Static Route Choice (congestion games)

Intro to traffic flow modeling and ITS

Raphaël Lamotte

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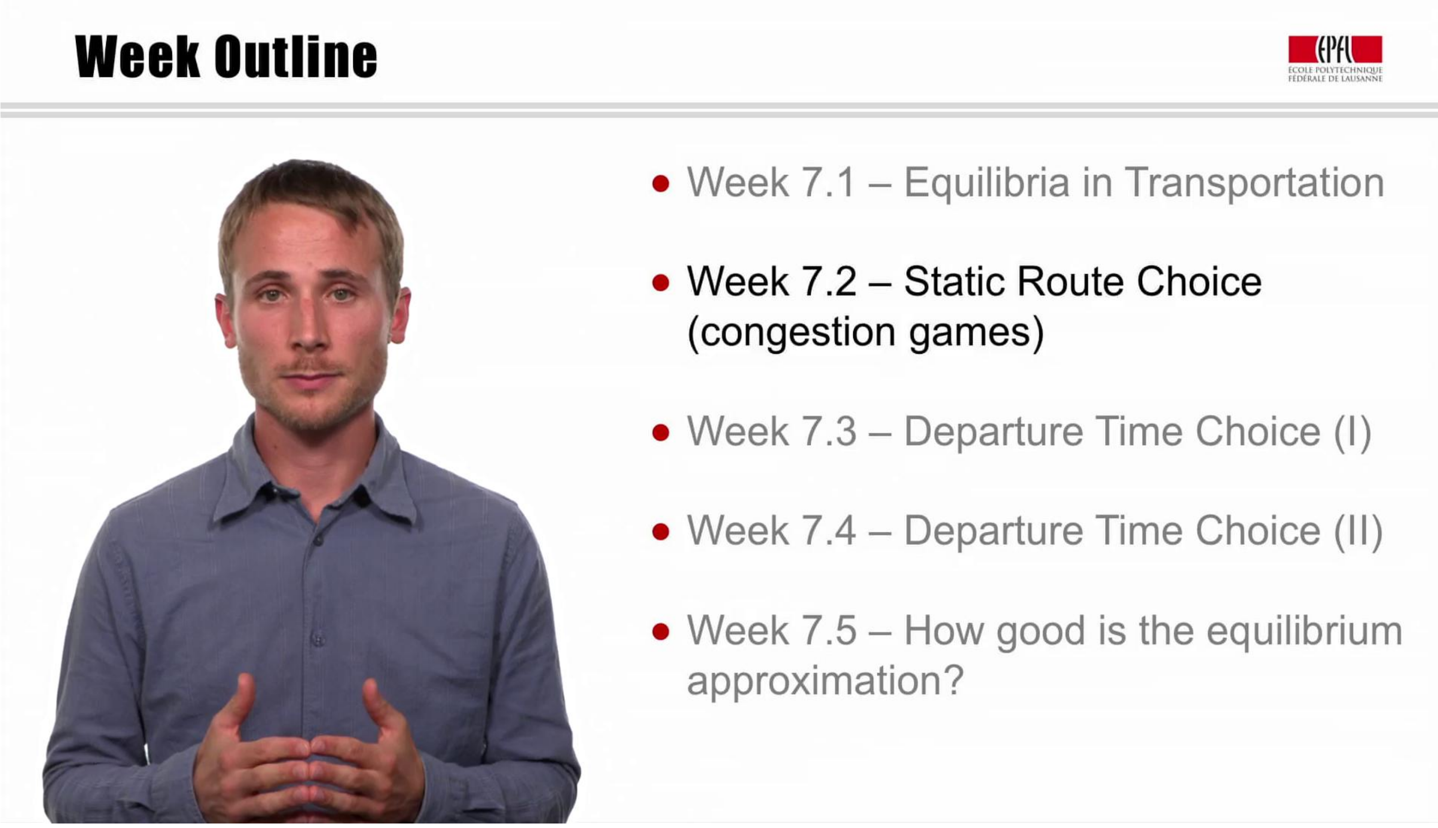
Video



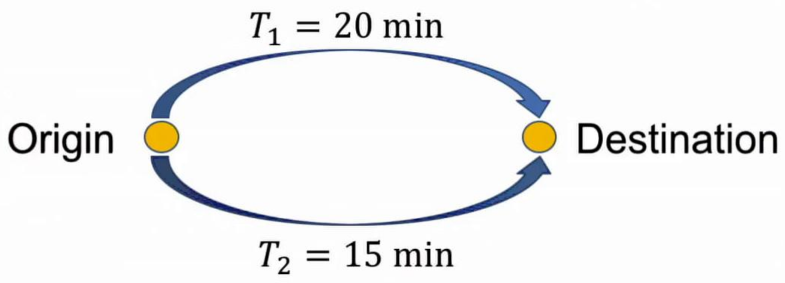
Hello, in this video explain how to find equilibria in simple road choice problems. The road choice problem consists in finding the routes that people use. Even that we know all the origins, all the destinations, and we have mathematical models that describe congestion in the network. In this video, we will only consider simple road choice problems that do not depend on time of day, and that have rather simple congestion mechanisms.

