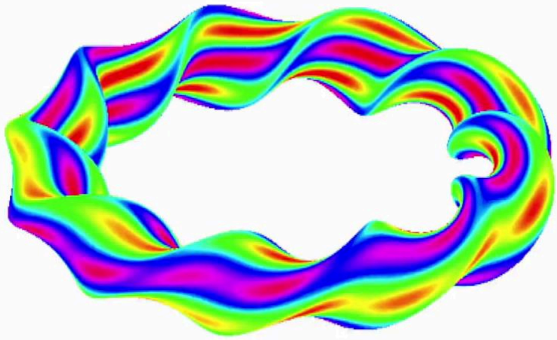
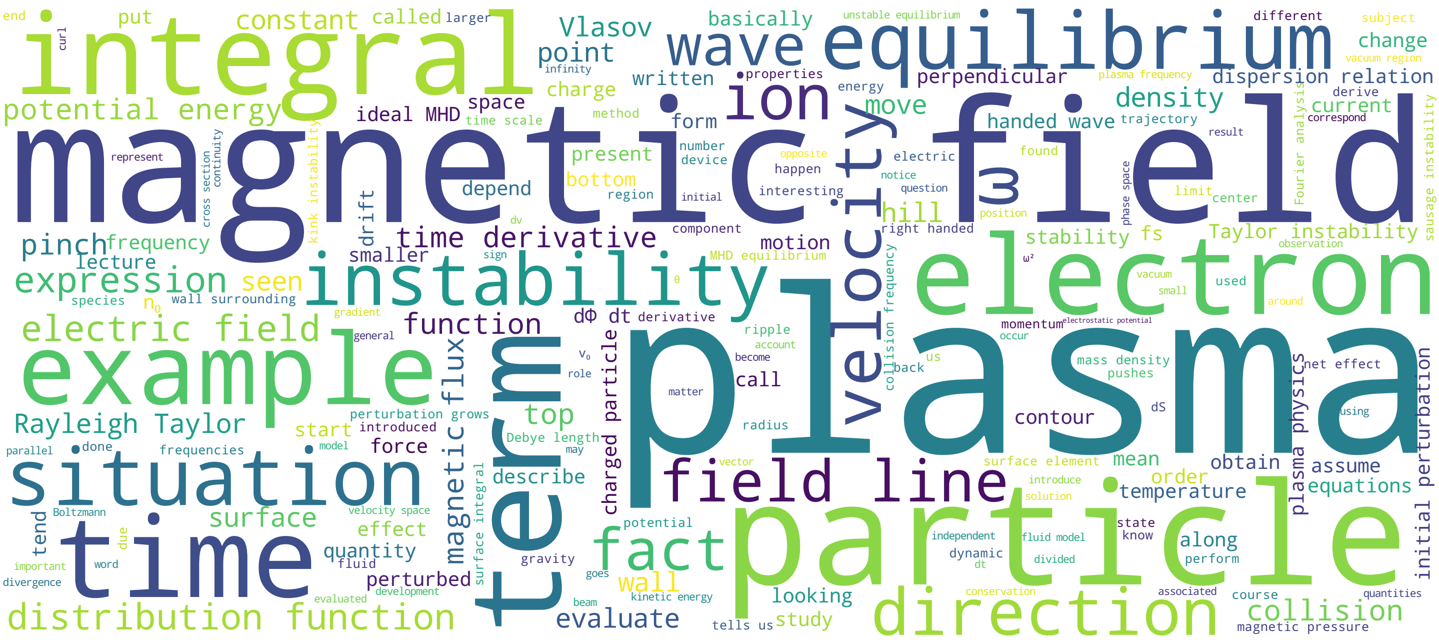


Stability of the MHD equilibrium

Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

Lecture 3g

Duccio Testa



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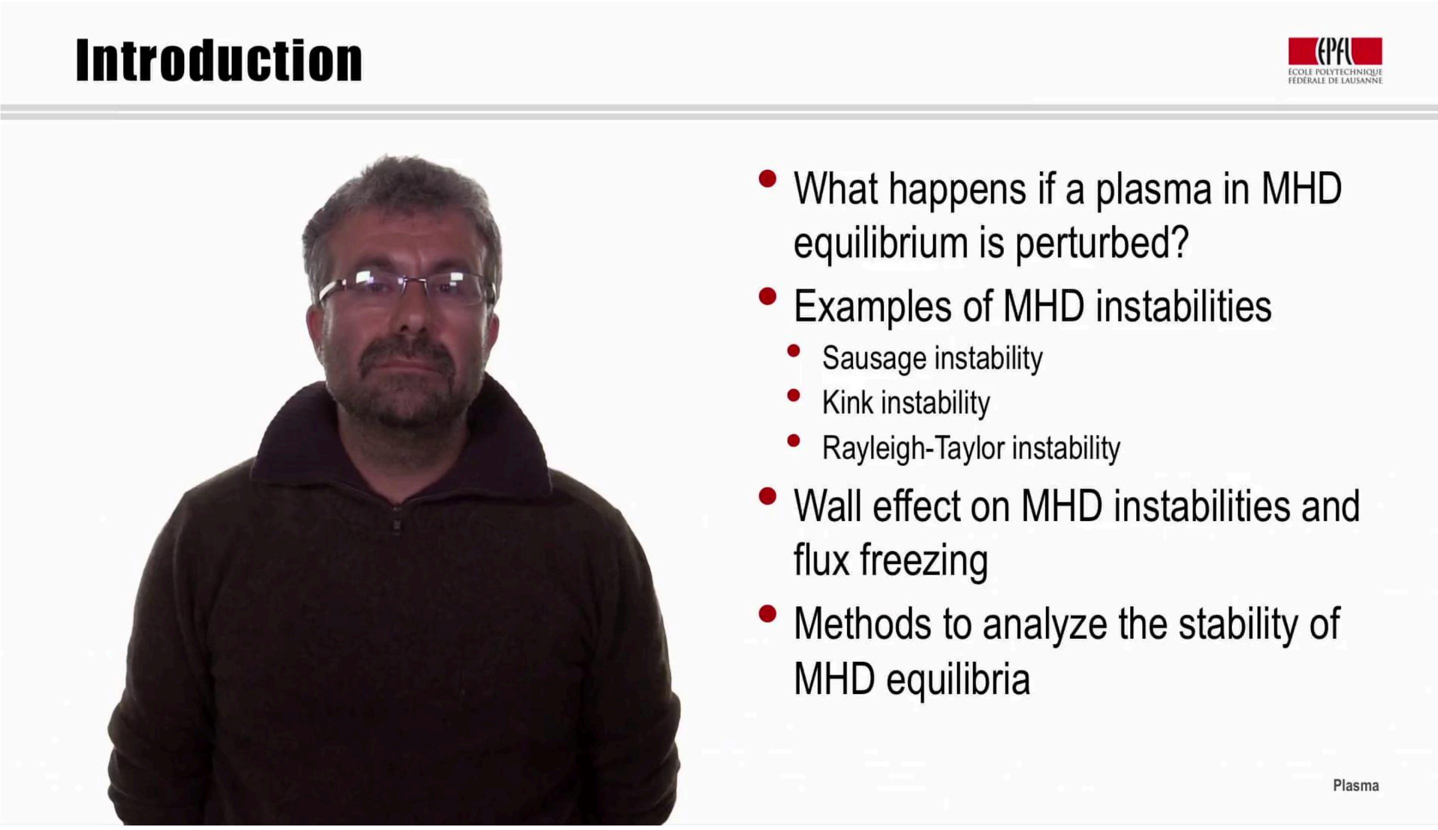


