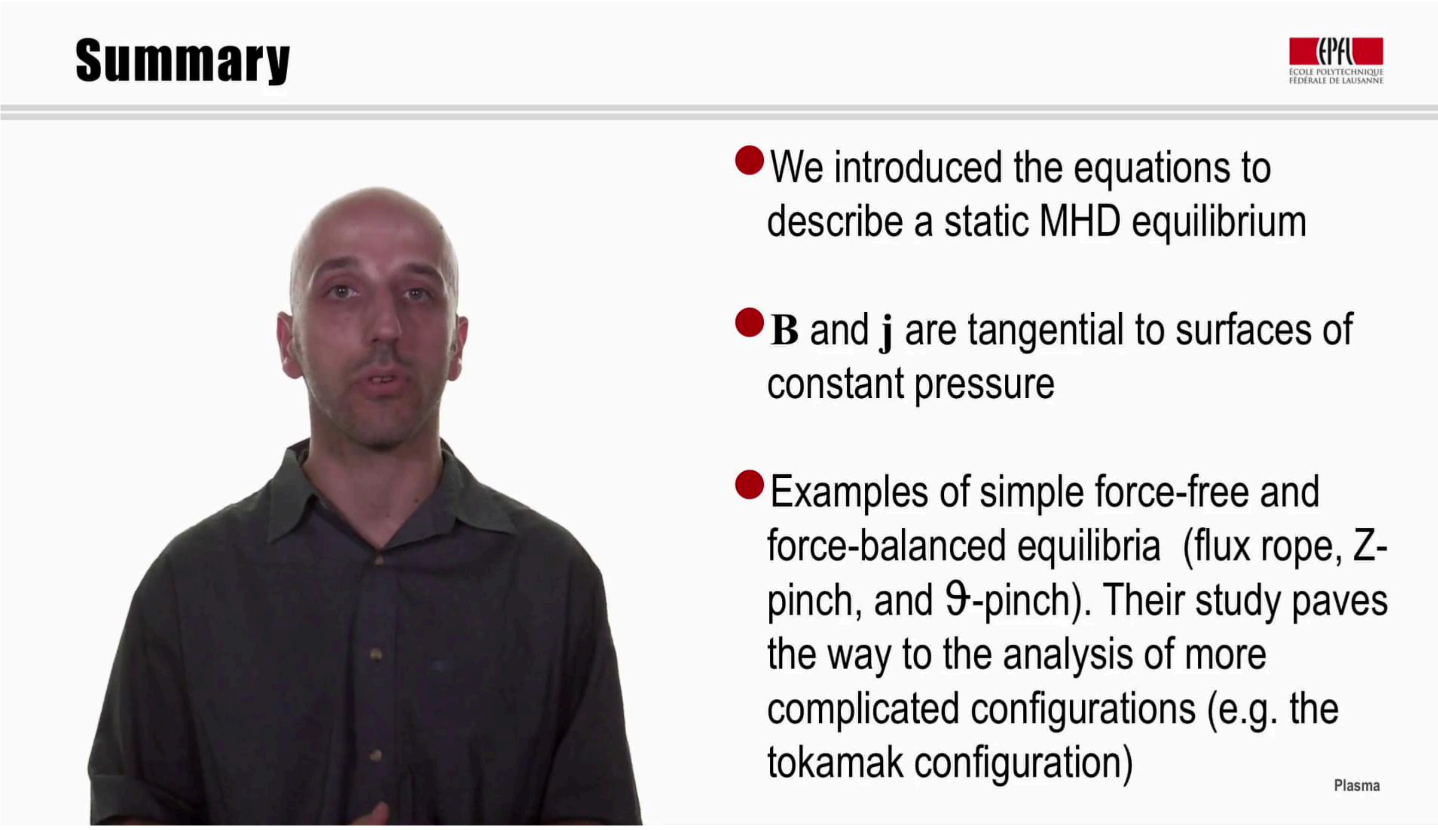
Plasma

We have seen the Z-pinch; there is another simple configuration that we can analyze in this cylindrically symmetrical case that is the θ-pinch. In the θ-pinch we impose that the Bθ component is equal to zero. Therefore, from this equation we find that j is in the eθ direction. If you want, the situation is reversed with respect to the Z-pinch. We have current that flows in the θ direction and a magnetic field that is in the z direction. The force-balance equation becomes something very simple where the contribution of the field line tension disappears as the magnetic field is straight.

Notes

Summary

28m 24s



Here we get to the conclusion of the present module where we have derived the equations that describe the equilibrium in plasmas within the ideal MHD model. The first thing that we have observed is that the vectors B and j are tangential to isobaric surfaces. Then we have observed that different kinds of equilibria can be present in a plasma — force-free and force-balanced equilibria. We have given some examples of these equilibria — the flux rope, the Z-pinch, and the θ-pinch. These are relatively simple scenarios, but let me say that they contain all the elements to look at the most complicated cases, such as the Tokamak, that are actually used nowadays to confine plasma and reach the conditions necessary for fusion reactions to occur.

Notes

Summary

29m 31s