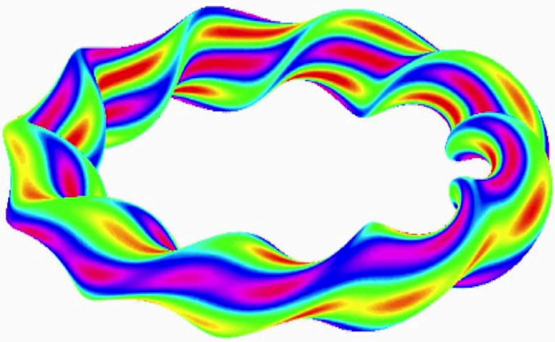
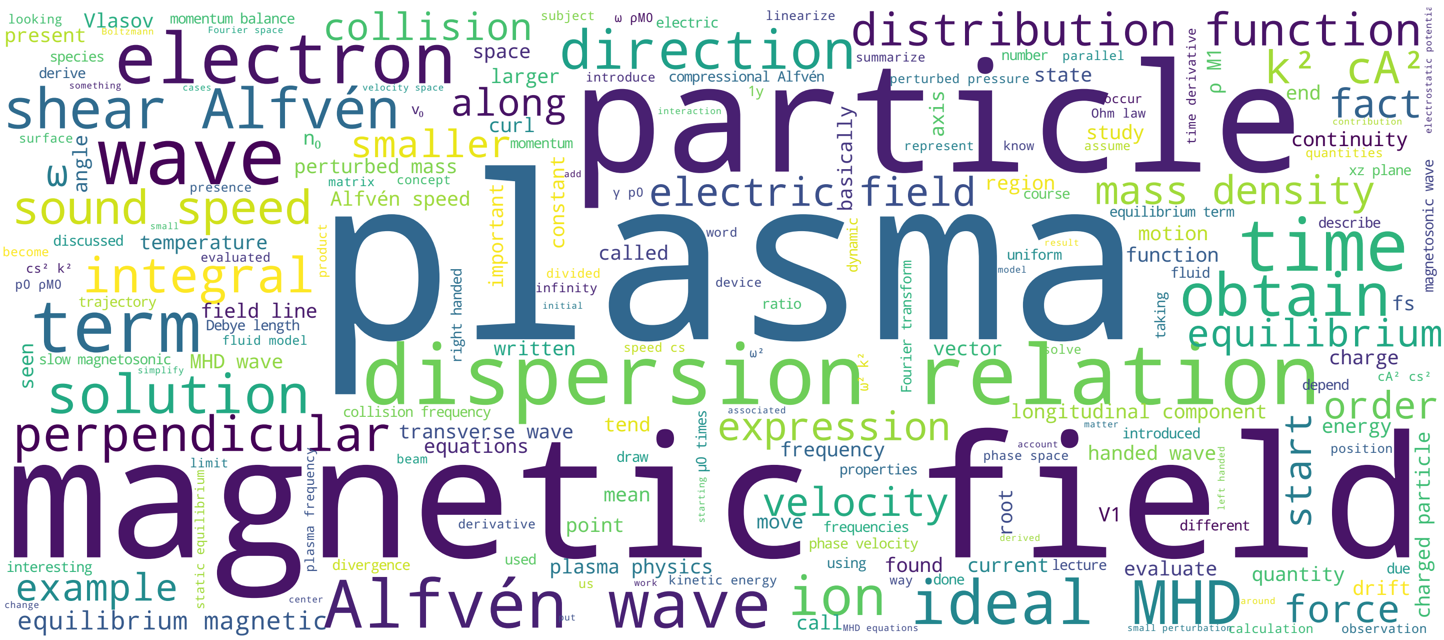


