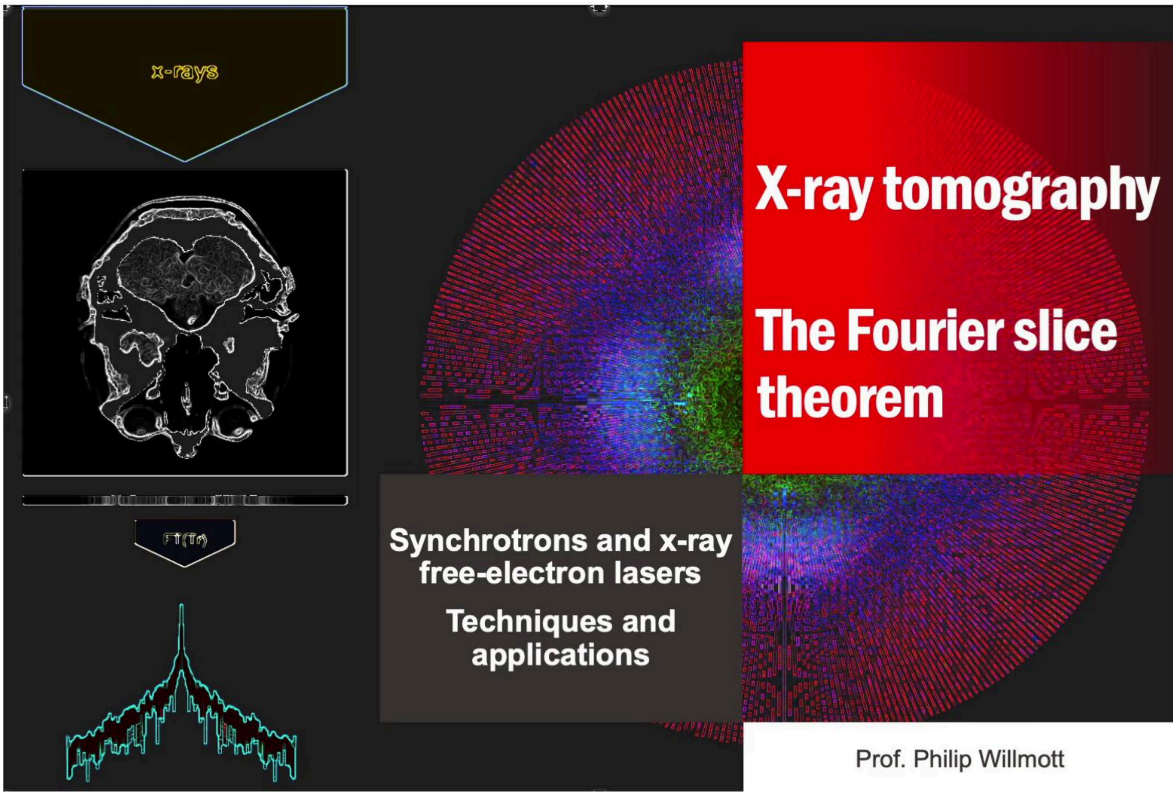
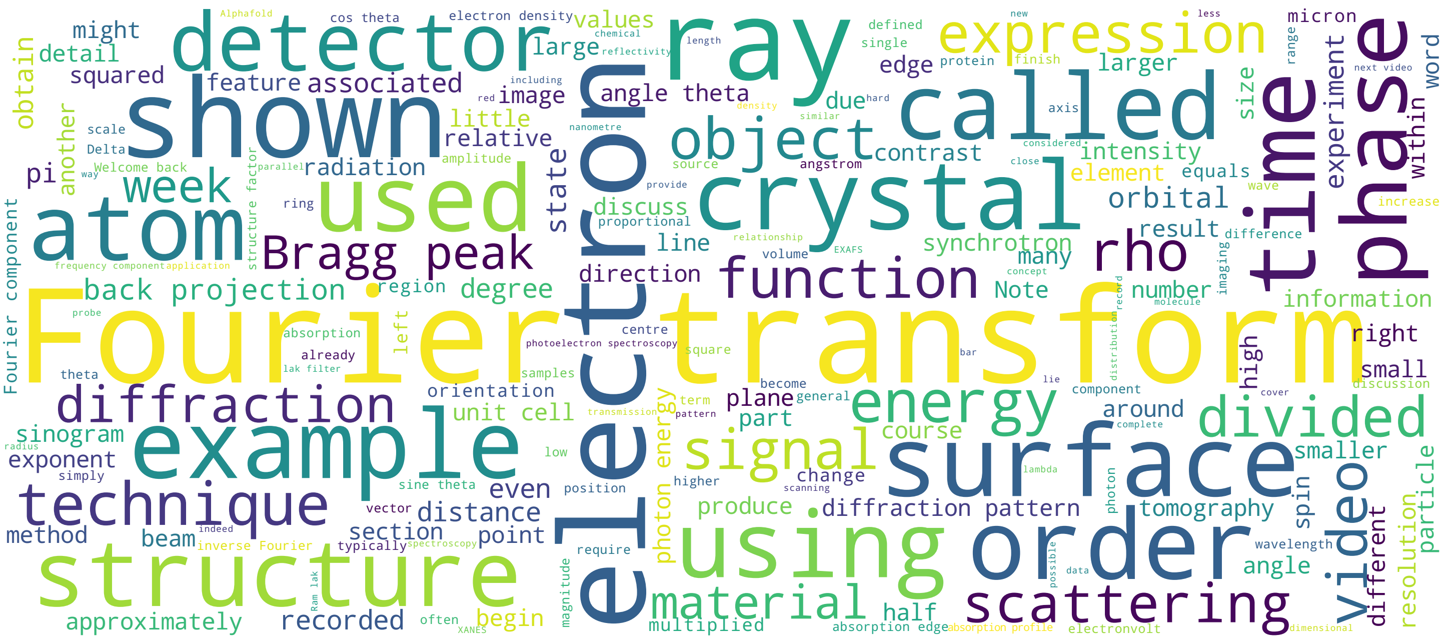
École polytechnique fédérale de Lausanne