My name is Duccio Testa and in this lecture we will be studying the stability of the MHD equilibrium. What happens if a plasma in MHD equilibrium is perturbed? We want to answer this question, and we will do this by working through some examples of MHD instability. We will be looking at the sausage instability, at the kink instability, and then at the Rayleigh-Taylor instability. We will then look at the effect of the wall surrounding the plasma on instabilities and the link to flux freezing. Then we will study methods to analyze the stability of MHD equilibria.

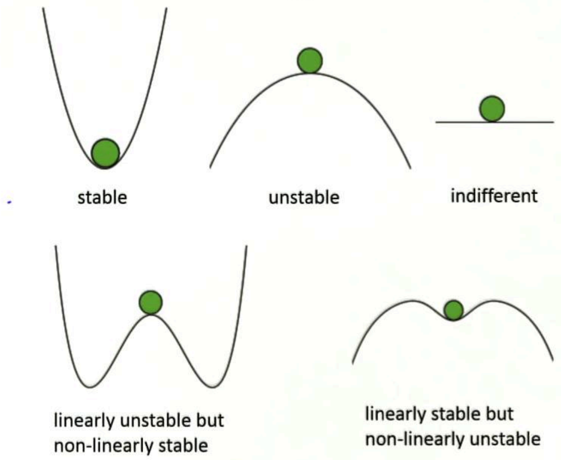
Notes

Summary

0m 05s

What if a plasma in MHD equilibrium is perturbed?

