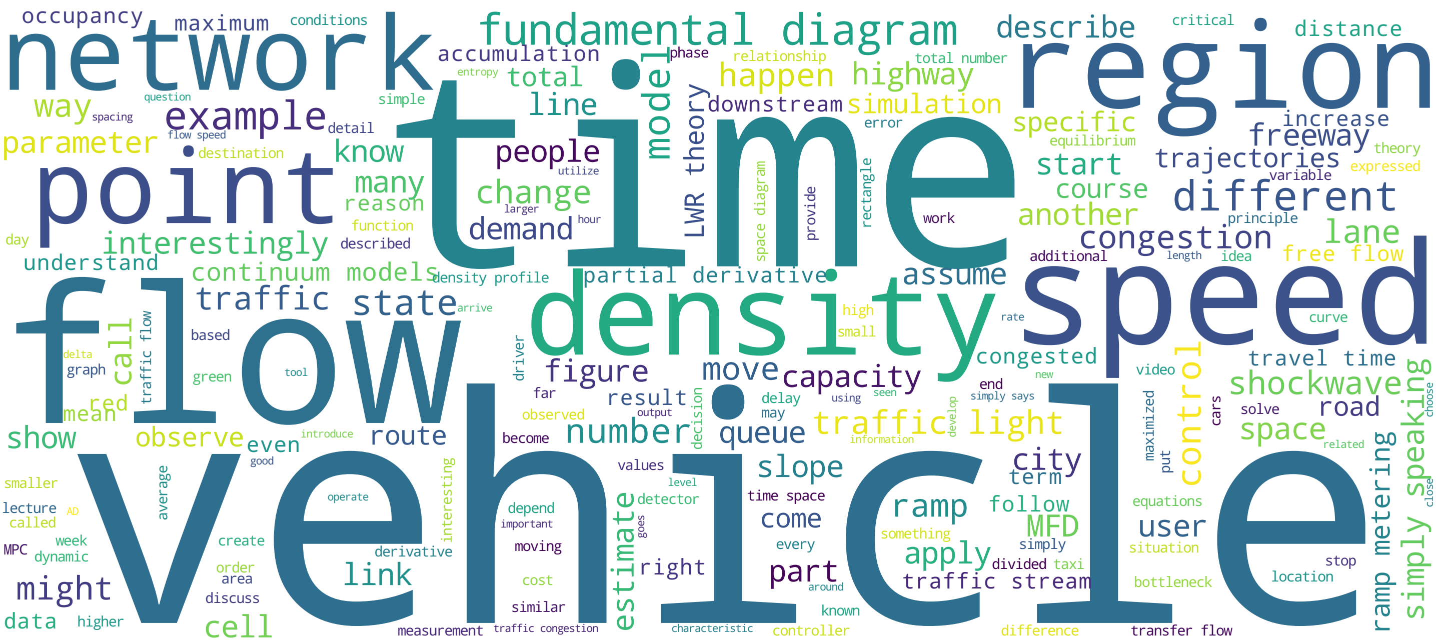
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Traffic flow modeling

