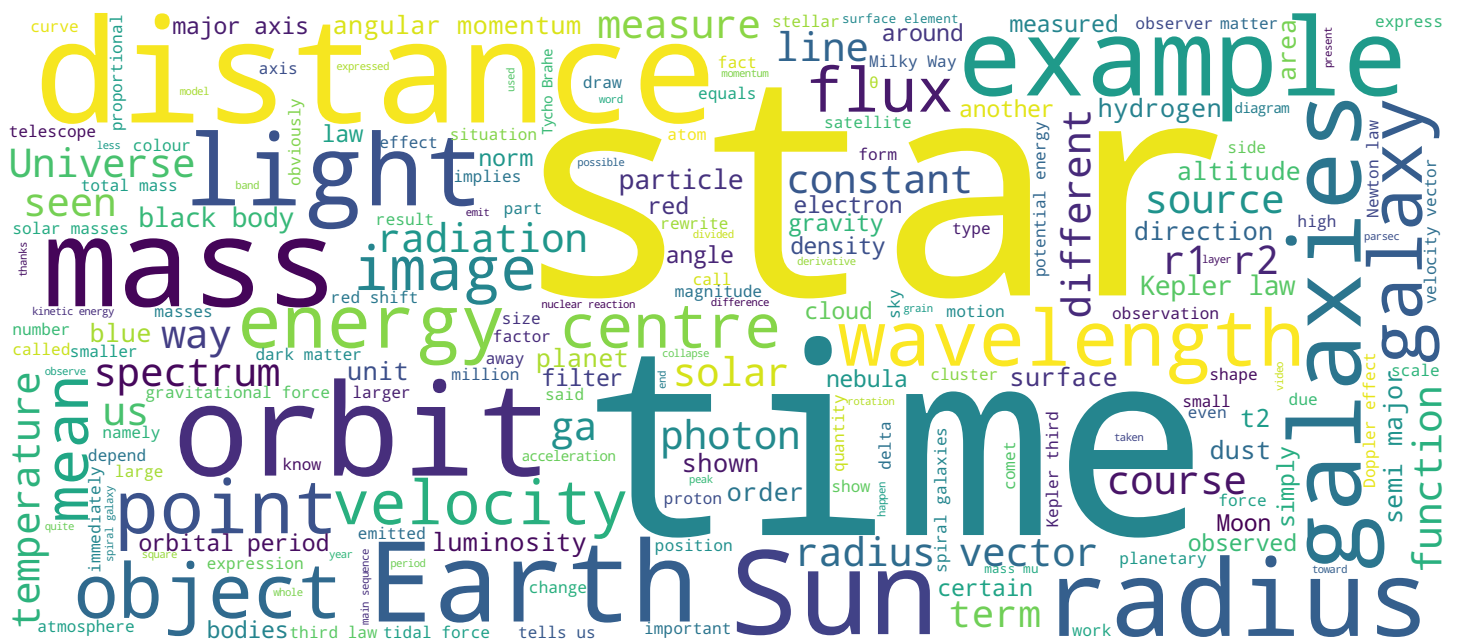


Kepler Laws

Introduction to Astrophysics; Chapter 1.2

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Video



Kepler Laws



Introduction to Astrophysics

This chapter presents the three Kepler's laws describing the motion of objects in the solar system and by extension in every planetary and stellar system. They related in a simple way the masses of the interacting bodies to the shape of their orbit and to the typical travel times on these orbits, namely the orbital periods. Kepler's laws are the result of the work of a great observer and mathematician. They were deduced from the observations of danish Tycho Brahe, an astronomer with quite a sturdy character, strong advocate of the geocentric view of the Universe Tycho Brahe worked from the island of Ven in front of Copenhagen, that king Frederick II bequeathed to him to create an observatory : Uraniborg, dedicated to the Muse of Astronomy. He was the last astronomer before the era of telescopes. He mapped the sky with an extreme precision for the time using sextants and astrolabes. He even took into account the atmospheric refraction, an apparent displacement of stars on the sky due to the refraction of light rays by the the Earth's atmosphere. In 1600, Kepler became Tycho Brahe's assistant who asked him to compute Mars' orbit.