Different possible scenarios, in analogy with classical mechanics



We consider examples of instabilities affecting MHD equilibria

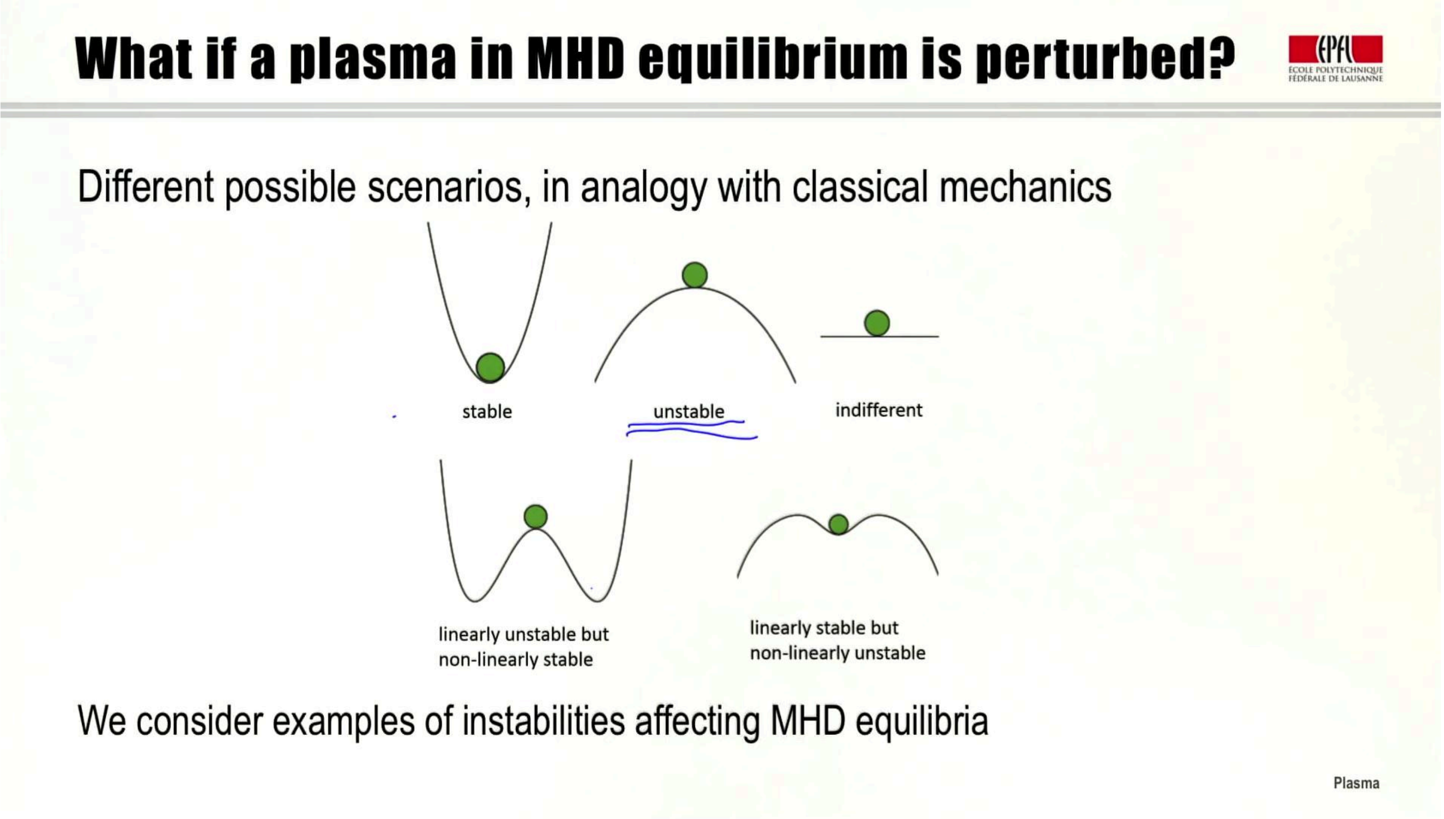
Plasma

What happens if a plasma in MHD equilibrium is perturbed? There are different possible scenarios and we will look at them in analogy with classical mechanics. The first example is this one here. We have a potential well — the black line — and our system, the green dot, sits at the bottom of the potential well. This is a situation of stable equilibrium because if we displace our system, its potential energy increases and then it will hold back to the bottom of the well in the starting equilibrium position. This is a stable situation. The second example here is actually the opposite of the first. We are in a situation of unstable equilibrium. In fact, we are at the top of the hill in potential energy, so if our system is displaced its potential energy will decrease and it will keep rolling away from the equilibrium point at the top of the hill. This is a situation of unstable equilibrium. We have a third situation here where the equilibrium of our system is indifferent because the potential energy is flat, so we can move it left or right. Potential energy doesn't change. This is a situation of indifferent equilibrium.

Notes

Summary

0m 52s



The situation can actually be more complex For instance, we have a case here where our system sits at the top of the hill in potential energy, but in fact, we notice [that] if our equilibrium is perturbed, this system will roll down the hill and it actually will not go very far because it will end up at the bottom of a potential well here. This is another situation of equilibrium. This system is really linearly unstable, but in the end it will be nonlinearly stable. The final case, here, is the opposite of our previous case. The system, in its initial condition of equilibrium, is at the bottom of a potential well, but if we move it, it may go up to the top of the hill. Then potential energy decreases by rolling down the hill. So this system is linearly stable, but nonlinearly unstable. We will consider the example of instability affected the MHD equilibria, and therefore we will basically work on the situation where we are in this condition, a situation of unstable equilibrium.

Notes

Summary

2m 17s

Sausage instability in the Z-pinch

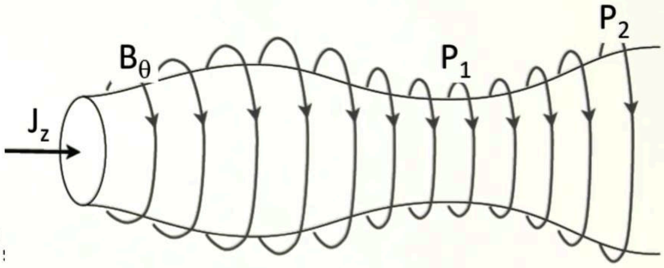
Assume initial perturbation in the form of compressions and bulges

- At a compression (P1),

$$B_\theta = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 \int_S J_z dS}{2\pi r}$$

increases, magnetic pressure increases, squeezing the plasma further

- At a bulge (P2), B_θ decreases, allowing the plasma to expand further



Net effect: sausage instability

Plasma

The first example of instability is the sausage instability in the Z-pinch. We remember from the previous lecture — [that] the Z-pinch is a confinement device, where we have an axial current, Jz, and an azimuthal magnetic field, Bθ. The equilibrium of the Z-pinch is described by the balance between the gradient in the plasma pressure P, the magnetic field pressure $B\theta^2/2\mu_0$ and the tension of the magnetic field lines being wrapped around the surface of the Z-pinch. Now we assume an initial perturbation in the form of compressions and bulges. P1 is a point where we have compressed the plasma. The total current in the Z-pinch, I, remains constant. At this compression point, the radius of the plasma cross-section becomes smaller and therefore Bθ increases. So the magnetic pressure increases, which squeezes the plasma further. So we are further compressing the plasma. The initial perturbation grows. Basically, at the point P2, which is a point of bulging, we have the opposite, Bθ decreases, which then allows the plasma to expand further. So the initial perturbation grows, and the net effect is the sausage instability.

Notes

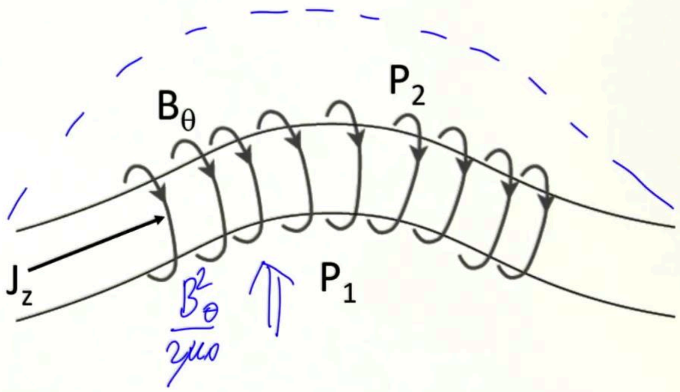
Summary

3m 34s

Kink instability in the Z-pinch

Assume initial perturbation in the form of a bending

- At P1, field lines come closer to each other, B_θ increases, therefore magnetic pressure increases
- At P2, field lines come apart, B_θ decreases and magnetic pressure decreases



Net effect: kink instability

Plasma

Now the second example of instability is the kink instability in the Z-pinch. We assume an initial perturbation in the form of bending. We've pushed up the plasma at P1. The field lines come closer to each other. Bθ increases, therefore the magnetic pressure increases, and this pushes the plasma up. At P2, the field lines comes apart. Bθ decreases, the magnetic pressure decreases, and the plasma cannot be pushed back down. So the net effect is the growth of the initial perturbation. The plasma is pushed up. The net effect is the kink instability.