An intro to LWR Theory

Intro to traffic flow modeling and ITS

Prof. Nikolas Geroliminis



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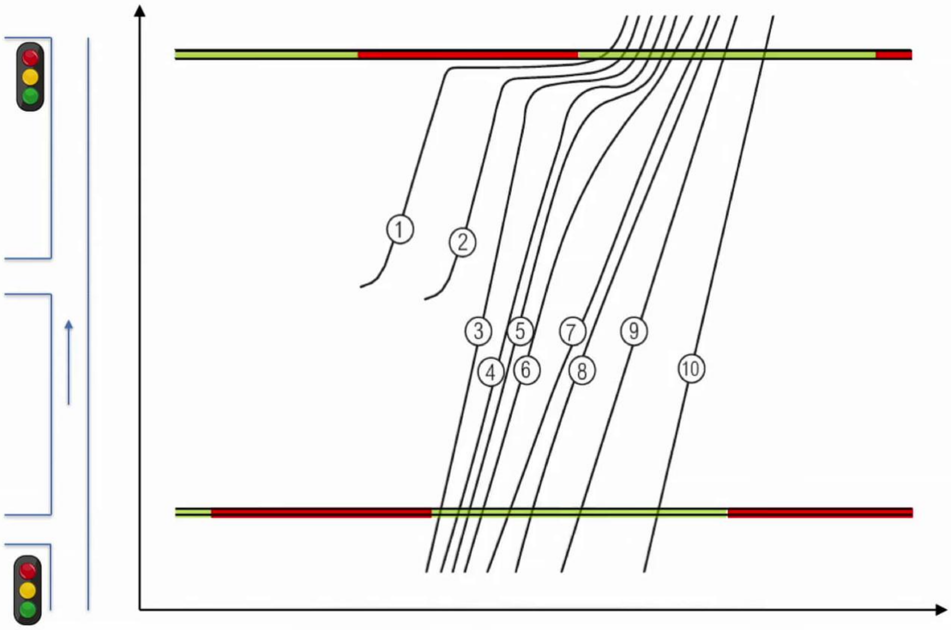
Video

EPFL

Trajectories of vehicles at a traffic light



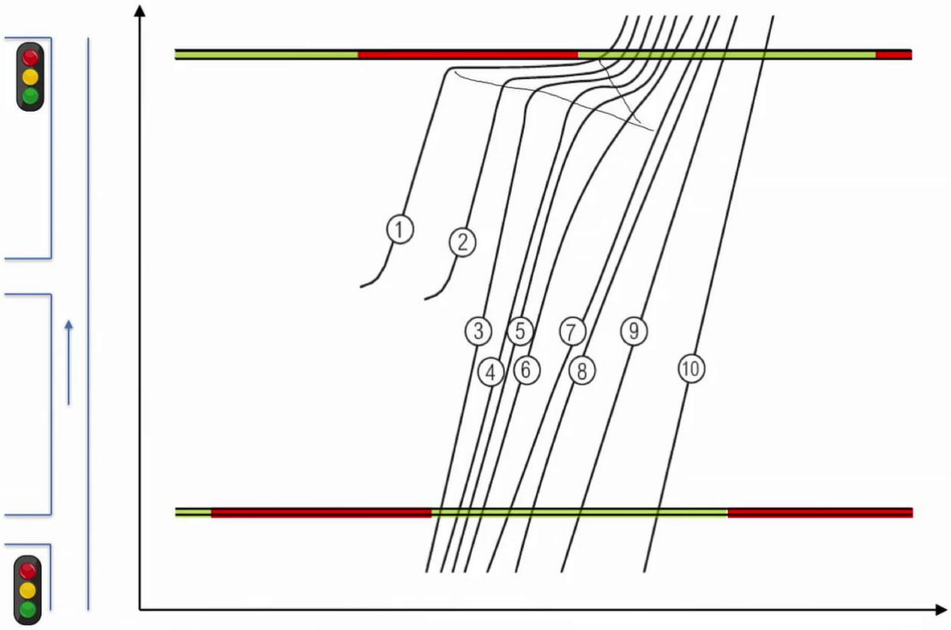
Welcome. Traffic congestion is a very dynamic process. The lecture of this week will focus on continuum models of traffic flow which is a way to understand how congestion changes over time and space by focusing on continuous variables like the flow of vehicles and the density of vehicles over time and space without giving emphasis in the trajectories of vehicles. Continuum models are rigorous mathematical models and in this lecture our emphasis will be more on the physical characteristics. Advanced readers can pay more attention in the class material for additional mathematical details. Let's give some intuition behind the continuum models. So let's look here. This is a simple example of a traffic light where we see different trajectories of vehicles, so our interest is not to model the exact movement of every vehicle but to create an approximation in an abstraction that looks at aggregated quantities like flow and density. So interestingly what we see in this figure is that vehicles are approaching the traffic light, because the traffic light is red they have to stop. They wait for some seconds and then they accelerate and they move again when the traffic light turns green.