Plasma waves in the ideal MHD model

Plasma Physics and Application to Fusion Energy, Astrophysics and Industry

Lecture 3h

Duccio Testa



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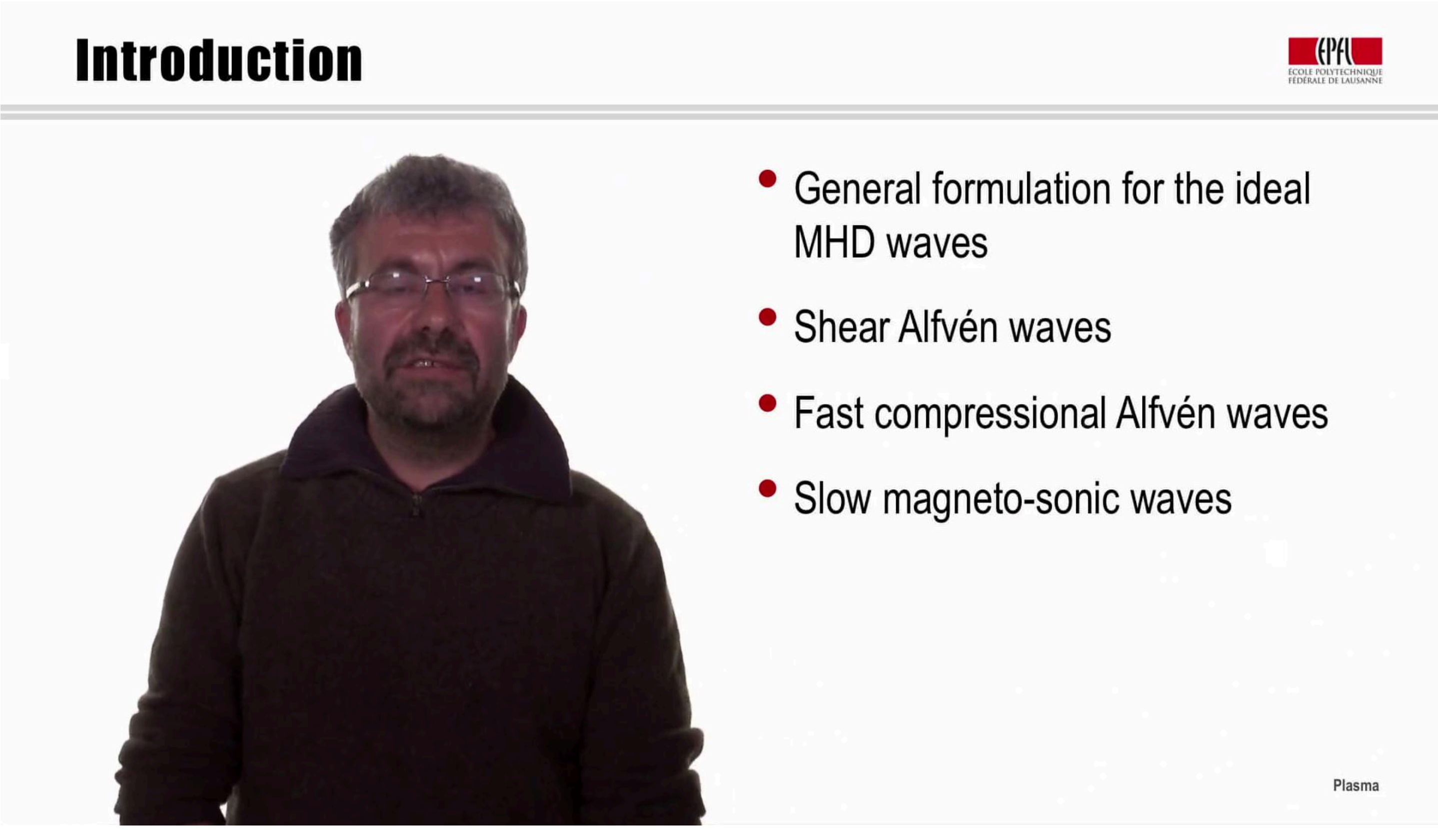
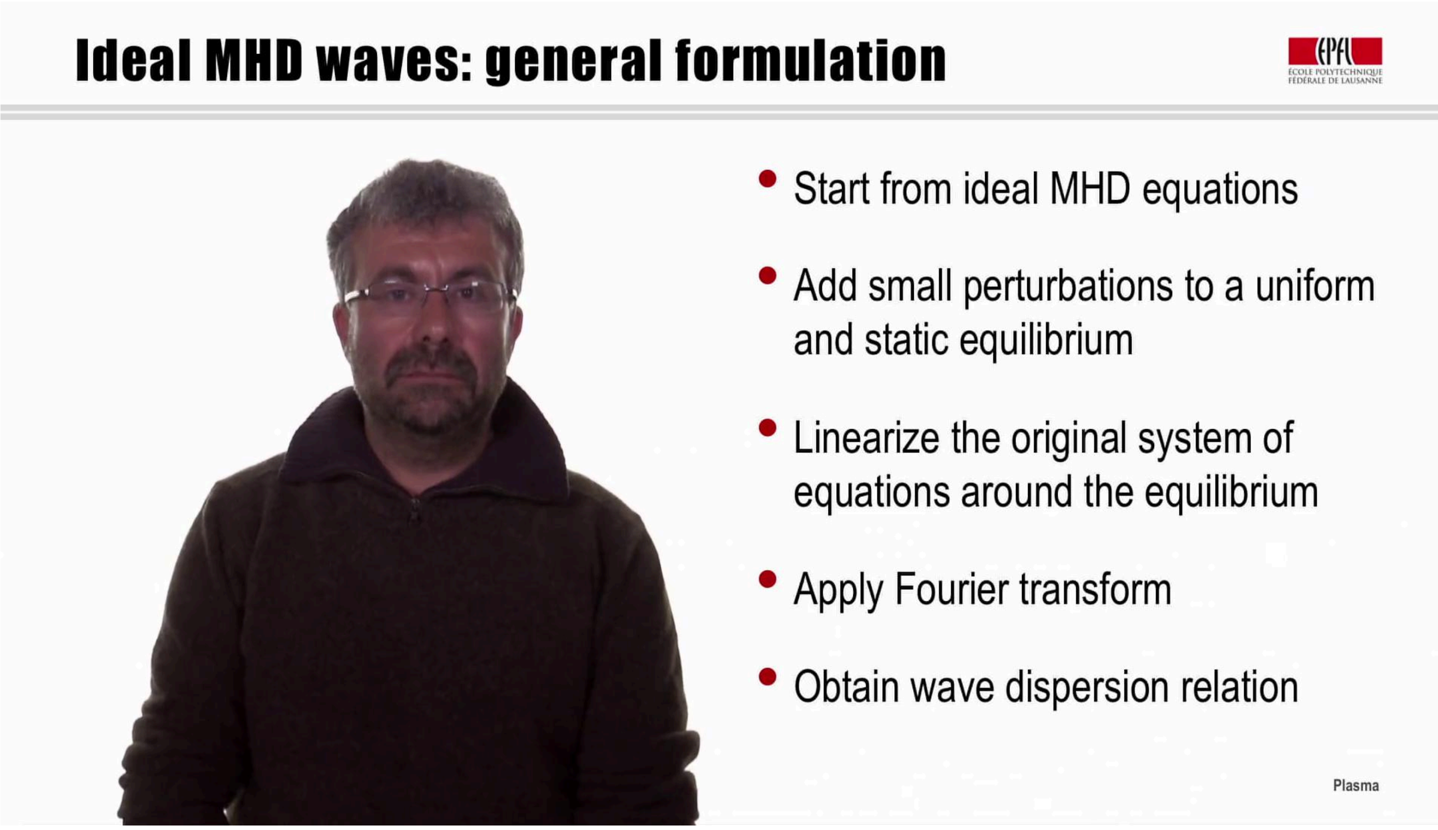


My name is Duccio Testa, and in this lecture we will be discussing plasma waves in the ideal MHD model. We will start with the general formulation for the ideal MHD waves and then, we will study the three classes of waves that are supported by the ideal MHD description of the plasma: the shear Alfvén waves, the fast compressional Alfvén waves, and the slow magneto-sonic waves.