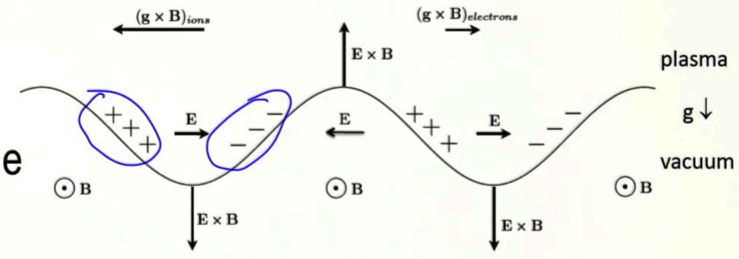
Notes

Summary

5m 09s

Rayleigh-Taylor instability in plasmas

- Equilibrium with plasma on top of a vacuum region, subject to gravity
- Interface between plasma and vacuum perturbed by a ripple
- Gravity-induced drift leads to charge separation, hence **E**
- The resulting **E**×**B** drift increases the original perturbation
- If vacuum region is on top, or if **g** is reversed, equilibrium is stable



$$\upsilon_{g-DRIFT} = \frac{m}{q}\frac{\bar{g}\times\bar{B}}{B^2}$$

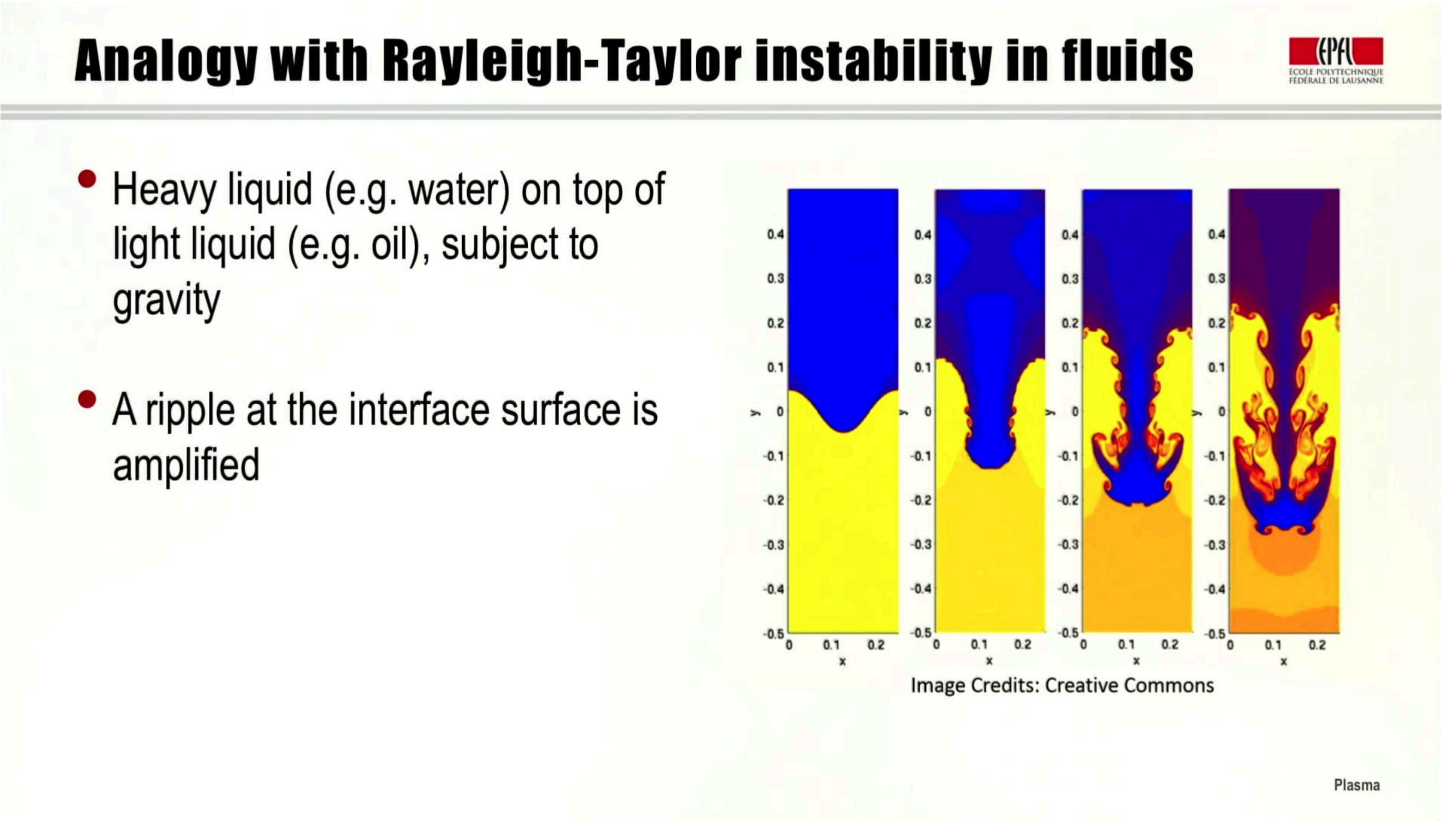
Plasma

After the example of instability is the Rayleigh-Taylor instability in plasmas We consider an equilibrium with the plasma on top of a vacuum region, subject to gravity. The interface between the plasma and the vacuum is perturbed by a ripple. There is a drift, induced by gravity, and this drift leads to charge separation — ions on one side, electrons on the other side. Charge separation leads to an electric field E, and the resulting (E × B) drift increases the original perturbation. It pulls the ripple down, it pulls the ripple up. So again, the initial perturbation grows. We are in a situation of instability. If we reverse this situation, so if we put the vacuum region on top, or if the direction of g is reversed, this equilibrium becomes stable.

Notes

Summary

6m 00s



The most common example of Rayleigh-Taylor instability is the Rayleigh-Taylor instability in fluids. Here we have a simulation that shows the development of the Rayleigh-Taylor instability when we have a heavy liquid — water, in blue — that sits on top of a light liquid — oil, in yellow, subject to gravity. We have an initial ripple at the interface surface and we see that as a function of time this ripple is amplified.

Notes

Summary

7m 08s

Rayleigh-Taylor instability in plasmas

Consider centrifugal force due to magnetic field curvature $F_c = \frac{mv_\parallel^2}{R_c^2}\mathbf{R}_c$

- If $\mathbf{R_c}$ points away from the plasma (from the heavier to the lighter fluid), instability
- If $\mathbf{R_c}$ points towards the plasma (from the lighter to the heavier fluid), the equilibrium is stable

bad curvature: magnetic field is **concave** towards the plasma

good curvature: magnetic field is **convex** towards the plasma

Plasma

We can go back to the Rayleigh-Taylor instability in plasma and the centrifugal force due to the magnetic field's curvature plays the role of gravity. This force, Fc, is expressed by this formula. You see that it depends on Rc. This is the radius of curvature of the magnetic field line. Here we have two drawings. In the first case, Rc points away from the plasma. So it goes from the heavier fluid to the light fluid. This leads to instability. In fact, this is a situation that we call a situation where the magnetic field has a bad curvature. The magnetic field is concave towards the plasma. The opposite situation is if Rc points towards the plasma, so from the lighter fluid, the vacuum, to the heavier fluid, the plasma. In this case, the equilibrium is stable. This is a region of good curvature. The magnetic field is convex towards the plasma.