Notes

Summary



Kepler Laws



Introduction to Astrophysics

As opposed to Brahe, Kepler supported the model of a heliocentric universe which was a source of disagreement between them and slowed down the work of Kepler. His work took him six years and was published in 1609. Later, Newton theories built a framework around Kepler's law which were until then only a purely mathematical description of Brahe's observations. Newton's law of gravitation thus became the theory behind the trajectories described by Kepler in the framework of heliocentrism. Note by the way that Newton was absolutely not a contemporary of neither Kepler nor Brahe, who therefore worked without any knowledge of the law of gravitation.

Notes

Summary



1m 23s

The 3 Kepler Laws



1. Planets follow elliptical orbits. The Sun is located at one of the foci of the ellipse.
2. A line that connects a planet to the Sun sweeps out equal areas in equal times.
3. The square of the orbital period is proportional to the cube of the orbit's semi-major axis.

Introduction to Astrophysics

What do Kepler's laws say ? The first one tells us that planets follow flat and elliptical trajectories with the Sun at one of the foci. The second one tells us that the radius vectors sweep out equal areas during equal intervals of time. Finally, the third one, maybe the most important in regard to astrophysics, tells us that the square of the orbital period is proportional to the cube of the semi-major axis of its orbit.

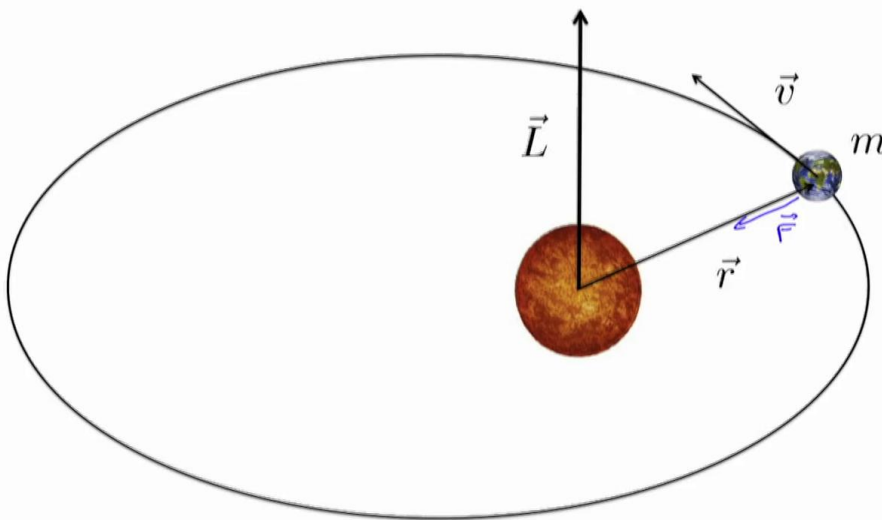
Notes

Summary



2m 09s

Orbites planes



$$\vec{L} = m \vec{v} \wedge \vec{r}$$

$$\frac{d\vec{L}}{dt} = m \underbrace{\frac{d\vec{r}}{dt} \wedge \vec{r}}_0 + m \underbrace{\vec{v} \wedge \frac{d\vec{r}}{dt}}_0 = 0$$

$$\vec{L} = c \vec{e}_z$$

Introduction to Astrophysics

Now back to the first Kepler's law. We will not prove here that the orbits are elliptical but we will quickly show that the orbits are flat. To this end, we will write the angular momentum : $m \vec{v} \wedge \vec{r}$ where \vec{v} is the velocity vector and \vec{r} the radius vector. Lets take the derivatives of this angular momentum with respect to time. We immediately see here the velocity vector. This vector is obviously colinear with itself and therefore this whole expression vanishes. As we have a radial force, we recognise here the derivative of a momentum with respect to time. We thus have a central force which is colinear with the radius vector. This means that \vec{F} being parallel to \vec{r} , this expression here vanishes. We have then shown that the time derivative of the angular momentum is equal to zero, so \vec{L} has a constant norm and direction with time. Therefore, the orbit lies on a plane.