Notes

Summary

0m 05s



To obtain the general formulation of ideal MHD waves, we start from the ideal MHD equations. Then we add a small perturbation to a uniform and static equilibrium, we linearize the original system of equations around the equilibrium, we apply Fourier transform, and then we obtain the wave dispersion relation.

Notes

Summary

0m 35s

Ideal MHD waves: general formulation

Start from ideal MHD equations: eliminate **j** and **E**

$$\nabla \times \bar{B} = \mu_0 \bar{j}$$

$$\rho_M \frac{d\bar{V}}{dt} = -\nabla p + \bar{j} \times \bar{B}$$

$$\Rightarrow \rho_M \frac{d\bar{V}}{dt} = -\nabla p + \frac{1}{\mu_0} (\nabla \times \bar{B}) \times \bar{B}$$

$$\frac{\partial \bar{B}}{\partial t} = -\nabla \times \bar{E}$$

$$\bar{E} + \bar{V} \times \bar{B} = 0$$

$$\Rightarrow \frac{\partial \bar{B}}{\partial t} = +\nabla \times (\bar{V} \times \bar{B})$$

$$\frac{\partial \rho_M}{\partial t} + \nabla \cdot \left(\rho_M \bar{V} \right) = 0$$

$$\frac{d}{dt} \left(p \rho_M^{-\gamma} \right) = 0$$

Plasma

We start from the ideal MHD equations and we use them to eliminate j and E. We have here Ampère's law and the momentum balance equation. We can take j from Ampère's law and plug it into the momentum balance equation. With this procedure, we eliminate j. So what we obtain is that ρM(dV/dt) is equal to -∇p +1/μ0 [times] the curl of B × B. We have then Faraday's law and the ideal Ohm's law. We can take the electric field from the ideal Ohm's law and put it into Faraday's law. So what we have: ∂B/∂t equals the curl of (V × B). We then have the continuity of the plasma mass and the equation of state.

Notes

Summary

0m 59s

Ideal MHD waves: general formulation

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Apply Fourier transform in space and time to linearized equations

$$-\omega\tilde{B}_1(\omega,\bar{k}) = \bar{k}\times\left(\tilde{V}_1(\omega,\bar{k})\times\bar{B}_0\right)$$

$$-\omega\tilde{\rho}_{M1}(\omega,\bar{k}) + \rho_{M0}\bar{k}\cdot\tilde{V}_1(\omega,\bar{k}) = 0$$

$$-\omega\rho_{M0}\tilde{V}_1(\omega,\bar{k}) = -\bar{k}c_s^2\tilde{\rho}_{M1}(\omega,\bar{k}) + \frac{1}{\mu_0}\left(\bar{k}\times\tilde{B}_1(\omega,\bar{k})\right)\times\bar{B}_0$$

$$\tilde{p}_1(\omega,\bar{k}) = \gamma\frac{p_0}{\rho_{M0}}\tilde{\rho}_{M1}(\omega,\bar{k}) = c_s^2\tilde{\rho}_{M1}(\omega,\bar{k})$$

Plasma

We can then consider a uniform and static MHD equilibrium, to which we add a small perturbation. The magnetic field, with an equilibrium term B0, plus its perturbation B1, for the fluid velocity V, since we consider a static equilibrium, the equilibrium term is zero. We only have the perturbation: V1. For the mass density, the equilibrium term is a constant, perturbation ρM1, and the same for the pressure. p0 is a constant. We can then linearize the initial system of equations around the equilibrium. What we obtain is that ∂B1/∂t is equal to the curl of (V1 × B0). Perturbation, perturbation first order. We do the same for the continuity equation: ∂ρM1/∂t + ρM0 [times] the divergence of V1 = 0. The momentum balance equation: ρM0 dV1/dt = -∇p1 + 1/μ0 [times] (∇ × B1) × B0. Then we have the perturbed pressure: p1 = γ (p0/ρM0) ρM1 — The perturbed mass density. This factor appears in the last equation: γ (p0/ρM0) is the sound speed, cs² — sound speed, cs = (γ p0/ρM0)^(1/2). So the perturbed pressure, p1, equals the sound speed squared, cs² times the perturbed mass density ρM1. Now we can apply Fourier transform in space and time to linearized equations.

Notes

Summary

2m 16s

Ideal MHD waves: general formulation

Apply Fourier transform in space and time to linearized equations

$$-\omega \tilde{B}_1(\omega,\bar{k}) = \bar{k} \times \left(\tilde{V}_1(\omega,\bar{k}) \times \bar{B}_0\right)$$

$$-\omega \tilde{\rho}_{M1}(\omega,\bar{k}) + \rho_{M0}\,\bar{k} \bullet \tilde{V}_1(\omega,\bar{k}) = 0$$

$$-\omega \rho_{M0} \tilde{V}_1(\omega,\bar{k}) = -\bar{k} c_s^2 \tilde{\rho}_{M1}(\omega,\bar{k}) + \frac{1}{\mu_0}\left(\bar{k} \times \tilde{B}_1(\omega,\bar{k})\right) \times \bar{B}_0$$

$$\tilde{p}_1(\omega,\bar{k}) = \gamma \frac{p_0}{\rho_{M0}} \tilde{\rho}_{M1}(\omega,\bar{k}) = c_s^2 \tilde{\rho}_{M1}(\omega,\bar{k})$$