Search MOOC

Video

Contents and objectives of this video

- The idea
- Use of Fourier transforms (oh no!)
- Generation of a 2-D FT via summation of 1-D FTs
- FT info explains why laminograms are rubbish
- How to un-rubbish laminograms

Hello, again, this video follows directly on from the previous one, where we take our expression for laminograms or unfiltered back projections and discover that there is a close relationship between a set of 1-D back projections at different angles and the 2-D object that produces them. Now, for this, we unfortunately have resort to Fourier transforms, but hey ho, that's life. Now, once we have achieved our goal, you will hopefully see from the Fourier transforms why, mathematically speaking, the halo arises and how we must proceed in order to remove it.

Notes

Summary

0m 05s

The idea

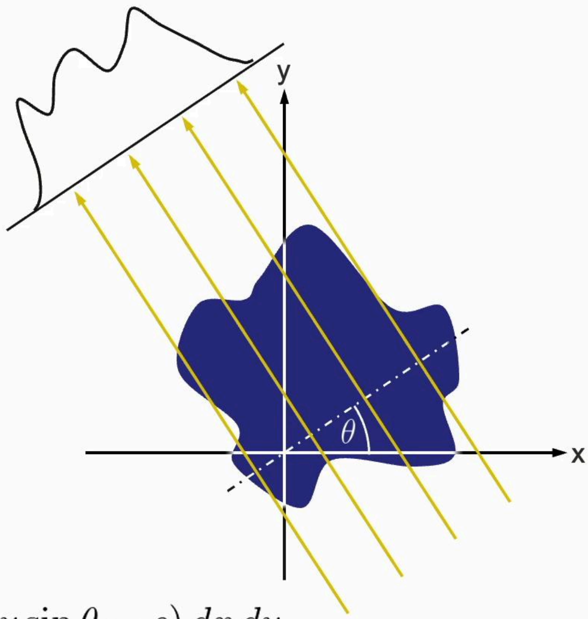
- Is there a relationship between the 1-D FT of the projection $g(\rho_j, \theta)$ and the 2-D FT of $I(x, y)$ describing the object itself?
- Begin with the FT of the projection

$$g(\rho, \theta) \overset{\text{FT}}{\Rightarrow} G(\omega, \theta)$$

$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(\rho, \theta) e^{-2\pi i \omega \rho} \, d\rho$$