Notes

Summary

0m 05s

2-route example: uncapacitated network



- UE: $q_1 = 0$; $q_2 = 1$
- SO: $q_1 = 0$; $q_2 = 1$

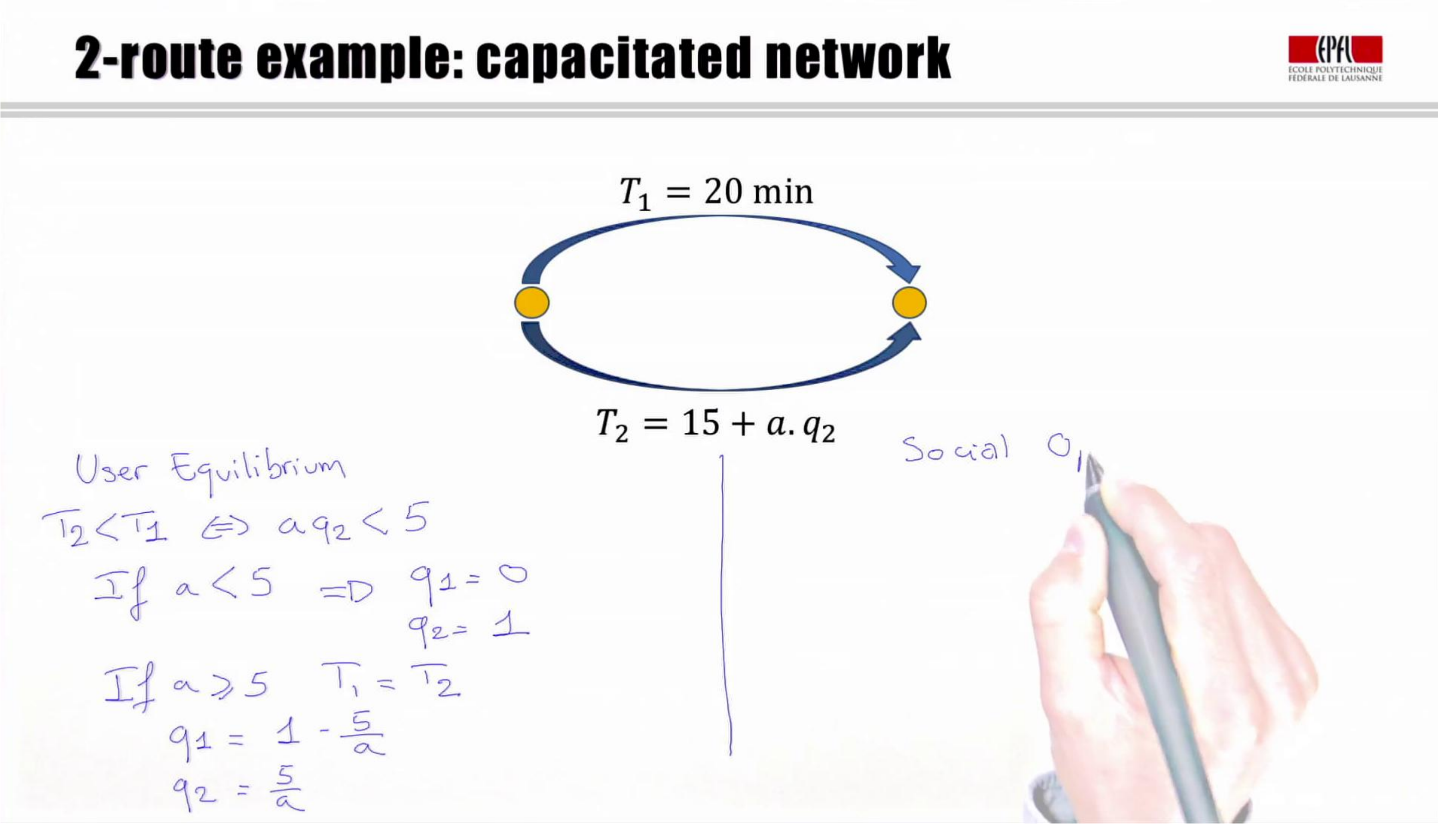
In the uncapacitated network, UE = SO

Let's start with a simple example with a unique origin, unique destination and two routes. Each route had the travel time that is indicated here and incapacitated in the sense that the traffic volume on each of these two routes does not influence the cost of the travel time. So in this context, people's cost is only influenced by the decision to take which route. It is not influenced by the decision of others. And as a consequence, maximizing their own utility is the same as maximizing welfare. So in this specific case, the user equilibrium and the social optimum are the same. And here are the users should use the route which has a smallest cost.