Notes

Summary

0m 05s

Trajectories of vehicles at a traffic light



But interestingly, vehicles do not move all simultaneously, so what we could see that vehicles follow some line when they stop that has a specific speed and another line when they start. So this is some of the basic characteristics of continuum models and we will describe and understand this in the lecture in more details.

Notes

Summary

1m 29s

Differential form of conservation law

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0$$

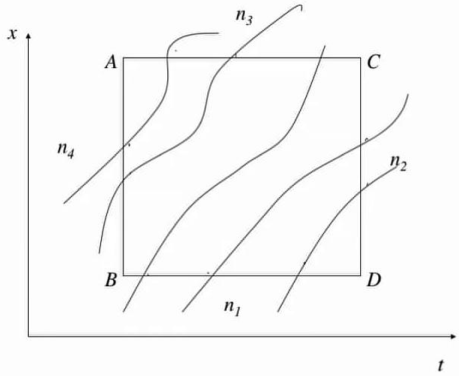
A simple proof

$$n_1 + n_4 = n_3 + n_2$$

$$q_{BD} * \Delta T + k_{AB} * \Delta L = q_{AC} * \Delta T + k_{CD} * \Delta L$$

$$0 = (q_{AC} - q_{BD}) * \Delta T + (k_{CD} - k_{AB}) * \Delta L$$

$$\frac{\Delta k}{\Delta T} + \frac{\Delta q}{\Delta L} = 0$$



The most important principle of continuous models is that they have a conservation law which simply says that flow cannot be lost and there is a mathematical way to describe that as we see in this equation which is a partial differential equation that simply says that the partial derivative of density with respect to time plus the partial derivative of flow with respect to space is equal to zero. So while at the moment this might look non-intuitive to you, we can understand this with a very simple example. So let's look here on the right where we can see a time-space diagram with the trajectories of some vehicles. So this differential form for conservation law, it simply says that the total number of vehicles that are entering this rectangle is equal to the total number of vehicles that are exiting the rectangle. So simply speaking, ‘n1’ plus ‘n4’ is equal to ‘n2’ plus ‘n3’, so one two one two three one two three one two, so two plus three is equal to three plus two. So let's see what this means mathematically.

Notes

Summary

1m 56s

Differential form of conservation law

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A simple proof

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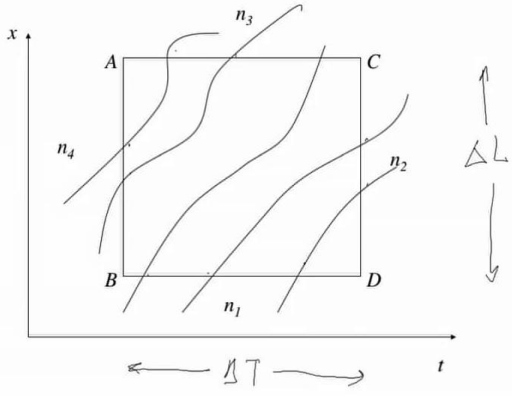
$$\frac{\Delta k}{\Delta T} + \frac{\Delta q}{\Delta L} = 0$$



So ‘n1’ which is what a stationary observer measures, can be expressed as the average flow between point B and D which is ‘qBD’ multiplied by the duration of time in this rectangle and ‘n4’ in a similar way is what an aerial photograph measures which is mainly the average density between point A and B multiplied by the distance between this point which is delta L and similarly ‘n3’ is the flow between A and C times delta T while ‘n2’ is the density between points C and D times delta L, so if we make some simple algebra over here, we come up with this equation and if we define as ‘qAC’ minus ‘qBD’ as the change in flow while ‘kCD’ minus ‘kAB’ the change in density, we come up with this equation. So if we take deltas to be extremely small then from here we can easily consider the proof for this partial differential equation.