But

$$g(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y) \, \delta(x \cos\theta + y \sin\theta - \rho) \, dx \, dy$$



What we rather pointedly are now going to ask is, is there a relationship between the 1-D Fourier transform of the back projection at one angle theta, g rho j theta, and the 2-D Fourier transform of Ixy describing the object. Needless to say, the answer is yes, or otherwise, I'd really be wasting your time. Now, the relationship between these two entities is called the Fourier Slice theorem. We'll begin by providing an expression for the Fourier transform of our single-angle back projection. The Fourier transform of little g as a function of rho theta is called big G as a function of omega theta, where omega is a set of spatial frequencies. As always with Fourier transforms, big G equals the integral across all values of rho of little g times the exponent of minus two pi I omega rho d rho. What this expression is saying mathematically, is what is the spectrum of spatial frequencies omega that contribute to make up our single-angle back projection. We should also remember that the radon transform little g is itself the double integral across the plane containing Ixy multiplied by our Delta function that selects only those values of x and y that lie on the line defined by rho, as we derived in the previous video.

Notes

Summary

0m 44s

The idea, continued

- Therefore

$$G(\omega,\theta) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x,y)\, \delta(x\cos\theta + y\sin\theta - \rho) dx\, dy \right] e^{-2\pi i \omega \rho} d\rho$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[I(x,y) \int_{-\infty}^{\infty} \underbrace{\delta(x\cos\theta + y\sin\theta - \rho)}_{\neq 0 \text{ only when } x\cos\theta + y\sin\theta = \rho} e^{-2\pi i \omega \rho}\, d\rho \right] dx\, dy$$

$$\Rightarrow G(\omega,\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x,y)\, e^{-2\pi i \omega (x\cos\theta + y\sin\theta)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x,y)\, e^{-2\pi i (ux+vy)}\, dx\, dy$$

whereby $u = \omega\cos\theta$, $v = \omega\sin\theta$

Thus, we obtain an increasingly frightening looking triple integral over x, y, and rho of the radon transform multiplied by the exponent of minus two pi I omega rho. We can quite legitimately rearrange this expression thus, as x and y in Rho are all independent of one another. Importantly, this term delta is only nonzero and equal to unity when x cos theta plus y sine theta are equal to rho. Hence, we can substitute for rho in the exponent term to obtain big G is equal to the double integral over x and y of the object's 2-D absorption profile Ixy multiplied by the exponent of minus two pi I omega times x cos theta plus y sine theta. Now we substitute omega cos theta for u and omega sin theta for v, and we obtain the relatively simple expression that big G the integral over x and y of Ixy times the exponent of minus two pi I multiplied by ux plus vy. u and v are frequencies in the horizontal and vertical direction, respectively.