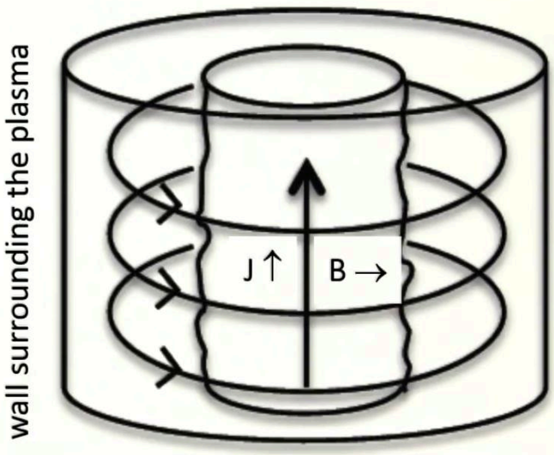
Notes

Summary

7m 40s

Wall effect on MHD instabilities

- Plasma with a current **J** and a magnetic field **B**
- An instability develops that pushes the plasma towards a surrounding ideal wall (η=0)
- The magnetic field cannot penetrate into the wall
- Can the magnetic field penetrate inside the plasma and the magnetic flux through the plasma change?



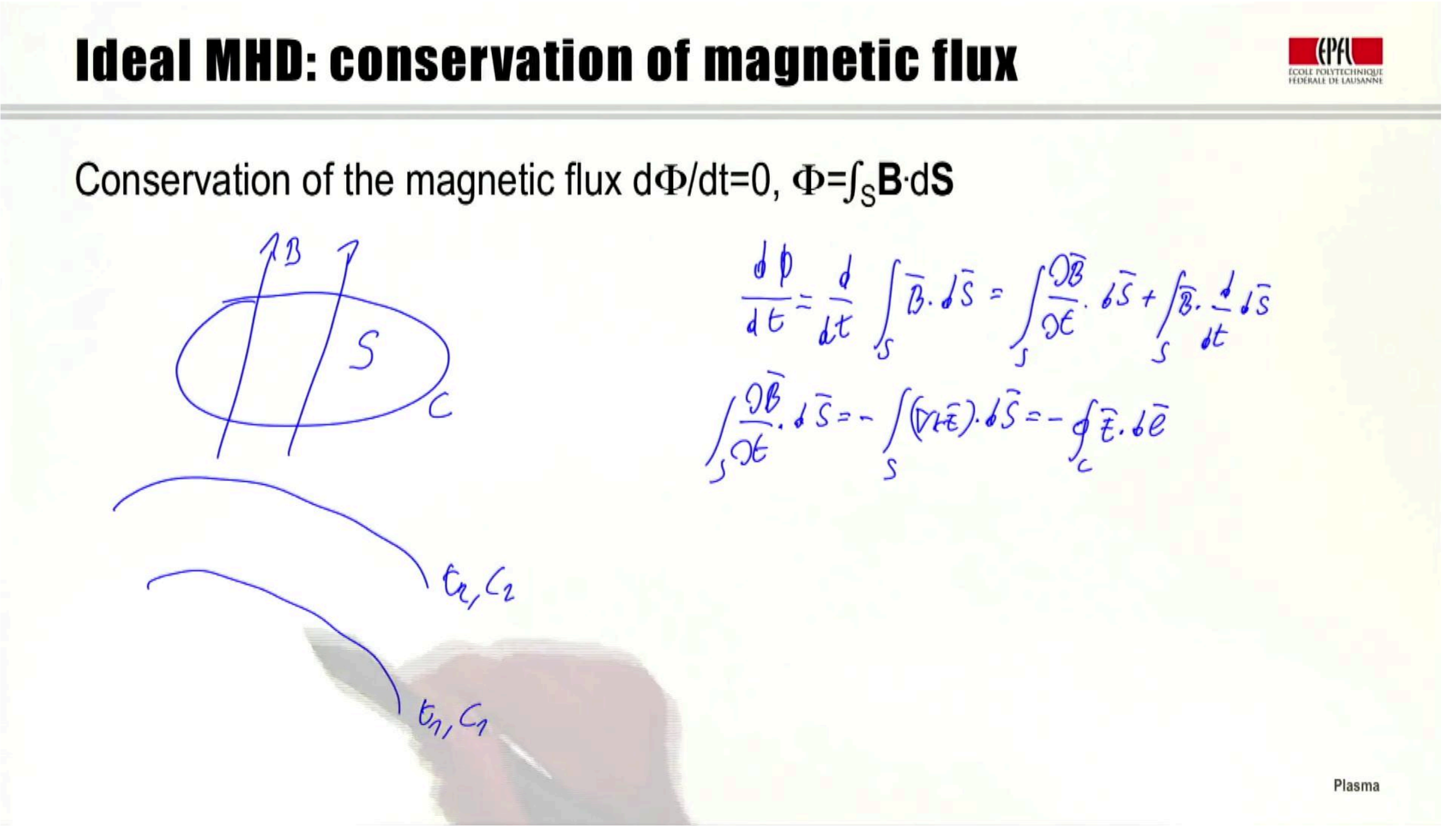
Plasma

Let's look at the role of the wall surrounding the plasma on MHD instability. You can see here a plasma column, which has a current J, and a magnetic field B. There is an instability that develops and pushes the plasma towards the surrounding ideal wall. The resistivity of the wall, η, is zero. The magnetic field cannot penetrate into the wall. So we ask ourselves: can the magnetic field penetrate inside the plasma and can the magnetic flux to [inside] the plasma change?