Notes

Summary

3m 10s

Shockwaves and conservation law

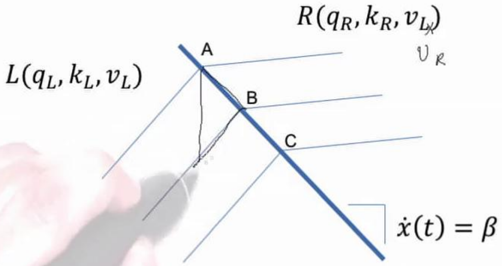
- A shock wave is a drastic change in traffic density that propagates through a traffic stream.
- Shocks only happen for increasing density profile $k_R > k_L$ (unique solution).
- Instantaneous speed adjustment (infinite acceleration)
- $\dot{x}(t) = \frac{q_R - q_L}{k_R - k_L}$ (Rankine-Hugoniot condition)

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \quad (1)$$

A simple proof of R-H:

$$(\beta + v_L) * t = s_L$$

$$(\beta + v_R) * t = s_R$$



So the second important principle of continuum models is the existence of shockwaves. Shockwave is a drastic change in traffic density which actually propagates through a traffic stream and an interesting characteristic of a shockwave is that they only happen for increasing density profile which is like the one described in the figure on the right. So what we have, we have some vehicles that are moving with a high speed ‘vL’ and for some reason these vehicles have to slow down in a slower ‘v’ speed which is ‘vR’, so there's a typo here. So this shockwave, actually it shows that there is an instantaneous speed adjustment between this traffic stream that changes speed from ‘vL’ to ‘vR’ and actually this happens with infinite acceleration. So in reality vehicles will have some time when they move from one speed to the other, so it might look like this, a trajectory but by making this simpler approximation, mathematically it will make our life much easier. So a very interesting condition is actually that we can estimate what is the speed of this shockwave and it is this line that separates the two different traffic streams and to do that we have to look at this time-space diagram and I would like from you to consider this specific triangle.

Notes

Summary

4m 28s

Shockwaves and conservation law

- A shock wave is a drastic change in traffic density that propagates through a traffic stream.
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- $\dot{x}(t) = \frac{q_R - q_L}{k_R - k_L}$ (Rankine-Hugoniot condition)

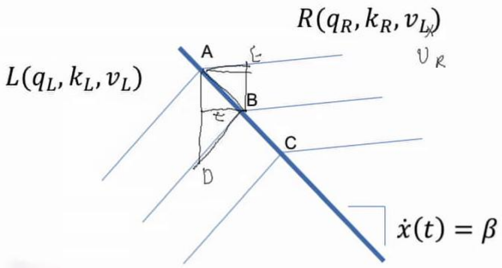
$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \quad (1)$$

A simple proof of R-H:

$(\beta + v_L) * t = s_L = \frac{1}{k_L}$

$(\beta + v_R) * t = s_R = \frac{1}{k_R}$

$k_L \cdot \beta + q_L = k_R \cdot \beta + q_R$



So in this specific triangle, let's call this point D, I would like to ask you what is ‘AD’. So ‘AD’ if you recall from the first week of this mooc, is the vertical distance between the trajectories of vehicles, so it is the spacing ‘SL’ while in the same way of thinking if we look at this triangle and we call this point E, the vertical distance ‘EB’ is the spacing between the trajectories on the right hand side of this figure. So if we also define this time ‘t’ and because ‘AD’ and ‘EB’ are parallel, this ‘t’ is equal also to that distance so this is another ‘t’. Then what we have, we have that the distance between A and D which is the spacing ‘SL’, is equal to this part plus that part but as we know what is the speed of the slope of these triangles, we can write the equation that beta which is the slope of the shockwave plus ‘vL’ which is the slope of the speed of these vehicles times ‘t’ is equal to ‘SL’. And in the same way of thinking we can write the same equation for the right hand side of the figure. So if we also consider that ‘SL’ is one over density and ‘SR’ is again one over density ‘KR’, we can do some manipulations in this equations by solving both equations for ‘t’ and what we will have, we will have that ‘KL’ times beta plus ‘QL’ is equal to ‘KR’ times beta plus ‘QR’.

Notes

Summary

6m 18s

Shockwaves and conservation law

- A shock wave is a drastic change in traffic density that propagates through a traffic stream.