Plasma

Our first equations become $-\omega\tilde{B}_1$ — the tilde indicates that we are now in Fourier space — $k \times (\tilde{V}_1 \times B_0)$. For the mass density we have $-\omega\, \tilde{\rho}_{M1} + \rho_{M0}\, (k \bullet \tilde{V}_1) = 0$. We have then, for the momentum balance equation, $-\omega\, \rho_{M0}\, \tilde{V}_1 = -k\, c_s^2\, \tilde{\rho}_{M1} + (1/\mu_0)\, (k \times \tilde{B}_1) \times B_0$. Finally, for the perturbed pressure: $\tilde{p}_1 = c_s^2\, \tilde{\rho}_{M1}$.

Notes

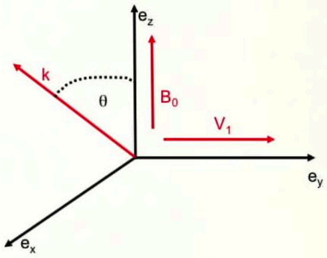
Summary

4m 17s

Ideal MHD waves: choice of geometry

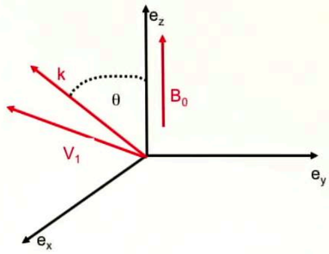
- Set $\mathbf{B_0} = B_0\mathbf{e_z}$ and $\mathbf{k} = (k_x,0,k_z)$ without loss of generality

Case 1: transverse waves $\mathbf{V_1} = (0,V_{1y},0)$



Case 2: waves with longitudinal component

$\mathbf{V_1} = (V_{1x},0,V_{1z})$



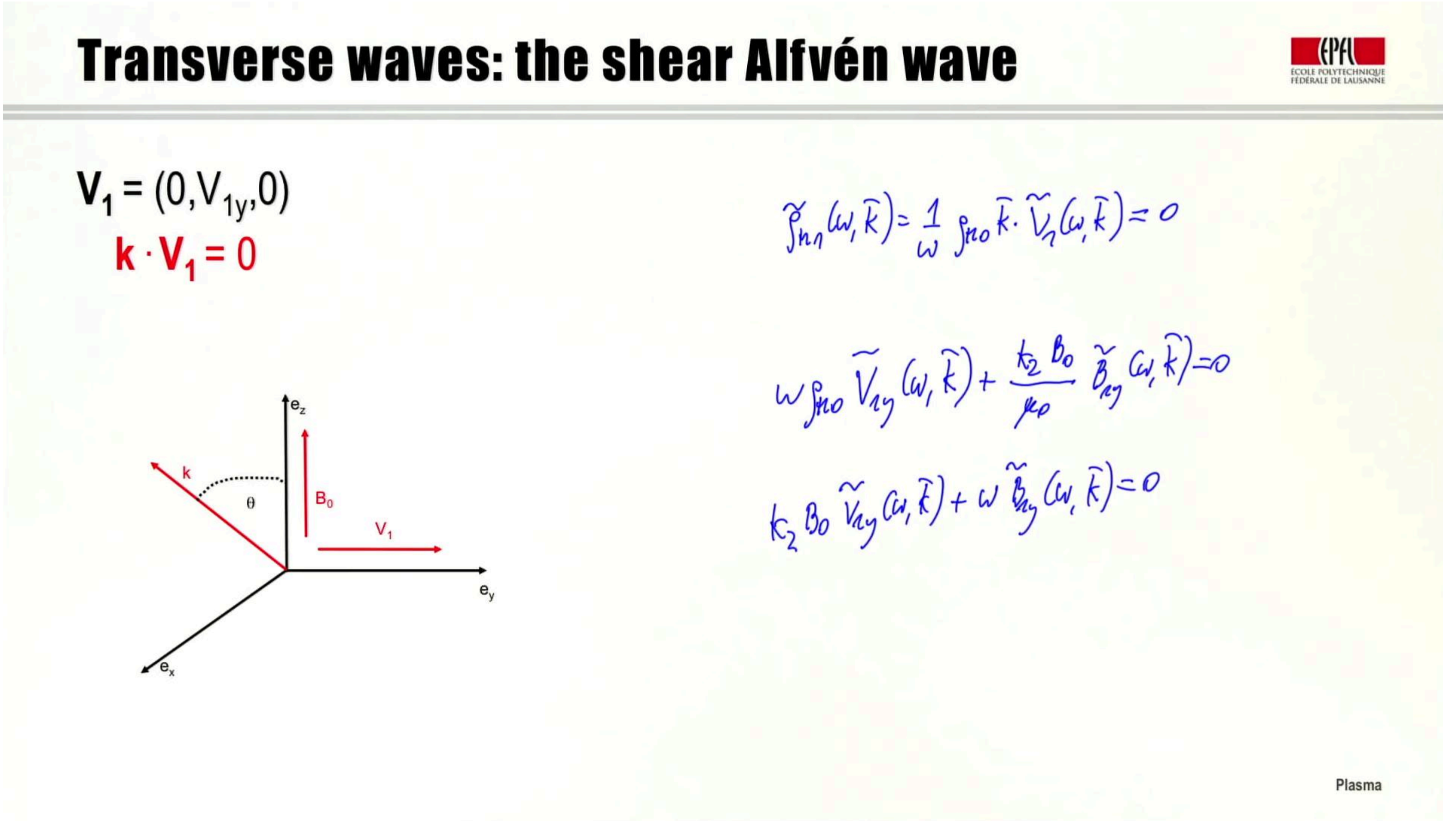
Plasma

We can now chose our geometry for the waves. Without loss of generality, we can take the equilibrium magnetic field to be along the z-axis and take the k vector in the xz-plane. We now have two cases. The first case is the case of transverse waves, where V1 is perpendicular to k. This case is depicted here. It is here, the equilibrium magnetic field B0 along the z-axis, the vector k in the xz-plane, and there is an angle, θ, between k and B0. The perturbed velocity: V1 in the y direction perpendicular to k. The second case is waves with a longitudinal component where V1 is also in the xz-plane, which is depicted in this second figure here.