Notes

Summary

0m 32s



So route two let's now consider a case that is a bit more complicated. So we again have two routes, but the cost of one of these two routes depend on the flow on that route. So here depends on the flow via this proportionality coefficient A, that we assume is positive. So in this case, and what is the condition to have user equilibrium? It is simply that either all the users take route two and the cost route two is smaller than route one. So when can this happen, we need to have T2 smaller than T1, which is equivalent to having AQ2 smaller than five. So if,A is smaller than five, even if we have the whole population on route two the population is assumed to have size one here, the cost on Route two is still smaller than the cost on Route one. So in this case, Q1 is equal to 0, the flow on route one and the flow on Route 2 is equal to 1. But if we have A, larger than 5, then equilibrium imposes that the cost and the two routes should be equal. So we have T1 equal T2, which means that q1, equal [1-5] of A, and Q2 equal 5 over A. So let's not consider the social optimum.

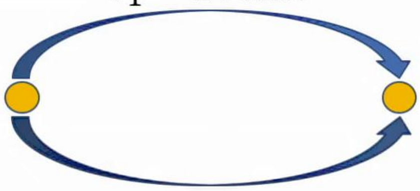
Notes

Summary

1m 17s

2-route example: capacitated network

$T_1 = 20$ min



$T_2 = 15 + a.q_2$

User Equilibrium

$T_2 < T_1 \iff a q_2 < 5$

If $a < 5 \Rightarrow q_1 = 0$, $q_2 = 1$

If $a \geq 5$ $T_1 = T_2$

$q_1 = 1 - \frac{5}{a}$

$q_2 = \frac{5}{a}$

Social Optimum

$TT = 20(1-q_2) + (15 + a q_2) q_2$

$\frac{dTT}{dq_2} = 0 \iff -20 + 15 + 2 a q_2 = 0$

$\iff q_2 = \frac{5}{2a}$

If $a < \frac{5}{2}$ then $q_2^* = 1$

$q_1^* = 0$

Else $q_2^* = \frac{5}{2a}$ $q_1^* = 1 - \frac{5}{2a}$

At the social optimum, what we want is to minimize the total travel time, the total travel time here is given by the flow, which Route 1 time the travel time and what one plus flown over to time to travel time or number two, so it's given by [21-Q2+15+AQ2] this is simply second order polynomial in Q2. So to maximize this, what we should do is simply we should try to find the Q2 search that the derivative of travel time, total travel time with respect to Q2 is equal to zero. That's equivalent to having -20+15+2AQ2= 0. So we have Q2= 5 over 2A, that however, that this is not always possible, because Q2 has to be between 0 and 1 it cannot be larger than the whole population. So if A is smaller than 5 over 2, then the social optimum is to have Q2=1, and Q1=0. On else, we have what we found here. So Q2 optimum = 5 over 2A]and Q1 optimum equal 1-5over 2A.

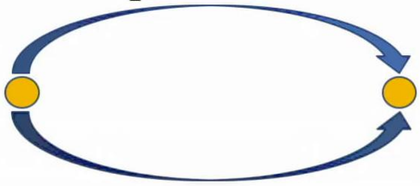
Notes

Summary

3m 09s

2-route example: capacitated network

$T_1 = 20$ min



$T_2 = 15 + a.q_2$

What is the difference in user behavior?

- UE: Choose Link2 if $T_2 - T_1 \leq 0$, i.e. if $15 + a\,q_2 - 20 \leq 0$.
- SO: Choose Link2 if $\frac{d\mathrm{TT}}{dq_2} \leq 0$, where $\mathrm{TT} = \mathrm{q}_2(15 + \mathrm{a}\,\mathrm{q}_2) + (1 - \mathrm{q}_2)20$

$\frac{d\mathrm{TT}}{dq_2} \leq 0 \Leftrightarrow 15 + 2aq_2 - 20 \leq 0$

- **Externalities**: influence of one's decision on others' utility.