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \quad (1)$$

- Shocks only happen for increasing density profile $k_R > k_L$ (unique solution).
- Instantaneous speed adjustment (infinite acceleration)
- $\dot{x}(t) = \frac{q_R - q_L}{k_R - k_L}$ (Rankine-Hugoniot condition)

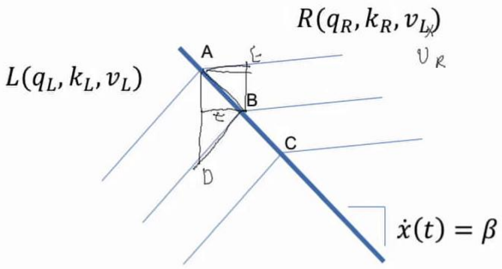
A simple proof of R-H:

$(\beta + v_L) * t = \underline{s_L} = \frac{1}{k_L}$

$(\beta + v_R) * t = \underline{s_R} = \frac{1}{k_R}$

$k_L \cdot \beta + q_L = k_R \cdot \beta + q_R$

$\beta =$



If we make some calculations over here and we solve for beta, beta is equal to the equation we see here. So this is a known condition as Rankine-Hugoniot and this is another important characteristic that we need to build this continuous models.

Notes

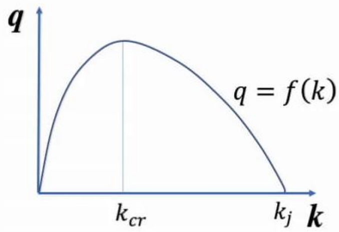
Summary

8m 48s

Basics of LWR Theory

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \quad (1)$$

$$\dot{x}(t) = \frac{q_R - q_L}{k_R - k_L} \quad (2)$$



$$q = f(k) \quad (3)$$

$$\frac{\partial k(x,t)}{\partial t} + f'(k)\frac{\partial k(x,t)}{\partial x} = 0 \quad (4)$$

$$k_t + c(k)k_x = 0 \quad (4')$$

$c(k)$:characteristic speed

What we have said so far is not only related to traffic but many dynamic systems in the physical environment are also governed by these equations, so for example, the propagation of waves if you throw a stone inside the water. So to define a traffic theory, we need one additional characteristic. So now we call this LWR theory where L, W and R are the initials of the authors that came up with this idea, we need a third characteristic, a third principle and this is the fundamental diagram that we see over here. So fundamental diagram is a steady state relationship between flow and density and if we assume that this is a unimodal concave curve with a value of 'k critical' where flow is maximized and also a value 'k jam' for which flow is equal to zero then by using all these three principles, we can come up with the description of the LWR theory. So simply speaking, if we replace in equation one 'q' with 'f of k' and we apply the simple chain rule of derivatives, we can come up with this equation. Then if we write it in a different form so 'kt' is simply speaking the partial derivative of density with respect to time, 'kx' is the partial derivative of density with respect to space and 'c of k' is the first derivative of the fundamental diagram.

Notes

Summary

9m 14s

Basics of LWR Theory

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \quad (1)$$

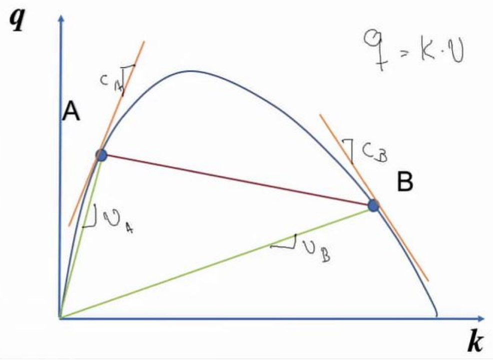
$$\dot{x}(t) = \frac{q_R - q_L}{k_R - k_L} \quad (2)$$

$$q = f(k) \quad (3)$$

$$\frac{\partial k(x,t)}{\partial t} + f'(k)\frac{\partial k(x,t)}{\partial x} = 0 \quad (4) \qquad k_t + c(k)k_x = 0 \quad (4')$$