Notes

Summary

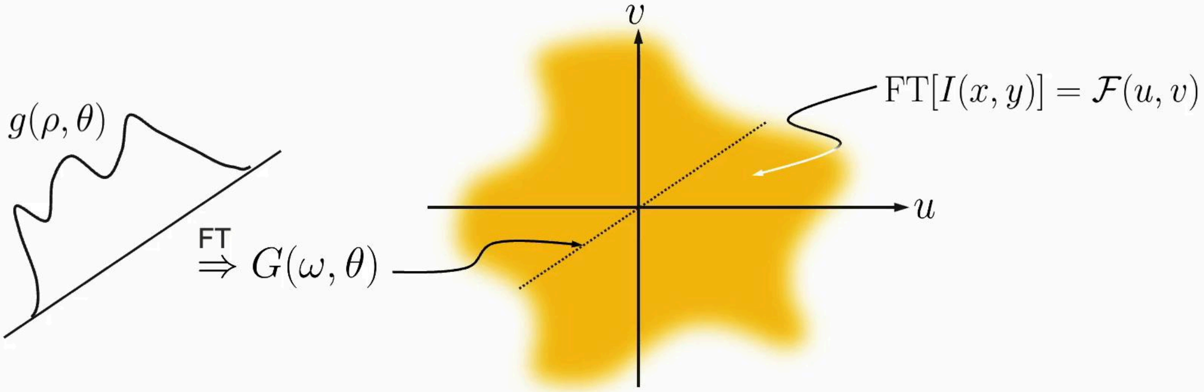
2m 30s

The idea, continued

$$G(\omega,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(x,y)\, e^{-2\pi i(ux+vy)}\, dx\, dy\Big|_{\substack{u=\omega\cos\theta \\ v=\omega\sin\theta}}$$

This is the slice of the 2D FT of the original image along the line

$$u = \omega\cos\theta\,,\ v = \omega\sin\theta$$



Our expression for g, omega theta is thus the Fourier transform of the original image Ixy along the line, u is equal to omega cos theta, and v is equal to omega sin theta. Now, what does this mean physically? In words, this means the 1-D Fourier transform of our projection equals the slice through the 2-D Fourier transform of the 2-D image at the angle theta. To repeat big G, the 1-D Fourier transform of little g, which is itself the projection of the 2-D absorption intensity profile of an object Ixy at a certain angle theta equals the components of the slice of the 2-D Fourier transform of Ixy at that same angle theta. This is shown schematically here.

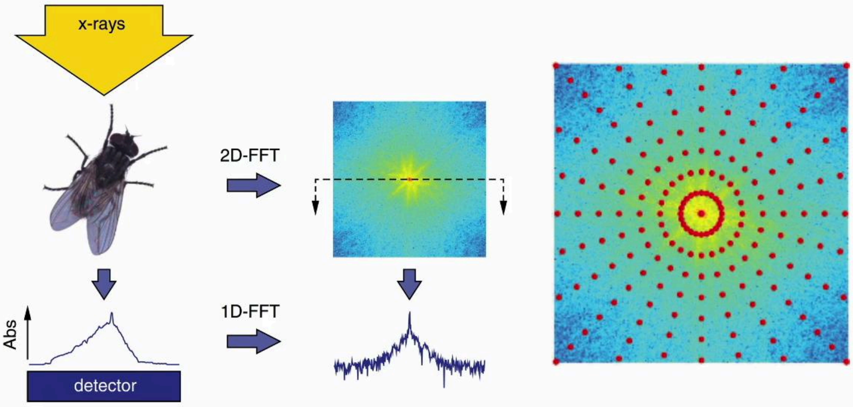
Notes

Summary

4m 01s

The Fourier slice theorem in summary

- $G(\omega, \theta)$, the 1-D FT of a projection $g(\rho, \theta)$ of $I(x, y)$ at an angle θ, is equal to the components of the 2-D FT of $I(x, y)$ along a line at the same angle θ
- After rotation by 180°, all the Fourier components of the 2-D FT are obtained
- Density of Fourier components higher in centre



If we repeat this for a set of angles between nought and pi radians, a complete set of Fourier components will be obtained. Note that the density of these Fourier components is highest in the centre and lowest at the edges. This is an extremely important observation and will help us in solving our halo problem.