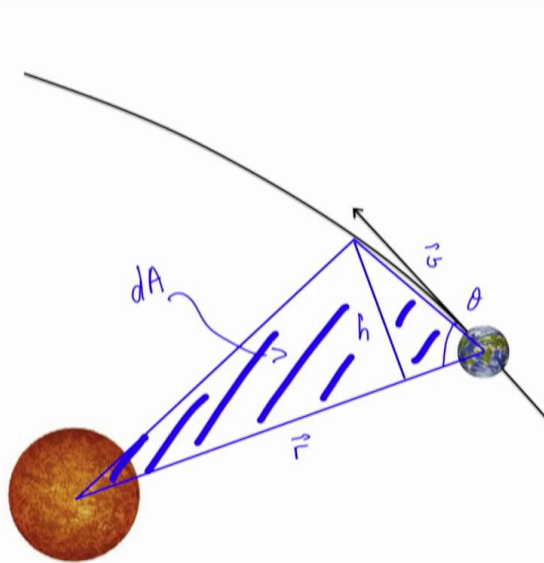
Notes

Summary



2m 38s

Area swept by the radius vector (law of areas)



$$r = \|\vec{r}\|$$

$$dA = \frac{r \cdot h}{2} = \frac{r}{2} \cdot v \cdot dt \sin \theta$$

$$\|\vec{L}\| = m v r \sin \theta$$

$$dA = \frac{\|\vec{L}\|}{2m} dt$$

$$A(t) = \frac{\|\vec{L}\|}{2m} t + K$$

Introduction to Astrophysics

Kepler's second law tells us that the radius vectors sweep out equal areas during equal intervals of time. Here is a small surface element between two instants while the Earth moves along its orbit. We have a triangle with its altitude here. We now seek to compute this area. We have here the altitude of the triangle, the radius vector here, Earth's velocity with a certain angle with respect to the tangent to its orbit. We will call this surface element dA . We then have that dA is equal to the radius vector, which is also the base of the triangle, times half the altitude. Let us express the altitude in terms of known quantities : the velocity, the angle and the time element to travel this small distance along Earth's orbit. We factor out the $r/2$. The altitude is $v \cdot dt$ times the sine of the angle. We immediately see that the norm of the angular momentum, which is $m v r \sin(\theta)$, shows up here up to a factor m . We can thus rewrite the surface element in this way. Here is the norm of the angular momentum which we proved previously to be constant, over $2m$ times dt . We now simply have to integrate this in order to get the area, which we will express as a function of time.

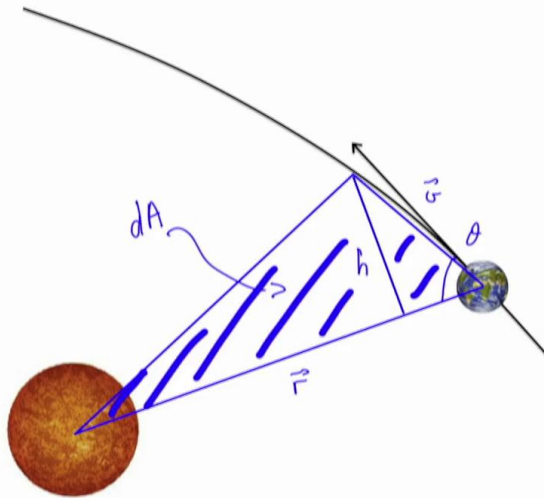
Notes

Summary



3m 54s

Area swept by the radius vector (law of areas)



$$r = \|\vec{r}\|$$

$$dA = \frac{r \cdot h}{2} = \frac{r}{2} \cdot v \cdot dt \sin \theta$$

$$\|\vec{L}\| = m v r \sin \theta$$

$$dA = \frac{\|\vec{L}\|}{2m} dt$$

$$A(t) = \frac{\|\vec{L}\|}{2m} t + K$$

$$\text{Pour } t = P \quad A(t) = \pi a b$$

$$A(t_2) - A(t_1) = \frac{\|\vec{L}\|}{2m} \cdot \Delta t$$

$$\Delta t = |t_2 - t_1|$$

Introduction to Astrophysics

We can derive the value of this constant with the boundary condition that, if t equals a full period, then $A(t)$ is the surface of the ellipse, πab where a and b are the semi-major, respectively the semi-minor axis. This allows to compute exactly the value of the constant. Between two times t_1 and t_2 , we have that $A(t_1) - A(t_2)$ is equal to the norm of the angular momentum over $2m$ times Δt , where $\Delta t = t_2 - t_1$. We have thus proved the second law. For any t_1 and t_2 , what is important is the value of Δt , namely the time interval between the two observations.