Notes

Summary

8m 51s



To answer this question, we have to study one important property of ideal MHD, which is the conservation of the magnetic flux. The magnetic flux Φ is the flux of the magnetic field B through the surface S that encloses the field line. These are our field lines. This is the surface S which is delimited by a contour C. Let's describe the time derivative of the magnetic flux: dΦ/dt. We can put the time derivative into the surface integral because the temporal and spatial coordinates are independent. So dΦ/dt is the sum of two terms: the time derivative of the magnetic field, ∂B/∂t • dS, plus B • the time derivative of dS, the surface element. We can study these two terms separately. We can start with the first one and we can use Faraday's law. So we have the integral over the surface S of ∂B/∂t is minus the integral [over] the surface S of the curl of E • ds. We can now use Stokes' theorem and express the surface integral in an integral over the contour C, delimiting the surface. Now we can work on to the second term. So we are looking at the time derivative of the surface element dS. So at time t1, the surface element is delimited by this contour, C1, which moves at time t2 to the front contour, C2.

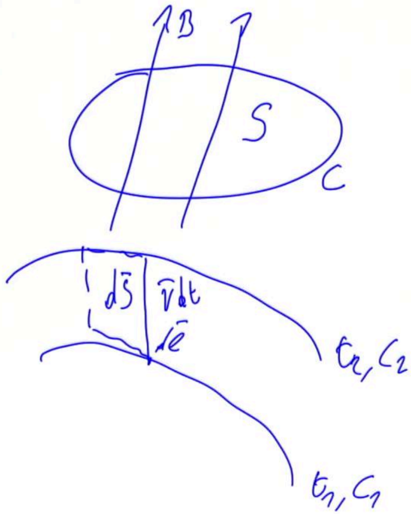
Notes

Summary

9m 27s

Ideal MHD: conservation of magnetic flux

Conservation of the magnetic flux $d\Phi/dt=0$, $\Phi=\int_S \mathbf{B}\cdot d\mathbf{S}$



$$\frac{d\Phi}{dt}=\frac{d}{dt}\int_S \bar{B}\cdot d\bar{S}=\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}+\int_S \bar{B}\cdot\frac{d}{dt}d\bar{S}$$

$$\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}=-\int_S (\nabla\times\bar{E})\cdot d\bar{S}=-\oint_C \bar{E}\cdot d\bar{\ell}$$

$$\int_S \bar{B}\cdot\frac{d}{dt}d\bar{S}=\oint_C \bar{B}\cdot(\bar{v}\times d\bar{\ell})$$

$$d\bar{S}=\bar{v}dt\times d\bar{\ell}$$

Plasma

So if you take a point on the first contour, we can have the vector, dl, and the normal, so the variation of the surface element, ds, we can simply see that it is the cross-product of (vdt × dl). So we can use this expression for [d/dt of] dS in the second term of our integral. So our surface integral, B • d/dt dS, becomes the integral over the contour C or B • (v × dl).

Notes

Summary

11m 23s

Ideal MHD: conservation of magnetic flux

$$\frac{d\Phi}{dt} = -\oint_C \bar{E}\cdot d\bar{\ell} + \oint_C \bar{B}\cdot(\bar{v}\times d\bar{\ell})$$

$$\frac{d\Phi}{dt} = -\oint_C (\bar{E} + \bar{v}\times\bar{B})\cdot d\bar{\ell} = 0$$

$$\Downarrow \quad \bar{E} + \bar{v}\times\bar{B} = 0$$

Plasma

We can now put the two terms together that we have obtained for the time derivative of the magnetic flux, dΦ/dt. So we have two integrals over the contour, C. We can rearrange the terms in the second integral and combine it with the first. What we have here is that dΦ/dt is minus the integral over the contour C of (E + v × B) • dl. This is the Ohm's and we know that the (E + v × B) in ideal MHD is equal to zero. And so this integral is equal to zero, which tells us that dΦ/dt is equal to zero. As dΦ/dt is equal to zero in the ideal MHD, the magnetic flux through every surface moving with the plasma is constant. The magnetic flux is therefore frozen into the plasma in the absence of plasma resistivity.

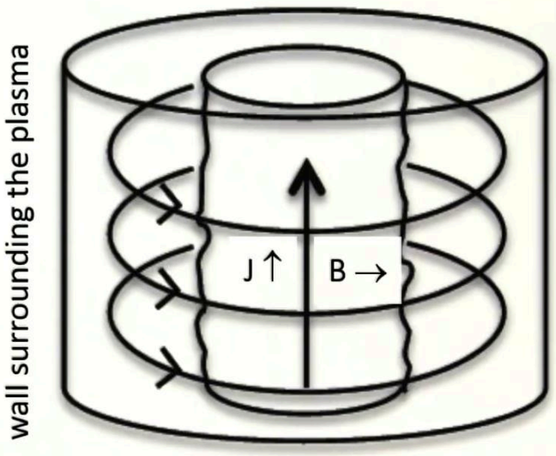
Notes

Summary

12m 09s

Wall effect on MHD instabilities

- Magnetic field is compressed in the vacuum region between the plasma and the wall
- The magnetic pressure is increased and pushes the plasma back towards the center of the device
- Plasmas can be stabilized by a surrounding wall
- Finite resistivity limits this effect to a finite time scale