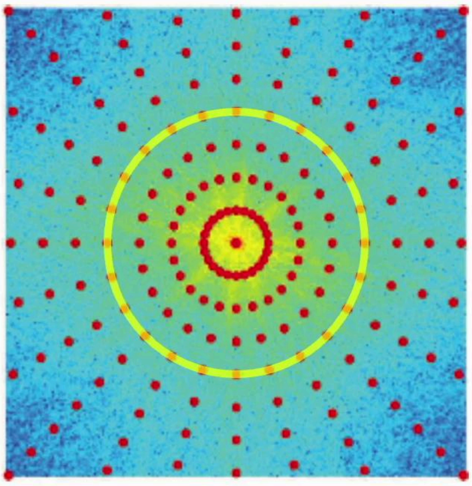
Notes

Summary

5m 00s

The Fourier slice theorem in summary

- Before we simply inverse FT the 2-D Fourier data, let's pause and think a moment…
- The non-homogeneous distribution of the FT components explains why simple back-projections ("laminograms") produce blurry images
  - Low-frequency Fourier components are **over-represented**
  - These are associated with smooth, large features (the blur, or "halo")
- For a given ring of components at radius ω, there is a constant number of those components equal to 2n, twice the number of recorded projections
  - Annular density of components = $2n/2\pi\omega = n/\pi\omega$
  - To make the density constant, we therefore need to multiply each component by $|\omega|$



This is called the **FILTERED BACK-PROJECTION**

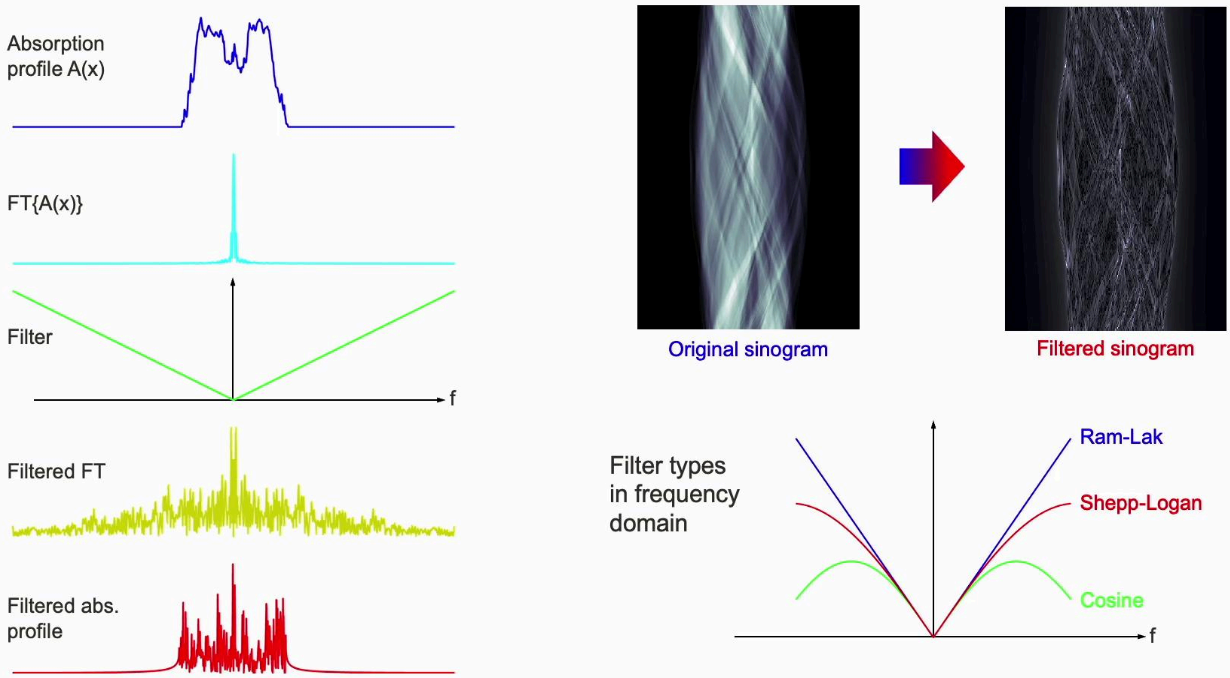
Before we simply inverse Fourier transform the 2-D Fourier data we have obtained via the preceding approach, in the hope of obtaining our real space reconstructed slice, let's pause just for a moment and think a little more about what this uneven distribution of Fourier components means physically. The low-frequency components close to the centre of the Fourier transform are overrepresented or oversampled, but it is exactly these low-frequency components that are associated with smooth, large features. This is the source, mathematically speaking, of our halo artefact. For a given ring of components at some radius omega, there is a constant number of Fourier components equal to 2n, where n is the number of recorded projections. Hence the annular or circumferential density, that is the density along the ring is equal to 2n divided by 2pi omega which is equal to n divided by pi omega. Hence, in order to make the weighting of each component equal, we need to multiply them by their absolute omega values. This action and similar variants to this is called the filtered back projection.