Notes

Summary

5m 13s

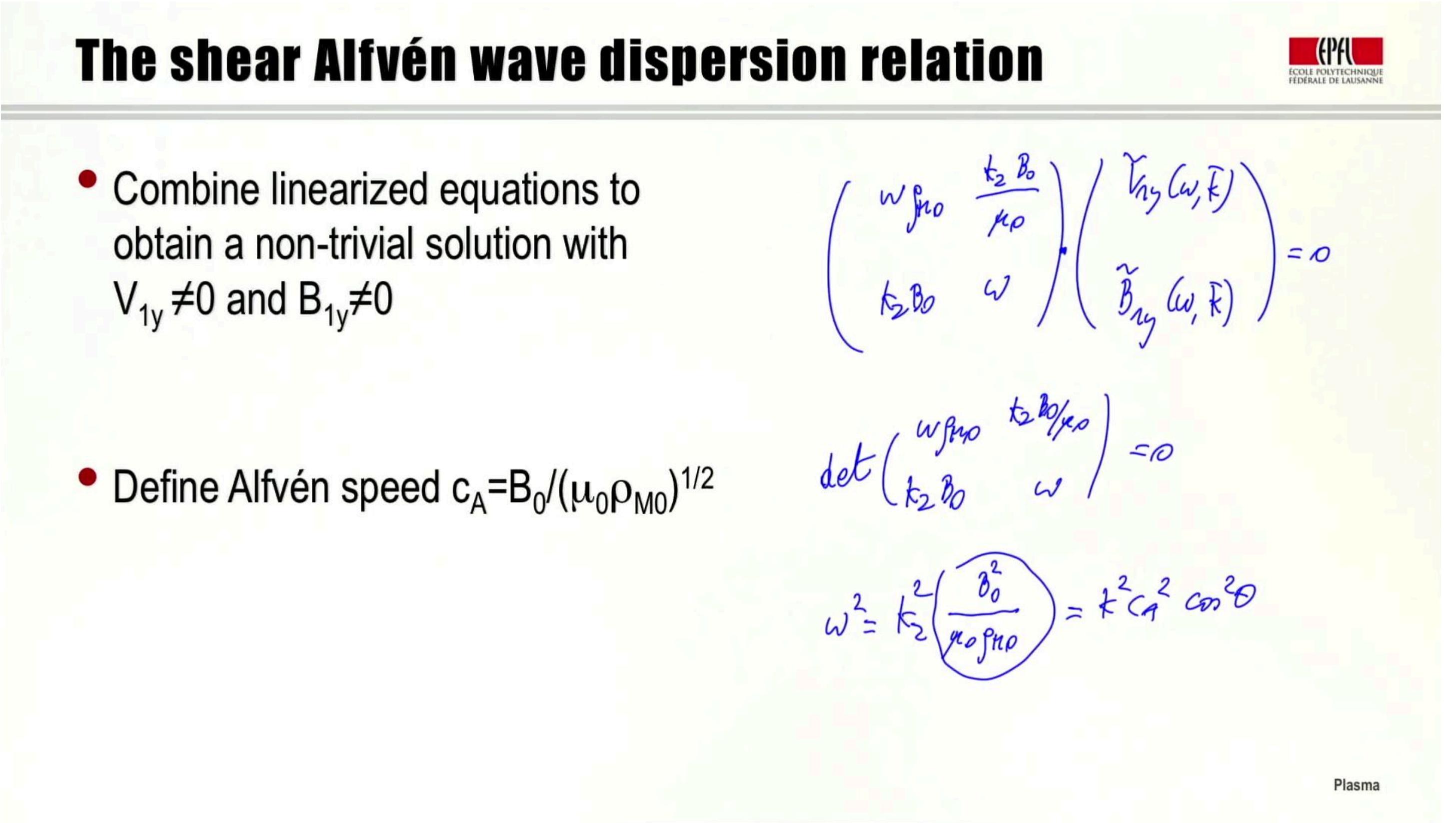


Let's look at transverse waves. V1 [is] only along the y direction, and therefore V1 is perpendicular to k, because k is in the xz-plane. We can now start with the equation in Fourier space for the perturbed mass density. Perturbed mass density: ρM̃1 = 1/ω ρM0 k • Ṽ1. But k and V are perpendicular to each other. Therefore, k and Ṽ are also perpendicular to each other. So ρM̃1 = 0. There is no perturbed mass density. Transverse waves are non-compressional. We then have two remaining equations that link B̃1y and Ṽ1y. So we end up with these two equations: ω ρM0 Ṽ1y + kz B0/μ0 [times] B̃1y and the second equation: kz B0 Ṽ1y + ω B̃1y = 0.

Notes

Summary

6m 17s



We can then combine these two linearized equations into a matricial form. Now we want to obtain a nontrivial solution which has B̃1y and Ṽ1y, both not equal to zero. We obtain this solution by taking the determinant of this two-by-two matrix to be equal to zero. This leads to the dispersion relation for the shear Alfvén wave. In this dispersion relation, there is a term that is very important. This quantity here, $B0^2/(\mu0\ \rho M0)$, this is the square of the Alfvén speed, cA. We can then cast this dispersion relation in a way that we see the Alfvén speed and the angle between k and the equilibrium magnetic field. This is the final form for the dispersion relation of the shear Alfvén wave: $\omega^2 = k^2\ cA^2\ \cos^2\theta$.