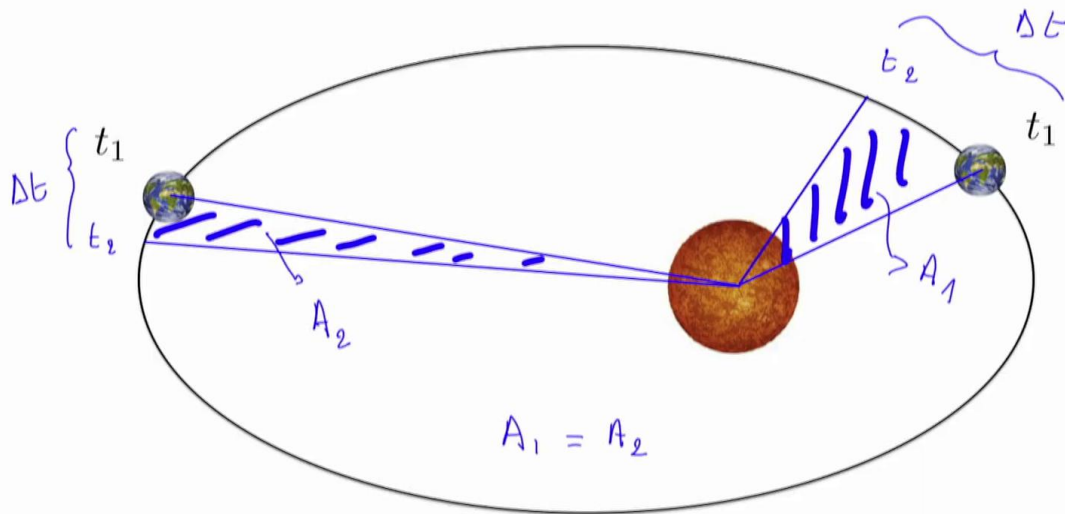
Notes

Summary



6m 01s

Area swept by the radius vector (law of areas)



Introduction to Astrophysics

We can see this here. If I draw a radius vector here and another one here, the surface swept out by this radius vector must be the same that the one swept out by the radius vectors between the same times t_1 and t_2 . I am here further away on the orbit so the covered angle will be smaller. Here is a time t_2 , and the other t_2 is here. We have Δt here, and the same Δt here. So the area swept out here is the same one as the area swept out there. This implies that the bodies further away on their orbit are slower. Their tangential speed along their orbit is smaller when further away from the body that attracts them, the Sun in this specific case, than when the Earth here is closer to the Sun. We finally see that the surface A_1 here is equal to the surface A_2 .