Notes

Summary

5m 20s

Filtering the back projection



Just to go through this for completeness's sake, if we begin with a given absorption profile A(x) for a fixed angle theta and take the Fourier transform of this, then modifying this using a simple ramp or so-called Ram-lak filter produces a modified Fourier transform in which the higher frequencies are much more visible than the original Fourier transform. The inverse Fourier transform results in the filtered absorption profile. The sinogram thus looks quite different. Note also that we have only considered the simplest case of a linear ramp filter. There are others which might be more appropriate depending on the detailed features of the sample, where too much enhancing of high-frequency components may result in ringing and artefactually sharp edges. Two alternatives called the Shepp-Logan and cosine filters are shown here.

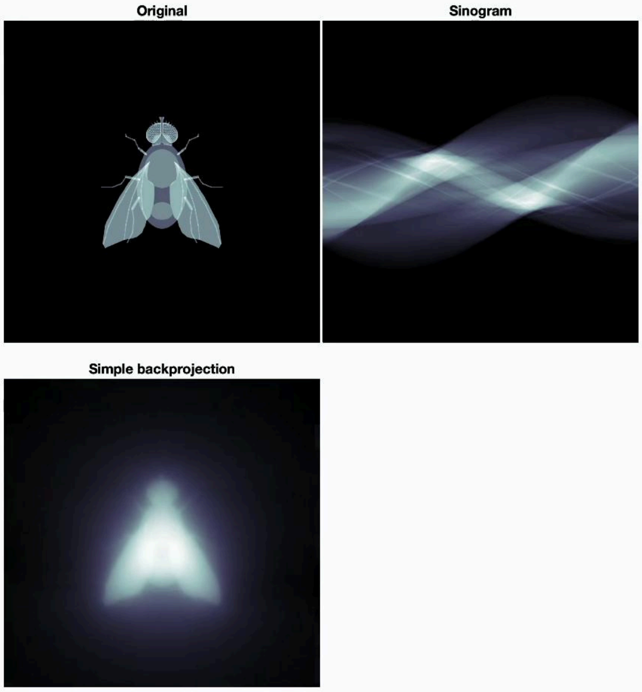
Notes

Summary

6m 46s

Filtering the back projection

- 0.5° steps, 0 – 179.5°
- With no filter, reconstruction is washed out, saturated, and blurred
- Frequency filter for back projections
  - “Ram-Lak” (Ramachandran-Lakshminarayan) filter
  - Weighting ∝ |frequency|
  - Suppresses low-frequency components
  - Acts on FT of each sinogram row
- Obtain filtered sinogram by inverse FT of filtered FT



Here is a simulated example of the cartoon fly we've already met, recorded in half-degree steps. The sinogram looks like this reconstructing the image from the sinogram without any filtering results in the blurry image with halo as expected.