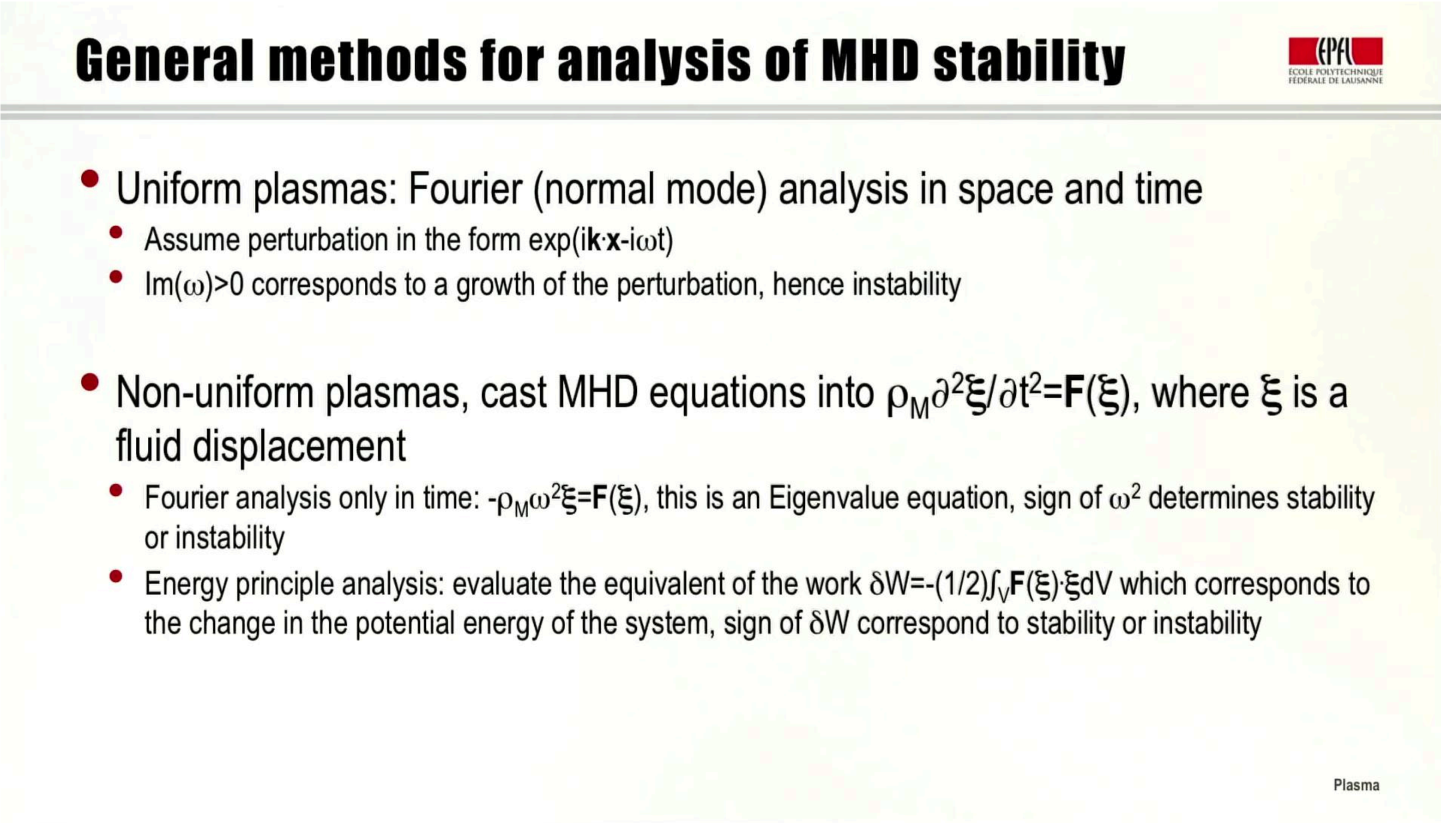
Plasma

We can now go back to the effect of the wall on MHD instabilities. The magnetic field, when the plasma is pushed towards the wall, cannot penetrate into the wall, so the magnetic field is compressed in the vacuum region between the plasma and the wall. The magnetic pressure is increased, and that pushes the plasma back towards the center of the device. This tells us that plasmas can be stabilized by a surrounding wall. However, this effect is limited by the finite resistivity of the plasma and of the wall, and so this effect only lasts over a finite time scale.