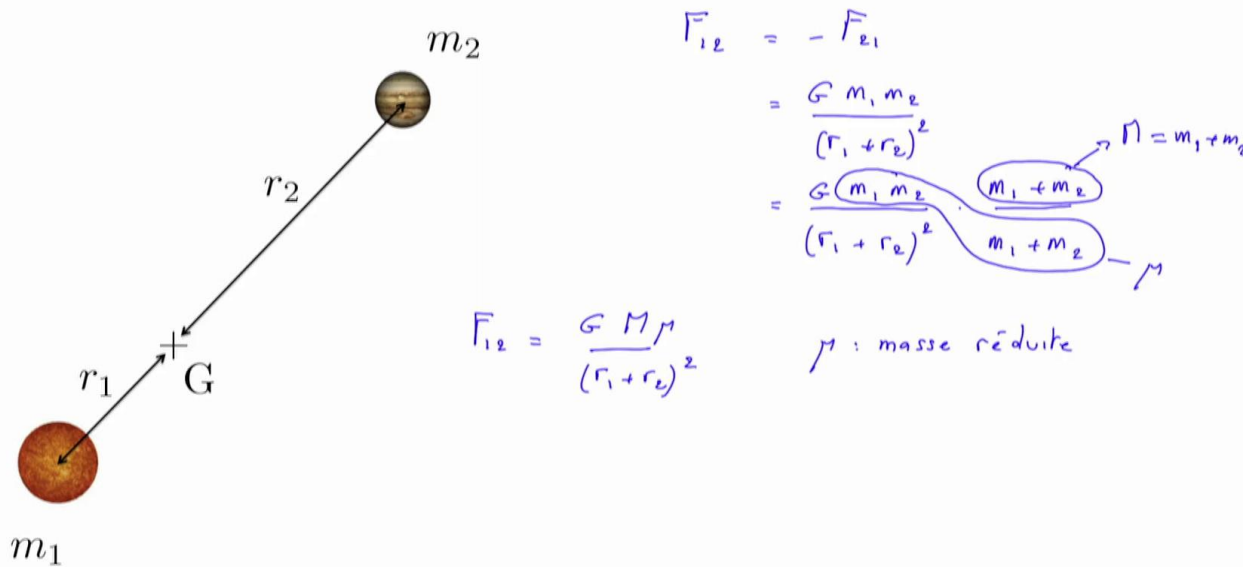
Notes

Summary



7/m 13s

Reduced Mass and 3rd Kepler Law



Introduction to Astrophysics

This is for the second law. Kepler's third law. This law states that the square of the orbital period is proportional to the cube of the semi-major axis of the orbit. We will now prove this law in the case of circular orbits. Let us consider a system of two bodies with masses m_1 and m_2 orbiting around a common centre of gravity at distances r_1 and r_2 . The gravitational force that 1 exerts on 2 is of course equal to minus the force by 2 on 1. Its value is $G m_1 m_2$ over the distance between the two bodies, that is $r_1 + r_2$ squared. This is simply Newton's law. I can multiply this equation by any quantity. I have here multiplied the top and bottom part by the same quantity. I can now rearrange the terms for example by grouping these terms here. This term is the total mass of the system and this is another mass, which we write μ , the reduced mass of the system. With these new definitions, the gravitational force is rewritten as $G M \mu / (r_1 + r_2)^2$ with μ being the reduced mass. This is equivalent to a mass μ orbiting around the mass M at a distance of $r_1 + r_2$.