Notes

Summary

7m 51s

The shear Alfvén wave - summary

- Transverse wave: $\mathbf{B_0}=B_0\mathbf{e_z}$, $\mathbf{k}=(k_x,0,k_z)$; $\mathbf{V_1}=(0,V_{1y},0)$: $\mathbf{V_1}\perp\mathbf{k}\perp\mathbf{B_0}$
- The shear Alfvén wave is non-compressional: $\rho_{M1}=\rho_{M0}(\mathbf{k}\cdot\mathbf{V_1})/\omega\equiv 0$
- Dispersion relation: $\omega^2=(k_zB_0)^2/(\mu_0\rho_{M0})=(kc_A)^2\cos^2\theta$
- The shear Alfvén wave is important in DT fusion plasmas as the velocity of fusion-born α's with E=3.5MeV is $>c_A$: α's become resonant during their slowing down
- Shear Alfvén waves were observed first in space, then in mercury plasmas

Plasma

We can now summarize what we have found for the shear Alfvén wave. The shear Alfvén waves are transverse waves. The perturbed velocity V1 is perpendicular to k and to the equilibrium magnetic field, B0. The shear Alfvén wave is noncompressional. The perturbed mass density, ρM1 = 0. The dispersion relation can be cast in a form: $\omega^2 = (k\ cA)^2 \cos^2\theta$. To give you a bit of a heads-up of topics that we will discuss later in this course, the shear Alfvén wave is important in DT fusion plasma because the velocity of fusion-born α particles whose energy at birth is 3.5MeV is super-Alfvénic: exceeds the Alfvén speed cA. Therefore, α's become resonant with shear Alfvén waves during their slowing down. Shear Alfvén waves are ubiquitous in plasma, and in fact they were observed first in space and then in mercury plasmas.

Notes

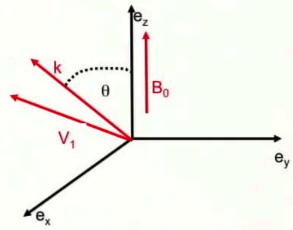
Summary

9m 08s

Ideal MHD waves with a longitudinal component

$\mathbf{V_1}=(V_{1x},0,V_{1z})$

$\mathbf{k}\cdot\mathbf{V_1}\neq 0$



- Repeat previous calculations to obtain

$$\mathbf{B_1} = (V_{1x}B_0/\omega)\ \mathbf{k}\times\mathbf{e_y}$$

- Dispersion relation
- As $(c_s/c_A)^2 = \gamma p_0/(B_0^2/\mu_0) << 1$, the dispersion relation can be simplified

$$\omega^4 - \omega^2 k^2 (c_A^2+c_s^2) + k^2 k_z^2 c_A^2 c_s^2 = 0$$

$$\omega^2 = \frac{1}{2}k^2(c_A^2+c_s^2) \pm \sqrt{\frac{1}{4}k^4(c_A^2+c_s^2)^2 - k^2 k_z^2 c_A^2 c_s^2}$$

$$\omega^2 \simeq \frac{1}{2}k^2 c_A^2\left(1+\frac{c_s^2}{c_A^2}\right)\left[1 \pm \left(1-\frac{2k_z^2 c_s^2}{k^2 c_A^2}\right)\right]$$

Plasma

Let's look at ideal MHD waves with a longitudinal component. V1 now has an x and z component, so k • V1 ≠ 0. We can repeat all the previous calculations to obtain an expression for the perturbed magnetic field: B1 = (V1x B0/ω) k × ey. We can continue the analysis, and we obtain a dispersion relation for the ideal MHD waves with a longitudinal component. ω^4 - ω^2 k^2 (cA^2 + cs^2) + k^2 kz^2 cA^2 cs^2 = 0 This is our dispersion relation for ideal MHD waves with a longitudinal component. We see that [we have] a term, that depends on the Alfvén speed cA and on the sound speed cs and it's in fact a quadratic relation for ω^2. We can solve this equation rather easily. So we immediately see that there are two roots of this dispersion relation for ω^2. We have a root with the plus and a root with the minus. We can simplify this dispersion relation because the ratio between the sound speed cs and the Alfvén speed cA, squared, is usually much smaller than one in typical plasma conditions. So we can take only terms that are first-order in the ratio between cs^2 and cA^2. So we end up with this formulation for the dispersion relation, ω^2, 1/2 k^2 cA^2 (1 +cs^2/cA^2), so a first [order] correction in cs, 1 ± the two roots for ω^2, (1 - (2kz^2 cs^2)/(k^2 cA^2)).

Notes

Summary

10m 15s

The *fast* compressional Alfvén wave

$$\omega_+^2 = \frac{1}{2}k^2 c_A^2\left(1+\frac{c_s^2}{c_A^2}\right)\left[1 + \left(1-\frac{2k_z^2 c_s^2}{k^2 c_A^2}\right)\right]$$

$$\omega_+^2 = k^2 c_A^2\left(1+\frac{c_s^2}{c_A^2}\right)\left(1-\frac{k_z^2 c_s^2}{k^2 c_A^2}\right)$$

$$\omega_+^2 \simeq k^2 c_A^2\left[1+\frac{c_s^2}{c_A^2}\left(1-\frac{k_z^2}{k^2}\right)\right]$$

$$\omega_+^2 = k^2 c_A^2\left[1+\frac{c_s^2}{c_A^2}\sin^2\theta\right]$$

Plasma

We can now consider the plus root of the dispersion relation. We can now proceed with this calculation by summing 1 and the brackets. We can now take the product between the two brackets and keep only the terms that are linear in the ratio $(cs/cA)^2$. Now we know $kz = k \cos\theta$, we can then recast this last term. So this is the final dispersion relation for the plus root of the dispersion relation of [the] ideal MHD wave with a longitudinal component. We call this wave the fast compressional Alfvén wave. We've seen that it's compressional — $k \bullet V(1) \neq 0$ — and it is the fast solution because the phase velocity is cA — the Alfvén speed, that is much larger than cs, the sound speed. We see the dispersion relation is simply $\omega_+^2 = k^2\, cA^2$ and that the sound speed and the angle between the k vector and the equilibrium magnetic field only enters as a small correction.