Lemma: An observer that travels at speed $c(k)$ sees no change in density

Proof: $\frac{dk(x,t)}{dt} = k_t + k_x \frac{dx}{dt} = k_t + k_x c(k) = 0.$



$c(k)$:characteristic speed

Simply speaking, this first derivative of the fundamental diagram for a specific point 'k' is the orange line that we see over here and this is called the characteristic speed or kinematic wave. And another point B has another characteristic speed with slope 'CB'. So an interesting lemma is that if an observer travels at the speed of the characteristic then he sees no change in the density profile and the proof actually is quite simple. So let's take the total derivative of the density with respect to time, so this total derivative as density is both a function of time and space, is the partial derivative with respect to time plus a partial derivative with respect to space multiplied by 'dx' over 'dt'. But 'dx' over 'dt' is the speed of this observer which is 'c of k'. Look here, this is the same with equation 4 prime. So this is a straightforward proof. What else do we see in this figure, so the orange lines are the speed of the characteristics, the green lines are simply speaking, the speed of the traffic stream, so this is the speed B and this is the speed A and please recall that the reason that this happens is because we have the fundamental relationship between flow, density and speed and what is the red line?

Notes

Summary

11m 04s

Basics of LWR Theory

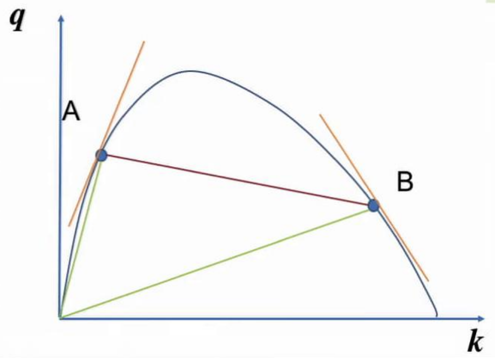
$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \quad (1)$$

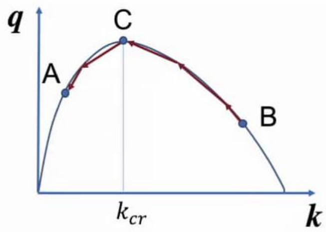
$$k_t + c(k)k_x = 0 \quad (4')$$

$$\dot{x}(t) = \frac{q_R - q_L}{k_R - k_L} \quad (2)$$

Entropy condition: If there is a decreasing density profile, there are multiple solutions. Entropy condition helps to choose a single solution. Flow is maximized for a given density, meaning that a path following FD states is chosen.

$$q = f(k) \quad (3)$$





The red line is the speed of the shockwave that we defined before. So graphically, it means that the speed of the shockwave is the slope of the line in a fundamental diagram that connects the two different states, so between A and B are R and L as we had in the previous slide. So I would like to define one more additional characteristic that this will complete the description of the LWR theory and this is the entropy condition. What we said before is that if there is an increasing density profile, we have a unique solution which is expressed by a shockwave but this is not the case if we have a decreasing density profile, so if density decreases with time downstream of a point. So in that case we need a condition which is called entropy condition, to choose a single solution and while this is an interesting mathematical property, I would like only to give some physical intuition here. So this entropy condition helps us to find a solution which is consistent with the physics of traffic and the idea there is that flow is maximized for a given density, meaning that if we move from state B to state A, we have to follow a path in the fundamental diagram which goes through all the states between B and A.