Notes

Summary

7m 44s

Filtering the back projection

- 0.5° steps, $0 – 179.5^{\circ}$
- With no filter, reconstruction is washed out, saturated, and blurred
- Frequency filter for back projections
  - "Ram-Lak" (Ramachandran-Lakshminarayan) filter
  - Weighting ∝ |frequency|
  - Suppresses low-frequency components
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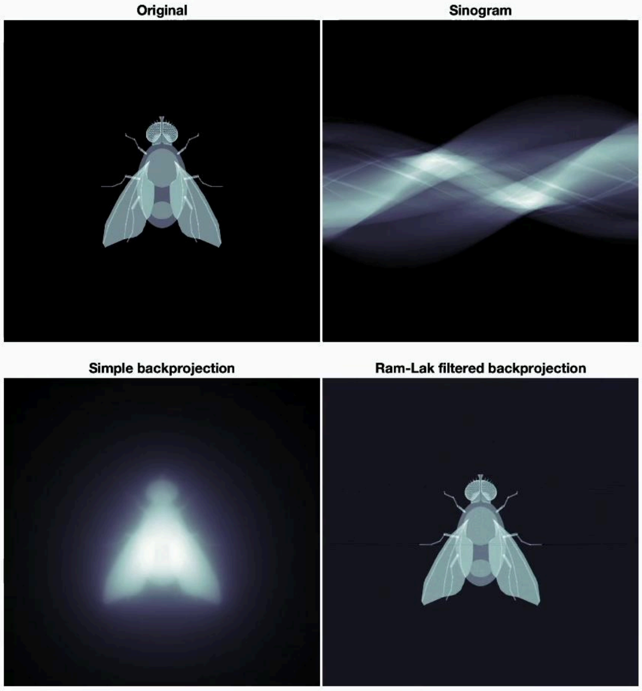


Filtering the sinogram with a Ram-lak filter, as shown in the previous slide, results in contrast with a much more faithful reconstruction with far higher fidelity and detail.

Notes

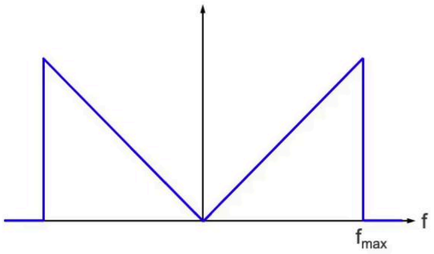
Summary

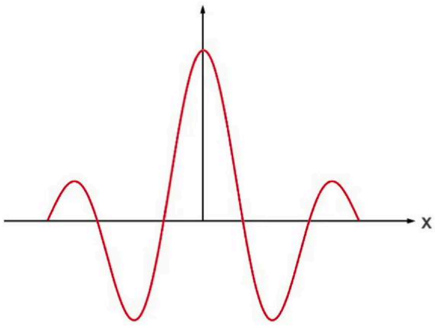
8m 05s

Spatial v frequency filtering

- So far:
  - Sinogram
  - FT each sinogram row
  - Frequency filter FT(sino) with e.g. Ram-Lak
  - Inverse FT of filtered FT(sino)
  - Back-project filtered sinogram
- Remember convolution theorem
  - If the FT(sinogram) is multiplied by a filter function G in frequency space, this is equivalent to the sinogram being convoluted with the FT of G
  - Spatial filtering: sinogram ⊗ FT(G)
    - e.g. FT(Ram-Lak)

$$A\left(\frac{\sin w}{w}+\frac{\cos w - 1}{w^2}\right), \quad w = 2\pi f_{\max} x$$