Notes

Summary

12m 43s

The *slow* magneto-sonic wave

$$\omega_-^2 \simeq \frac{1}{2} k^2 c_A^2 \left(1 + \frac{c_s^2}{c_A^2}\right)\left[1 \ominus \left(1 - \frac{2k_z^2 c_s^2}{k^2 c_A^2}\right)\right]$$

$$\omega_-^2 = \frac{1}{2} k^2 c_A^2 \left(1 + \frac{c_s^2}{c_A^2}\right)\left(\frac{2k_z^2 c_s^2}{k^2 c_A^2}\right)$$

$$\omega_-^2 = k_z^2 c_s^2 = k^2 c_s^2 \cos^2\theta$$

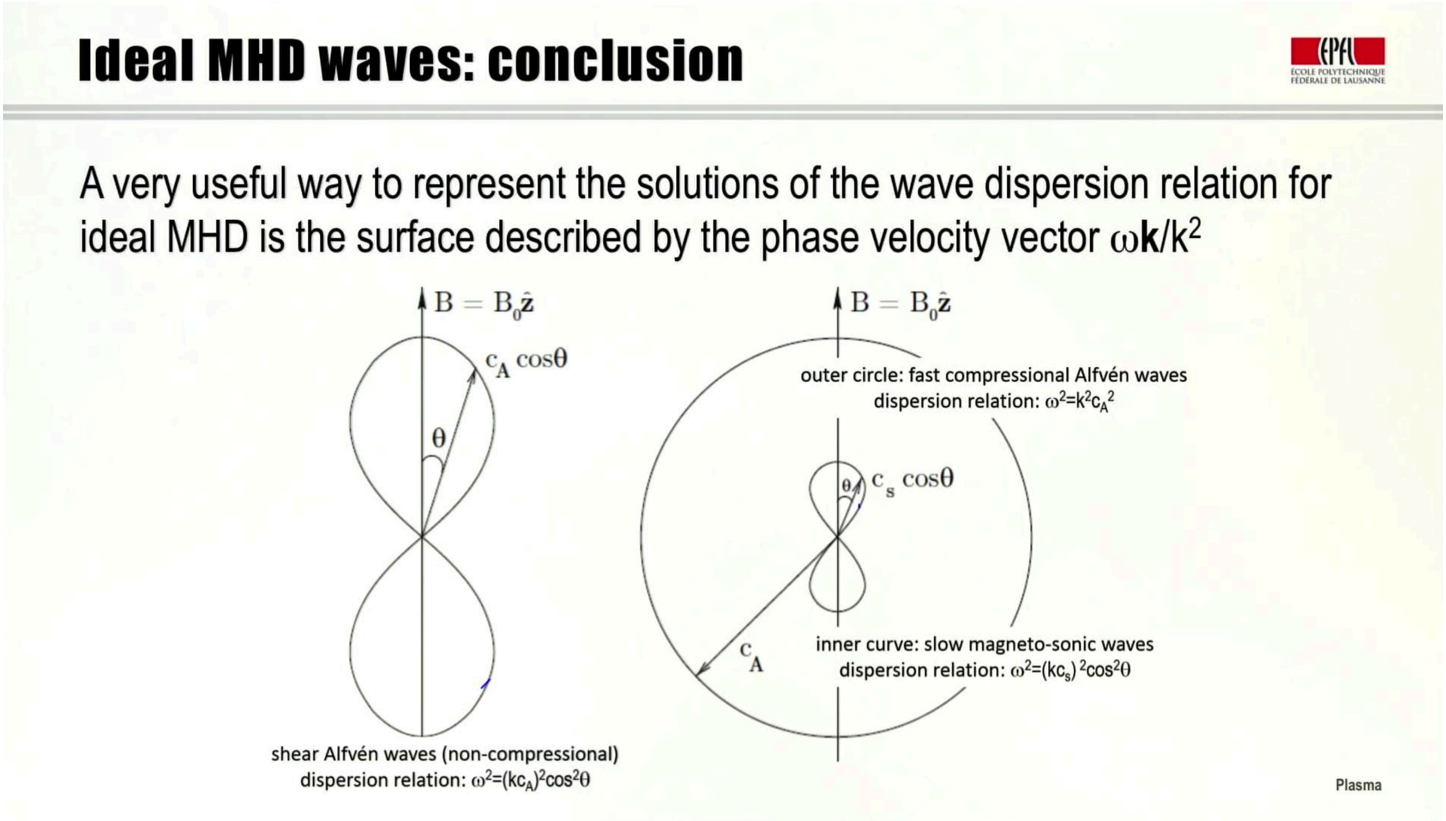
Plasma

We have worked through the fast wave solution, the plus solution. Now we can take the minus solution. This will lead us to the slow wave. We've taken the minus solution here. We can now work through the algebra. The 1 - 1 term cancels out, so we only remain with 2 k_z^2 c_s^2 /(k^2 c_A^2). (k^2 c_A^2) and (k^2 c_A^2) simplifies out. So what we end up with: ω_-^2 = k_z^2 c_s^2 and this is why we call this wave the slow wave, because the phase velocity is cs. That is much smaller than cA. The wave frequency depends on the sound speed and the angle between the k vector and the equilibrium magnetic field. So we call this the slow magnetosonic wave — a sound wave that exists in the presence of a magnetic field.