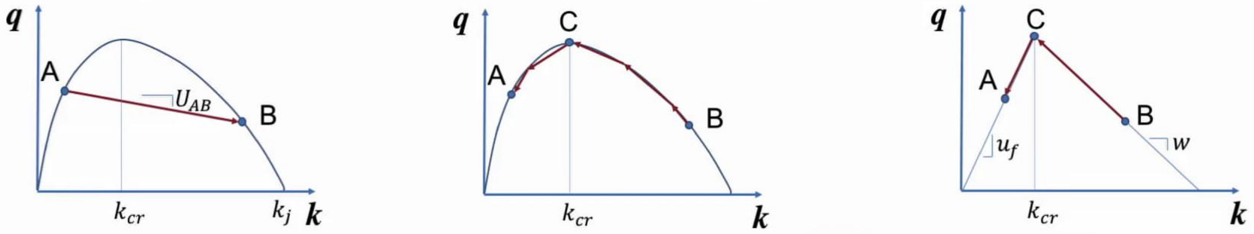
Notes

Summary

12m 50s

Principles of LWR theory

- All possible states that can be observed in a road segment should belong in a FD (stationary conditions)
- When a traffic stream is moving from A→B with $k_A < k_B$ it follows a linear interface between A and B in the FD
- When a traffic stream is moving from B→A with $k_A < k_B$ follows a path **on the FD** (entropy conditions)
- If additionally $k_A < k_{cr} < k_B$ then the path passes through capacity state C (queues at active bottlenecks discharge at capacity)
- The speed of a shockwave between two states is $U_{AB} = \frac{q_B - q_A}{k_B - k_A}$
- If FD is triangular, then from B→A, there are only characteristics with speeds u_f and/or w



I would like to summarize here simple principles of LWR theory that even if we put on the side the mathematical details, we can utilize them to solve traffic problems when conditions change over time and space. So the first principle is that all possible states that can be observed in a road segment belong in a fundamental diagram between flow and density and these stationary conditions hold also in the dynamic case. So when we have a traffic stream which is moving from lower density to higher density profile, it will follow a shockwave which is given by the equation we described before. Well now, we have the other way around, a situation where we move from higher density to lower density, so we need additional the entropy condition and what we learn is that we have to move on the state that belong in the fundamental diagram. I would like now to introduce a very common fundamental diagram which actually is very reasonable from an empirical point of view so it has been observed with empirical data but also make the derivations of LWR theory much simpler and this is a triangular fundamental diagram where can be expressed by three parameters.

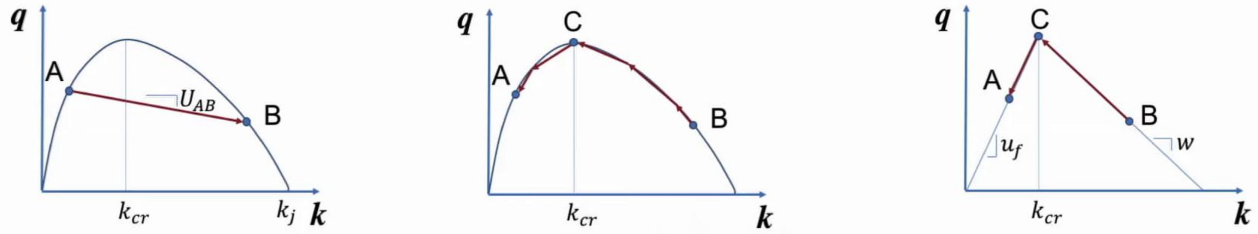
Notes

Summary

14m 26s

Principles of LWR theory

- All possible states that can be observed in a road segment should belong in a FD (stationary conditions)
- When a traffic stream is moving from A→B with $k_A < k_B$ it follows a linear interface between A and B in the FD
- When a traffic stream is moving from B→A with $k_A < k_B$ follows a path **on the FD** (entropy conditions)
- If additionally $k_A < k_{cr} < k_B$ then the path passes through capacity state C (queues at active bottlenecks discharge at capacity)
- The speed of a shockwave between two states is $U_{AB} = \frac{q_B - q_A}{k_B - k_A}$
- If FD is triangular, then from B→A, there are only characteristics with speeds u_f and/or w