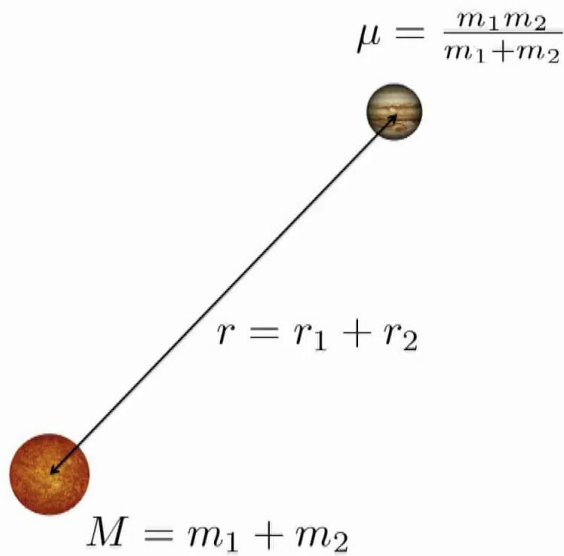
Notes

Summary



8m 34s

Reduced Mass and 3rd Kepler Law



$$a_\mu = \omega_\mu^2 (r_1 + r_2) = \frac{G M}{(r_1 + r_2)^2}$$

$$\omega_\mu = \frac{2\pi}{P_\mu}$$

$$\frac{4\pi^2}{P^2} = \frac{G M}{(r_1 + r_2)^3}$$

$$\Rightarrow \frac{P^2}{(r_1 + r_2)^3} = \frac{4\pi^2}{G M}$$

$$\frac{P^2}{r^3} = \frac{4\pi^2}{G M}$$

$m_1 \gg m_2$
 $P \approx m_1$
 $r \approx r_1$

Introduction to Astrophysics

This situation is represented here. We have here μ that orbits at a distance $r_1 + r_2$ around a virtual mass M that would have the total mass of the planetary system. We have thus reduced problem of the motion of two bodies around a common centre of gravity to the motion of a body μ , with virtual mass μ , around another virtual body which has a mass equal to the sum of the masses of the two bodies in the planetary system. The distance between these two virtual bodies is the same as here : $r_1 + r_2$. Let us now consider the acceleration of the mass μ due to mass M in the case of a circular orbit. We call it a_μ . It is equal to the angular velocity squared times $r_1 + r_2$. According to Newton's law, it is also equal to $G M / (r_1 + r_2)^2$. On the other hand, ω is equal to 2π over the orbital period and we can thus substitute ω here, ω_μ and P_μ . I therefore get $4\pi^2 / P^2 = G M / (r_1 + r_2)^3$. In other words, $P^2 / (r_1 + r_2)^3 = 4\pi^2 / G M$. This is the distance r . In the case where the mass m_1 is much larger than m_2 , then the total mass M is approximately m_1 . The centre of gravity of the system will be inside m_1 , almost at the centre of m_1 . We have thus that r is approximately equal to r_1 . We can therefore rewrite this as $P^2/r^3 = 4\pi^2 / G M$ which is Kepler's third law.

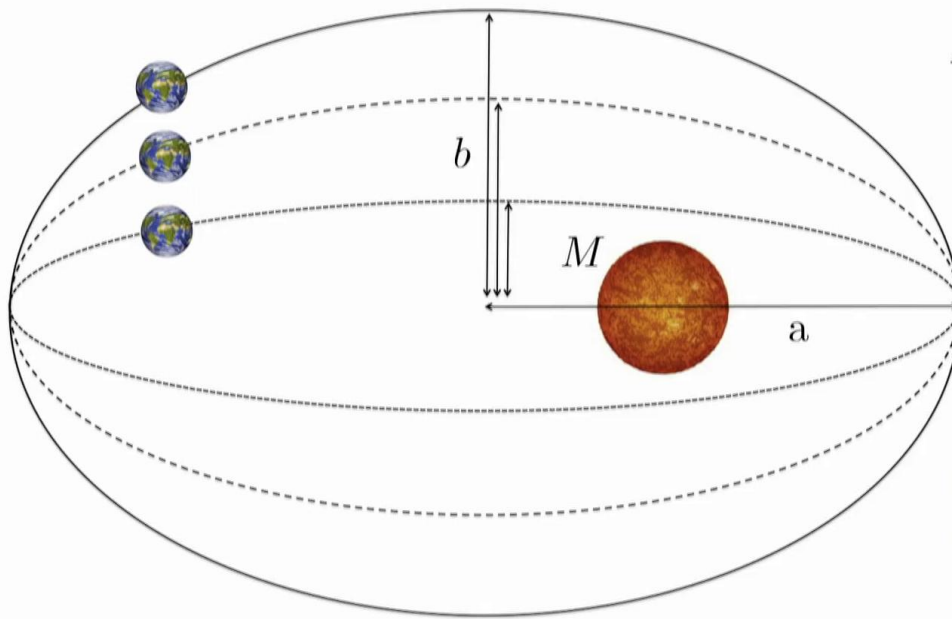
Notes

Summary



10m 35s

The Orbital Period is Independent of Eccentricity



$$\frac{P^2}{a^3} = \frac{4\pi^2}{GM}$$

$$e^2 = 1 - \left(\frac{b}{a}\right)^2$$

a en UA

$a = 1$

P en années

$P = 1$

$$\boxed{\frac{P^2}{a^3} = 1}$$

Introduction to Astrophysics

We note that this law does not depend on the eccentricity of the orbit, defined as $1 - (b/a)^2$ where b , a are the semi-minor, respectively the semi-major axis of the orbit. This means that orbits with different eccentricities have the same orbital period as long as the semi-major axis of the orbit and the mass of the main body are set. As a final note, if we express the semi-major axis of the orbit in astronomical units, that is 150 millions of kilometres, and in the case of the Earth, $a = 1$, since 150 million of kilometres is the distance from Earth to the Sun, and if the period is expressed in terrestrial years, then $P = 1$. We have thus renormalized the units of Kepler's third law in such a way that $P^2/a^3 = 1$ in the appropriate units. This is a very useful relation in the case of the solar system.

Notes

Summary



13m 16s



Images: Wikipédia

This ends our chapter on Kepler's laws. We will meet them many times during this course, in particular in the description of the shape of cometary tails, in the expression of tidal forces acting on the Earth and the Moon for example, and of course in the search for exoplanets.

Notes

Summary



14m 24s