Notes

Summary

14m 27s

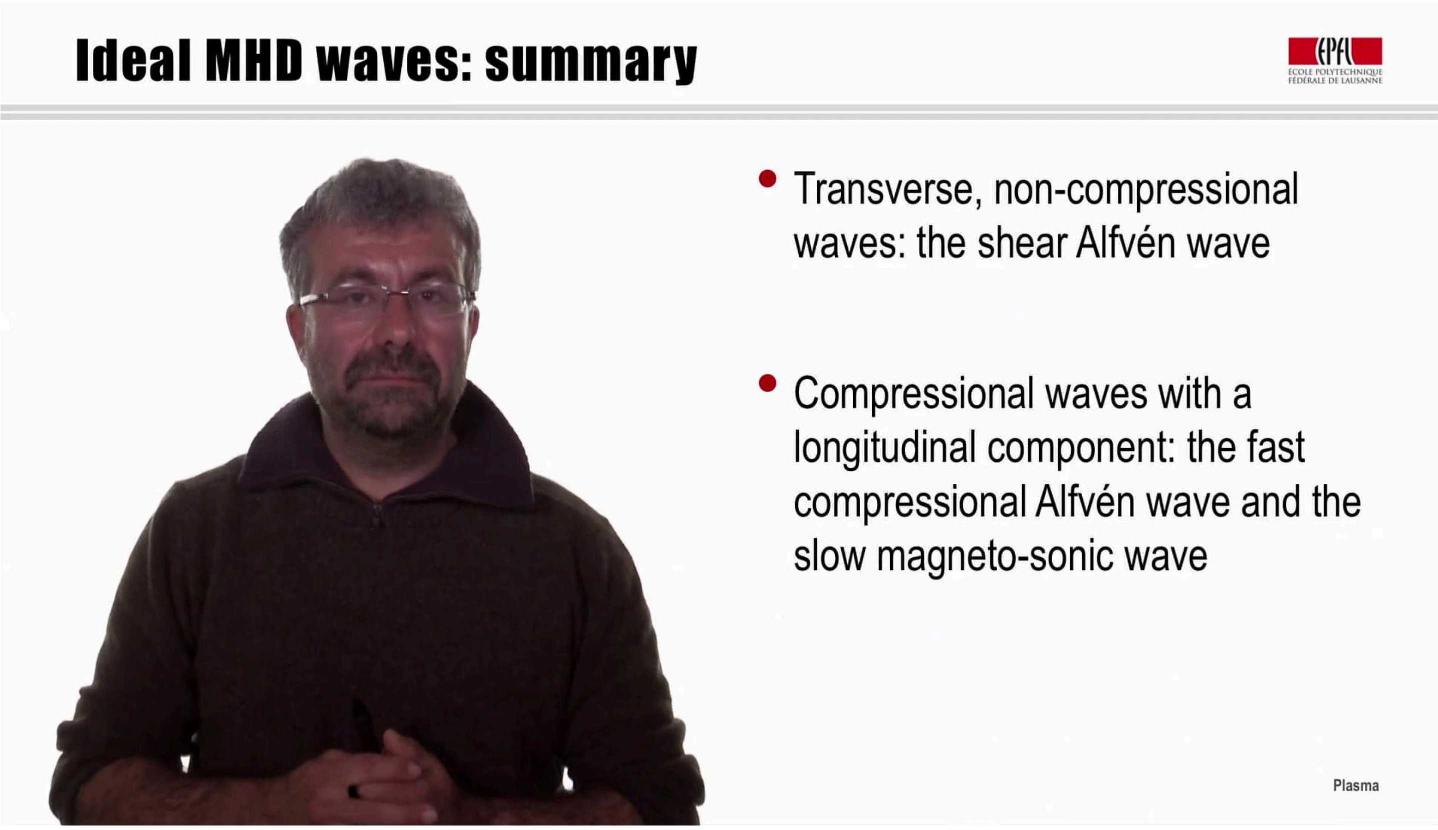


A very useful way to represent the solution of the wave dispersion relation for ideal MHD is the surface described by the phase velocity vector ω k /k². We have here on the left the case for the shear Alfvén wave, which is noncompressional, the dispersion relation, $\omega^2 = (k\ cA)^2 \cos^2\theta$. So if we draw this curve, what we see is that we obtain an eight along the axis of the magnetic field and the effective radius of this eight is cA cos θ. In the second case, we have the dispersion relation for waves which have a longitudinal component. We have an inner figure that is again another eight. This corresponds to the slow magnetosonic wave for which the dispersion relation is $\omega^2 = (k\ cs)^2 \cos^2\theta$. The radius is (cs cos$\theta$), so it's much smaller than (cA cosθ). The outer circle corresponds to the fast compressional Alfvén waves, for which the dispersion relation is $\omega^2 = k^2\ cA^2$ when we neglect all correction due to the sound speed. This is a circle whose radius is cA.

Notes

Summary

15m 44s



We can now summarize what we have discussed on ideal MHD waves. The first wave that we discussed [is the] transverse wave. This is a noncompressional wave, the shear Alfvén wave. This wave will be discussed in the concept of burning plasmas. Then we have discussed compressional waves which have a longitudinal component, and we found two solutions: the fast compressional Alfvén wave and the slow magnetosonic wave.

Notes

Summary

17m 09s