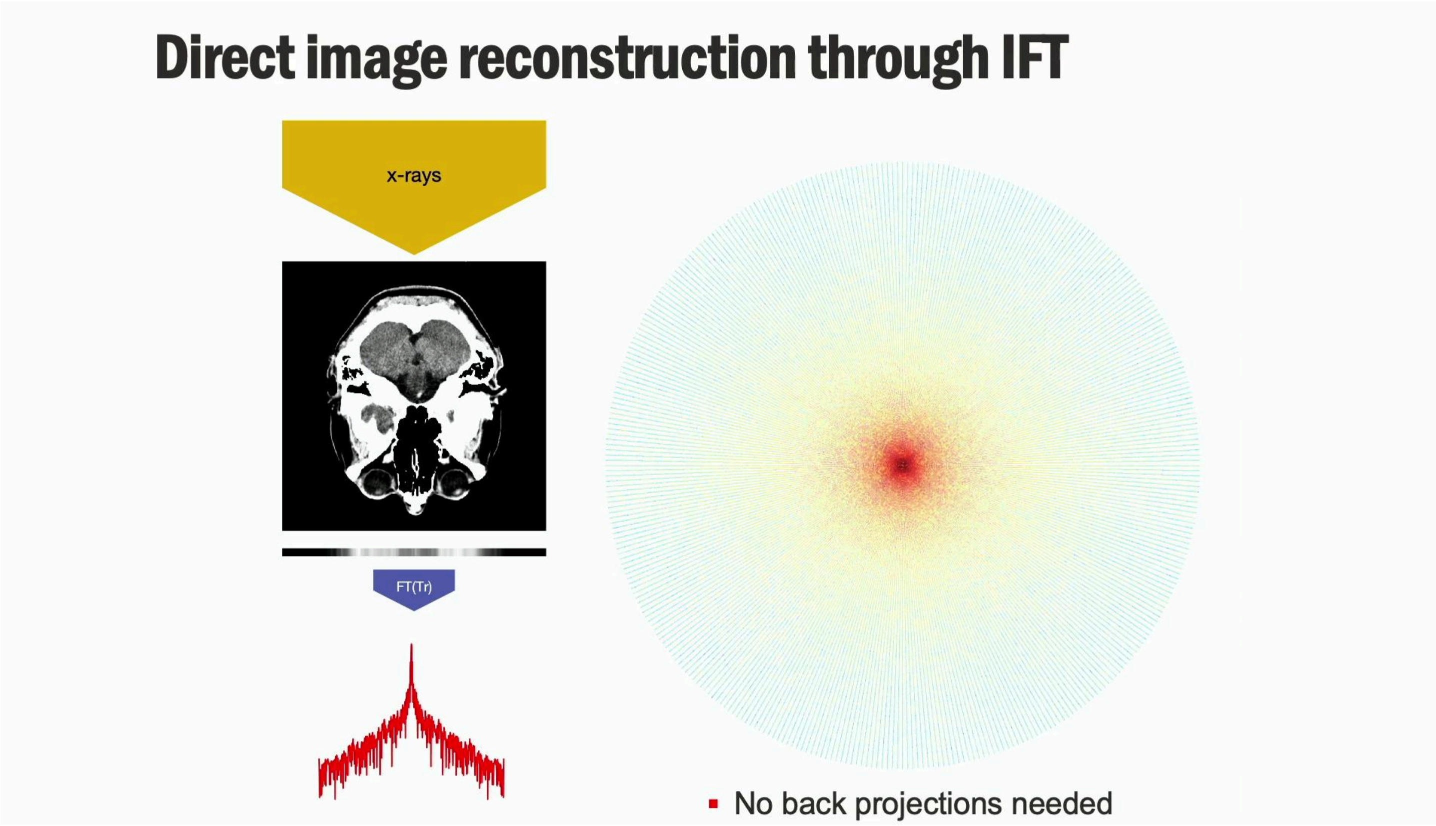


Thus far, our procedure has been to take the sinogram, take the Fourier transform of each of its rows, filter this with some frequency filter, and then back-project the filtered sinogram. However, from the convolution theorem, we know that if the Fourier transform of a function, in this case, our sinogram is multiplied by a filter function in frequency space, this equates to the sinogram itself being convoluted by the Fourier transform of the filter function, which, in the case of a Ram-lak filter, produces the function shown here in red.

Notes

Summary

8m 17s



With the computing and number-crunching power of modern computers, direct image reconstruction without filtered back projections can be achieved today through the inverse Fourier transform and the Fourier slice theorem as shown here in this animation.

Notes

Summary

8m 54s

In the next video...

We complete our basic background knowledge of tomography in the last video of this first section, in which we consider some practical aspects related to tomography beamlines from the source to the x-ray optics, all the way to sample manipulation requirements.

Notes

Summary

9m 14s