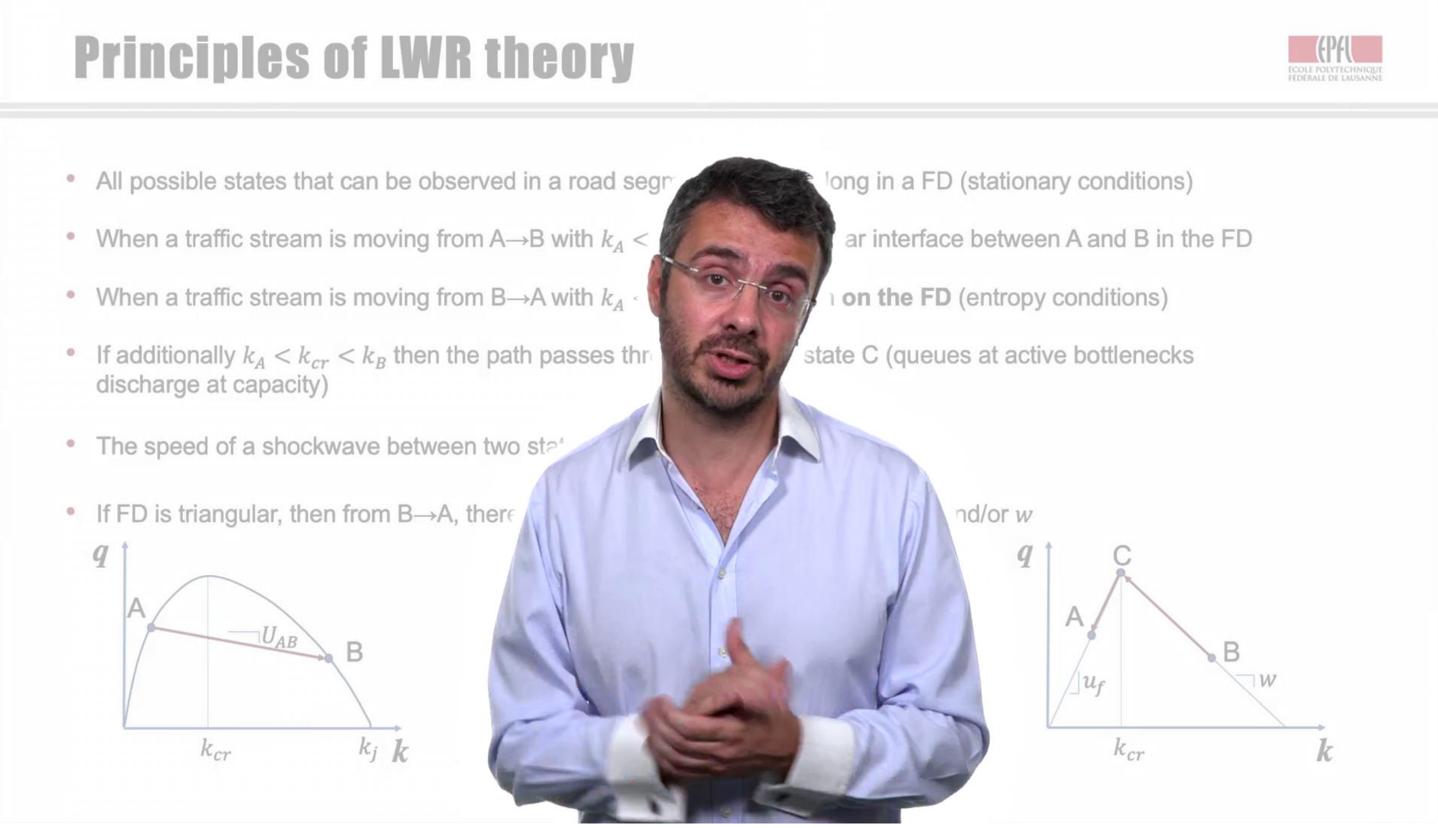


So this is the free flow speed which is the flow the slope of the increasing part, is the critical value of the density for which capacity is maximized and then is speed ‘w’ which is the maximum shockwave speed which mainly connects the jump state with a capacity state. And interestingly what we know, we know that if a fundamental diagram is triangular then when we move from B to A, so from high density to low density, there are only characteristics that have speed either free flow and or the maximum wave speed ‘w’.

Notes

Summary

15m 54s



So what we have learned so far is that continuum models contain a number of mathematical and physical principles and can be utilized to describe the evolution of congestion over time and space. What we do now in the next lecture is that we will see some concrete examples and how can we apply this theory so to obtain specific characteristics of traffic congestion.

Notes

Summary

16m 38s