Notes

Summary

13m 23s

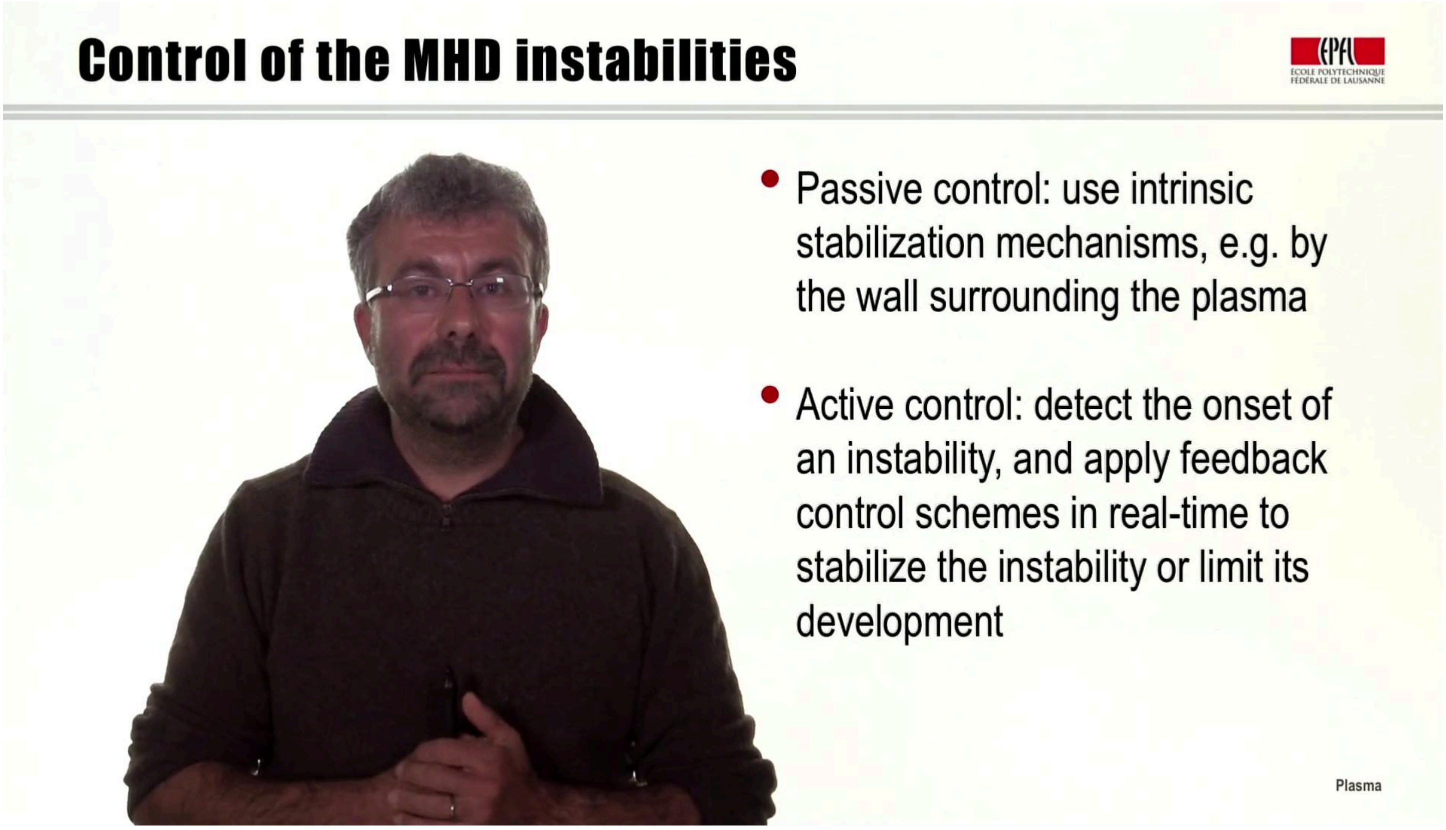


Let's look at some general methods for general analysis of MHD stability. For uniform plasmas, we can perform a Fourier analysis in space and time. This corresponds to a normal mode analysis. So we assume perturbation in the form of an exponential ($ik \cdot x - i\omega t$) and then we look at the imaginary part of ω, the wave frequency. If the imaginary part of ω is positive, we are in a situation where the perturbation grows, so we are in a situation where an instability develops. For non-uniform plasmas, we can cast the MHD equations into a force balance equation. We have the mass density of the fluid element ρM times $d^2\xi / dt^2$. ξ is a fluid displacement, and this is equal to a force operator $F(\xi)$, that represents the force acting on the fluid element whose mass density is ρM, producing a displacement ξ. We can then perform a Fourier analysis only in time. This tells us that $-\rho M\omega^2 \xi$ is equal to F — this is an Eigenvalue equation; the sign of ω^2 determines the stability or instability. We can also perform an energy principle analysis. We can evaluate the equivalent of the work δW which is -1/2 the integral over the plasma volume V of $F \cdot \xi$. This corresponds to the change in the potential energy of the system. The sign of δW, corresponds to stability or instability.

Notes

Summary

14m 10s

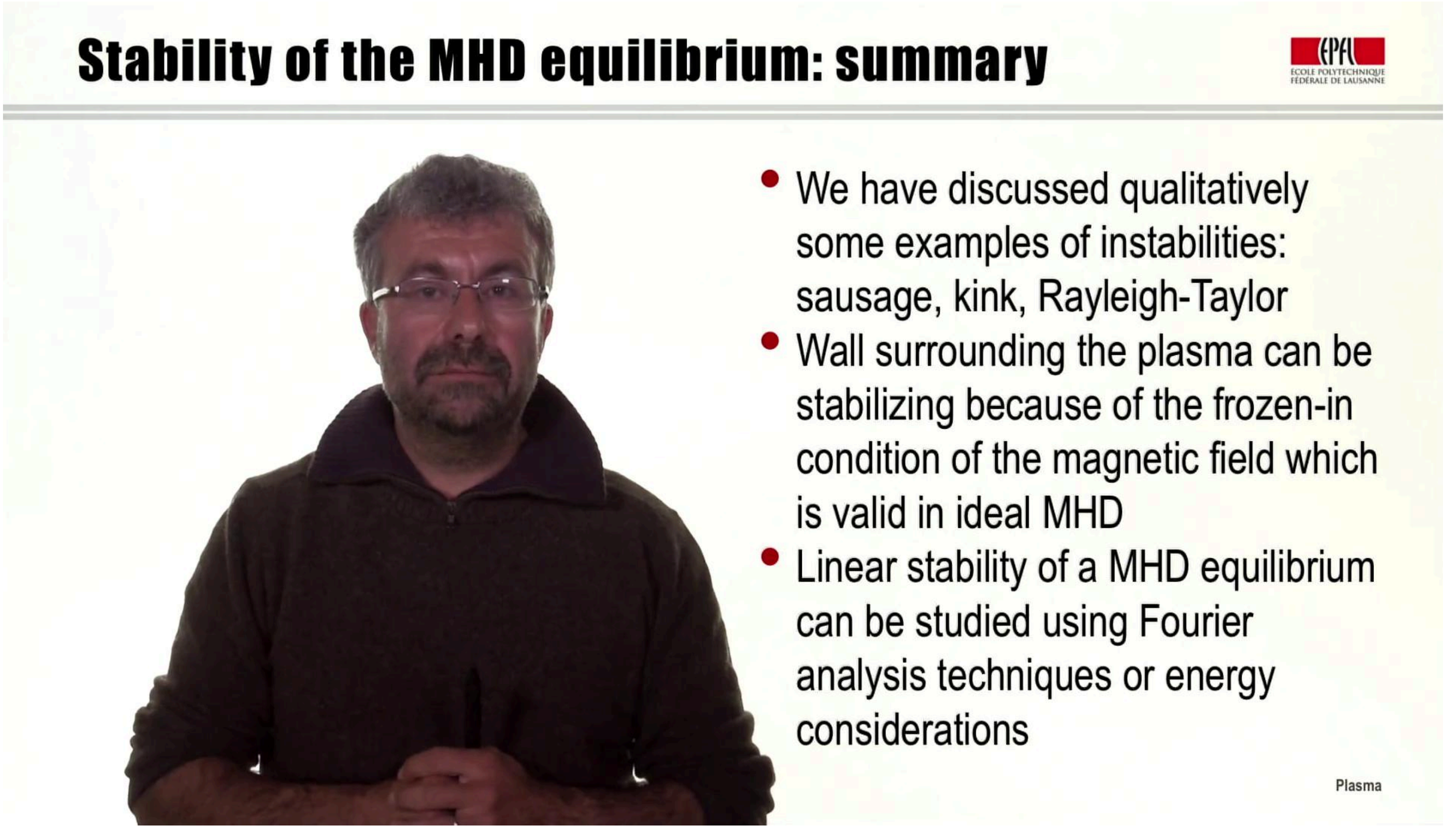


We've seen that the plasma in an MHD equlibrium and in general can be subject to many instabilities. How can we control this instability? We have two methods. The first method is a passive control system. We use intrinsic stabilization mechanisms as we have seen in the case of a wall surrounding the plasma. The second method is an active control system. We detect the onset of an instability and apply feedback control schemes in real-time that act on plasma parameters and with this feedback control scheme we can try to stabilize the instability or limit its development.

Notes

Summary

16m 06s



Let's summarize what we've discussed in this lecture. We've qualitatively discussed some examples of instability: the sausage, the kink, and the Rayleigh-Taylor. We've see that the wall surrounding the plasma can be stabilizing because of the frozen-in condition of the magnetic field, which is valid in ideal MHD. Then we've seen that the linear stability of an MHD equilibrum can be studied using Fourier analysis techniques or energy considerations.

Notes

Summary

16m 49s