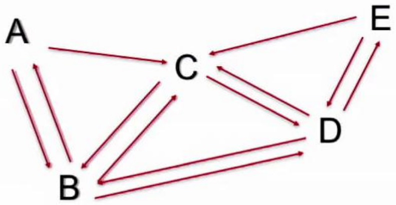
So what is the difference in user behavior between the user equilibrium and the social optimum at the user equilibrium, users choice lane 2 if it has a shorter travel time, that is to say if 15+AQ2 is smaller than 20, at the social optimum, users choose link two if when more people go a link to it decreases the total travel time. So that's to say that we have this derivative that would be negative. If we look at the conditions in the end, the only difference is this factor 2, why it's because when a user chooses to go a link two, he has an influence on all the users because he will slow down the Q2 users that travel on this link by A. So that's the why we have one more AQ2 in this difference. This AQ2 is called an externality, it's an influence that people have and others here it's a negative externality because it decreases utilities.

Notes

Summary

4m 53s

Static route choice in general networks



- <u>Notations</u>

N: set of nodes
A: set of arcs
q_{od}: demand for OD pair $o \to d$, $(o, d \in N)$
R_{od}: set of routes connecting pair $o \to d$
f_r^{od}: flow on the route $r \in R_{od}$
$\delta_{r,a}^{od}$: binary variable equal to 1 iff arc a belongs to route r between o and d.

- <u>Arc cost:</u> $c_a(x_a)$
- <u>Feasible route flows:</u>

$$\left\{ \left(q_r^{od} \right)_{r \in R_{od}} \in \mathbb{R}^+, \sum_{r \in R_{od}} q_r^{od} = q_{od} \right\} \quad \textbf{(1)}$$

- <u>Arc flow:</u>

$$x_a = \sum_{o \in N} \sum_{d \in N} \sum_{r \in R_{od}} q_r^{od} \, \delta_{r,a}^{od} \quad \textbf{(2)}$$

- <u>Route cost:</u> $\forall o, d \in N, \forall r \in R_{od}$,

$$C_r^{od} = \sum_{a \in A} \delta_{r,a}^{od} c_a(x_a)$$

- <u>Equilibrium:</u> $\forall o, d \in N, \forall r, s \in R_{od}$,

$$q_r^{od} > 0 \Rightarrow C_r^{od} \leq C_s^{od}$$

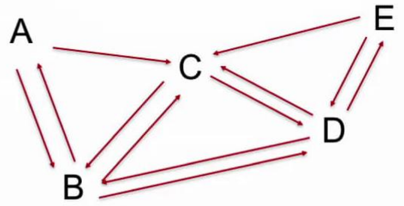
So let's know see how we can find the user equilibrium. In more general networks, these networks are made of nodes, here we have a set of nodes N, and of arcs that are directed connecting path of nodes, the demand is specified for pairs of nodes as well and denoted, Q. The costs are specified for arcs. So, the cost and they are connecting B to D is a function of the flow on this same arc B to D. So, here we have a program that is specified in terms of demand between origin and destination, and we want to find the equilibrium with cost that are specified per arc. So the method to do this is to solve an equivalent optimization problem that works with the Arc flows. To do this, we need to specify the set of UK flows that are feasible given this demand. So to do this, and we will use route flows, the set of physical workflows is a set of roots flows, which are positive and such that the sum of root flows from about from an Node O to a node D, all roots should be equal to that demand from that same Node O to that same Node D. And then the arc flows expressed at the function of the road flows just by summing of all the origin of all the destination and of all the routes the flows.

Notes

Summary

5m 54s

Static route choice in general networks



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- Route cost: $\forall o, d \in N, \forall r \in R_{od}$,

$$C_r^{od} = \sum_{a \in A} \delta_{r,a}^{od} c_a(x_a)$$

- Equilibrium: $\forall o, d \in N, \forall r, s \in R_{od}$,

$$q_r^{od} > 0 \Rightarrow C_r^{od} \leq C_s^{od}$$

And we have this additional very variable Delta that is equal to one if and only if the arc A belongs to the root are connecting origin O, to destination D. Ok. So now what is an equilibrium, it's a situation such that users cannot reduce their costs by attending decision. So here we express this in terms of root cost. So the root cost is simply the sum of the arc costs, multiplied by the same identifiable data that indicates whether this arc belongs to that route. And then equilibrium condition is simply that if the route as a positive flow, then its cost CR should not be larger than the cost of any other route connecting the same origin to the same destination.

Notes

Summary

7m 30s

Static route choice in general networks

- Congestion games:
  - $c_a = c_a(x_a)$
  - $x_a \rightarrow c_a(x_a)$: positive and monotonically increasing.
- Beckmann's transformation for congestion games,

$$\boxed{x \text{ is a UE} \Leftrightarrow x \in \arg\min\{\textstyle\sum_{a\in A} \int_0^{x_a} c_a(u)\mathrm{d}u \text{, subject to (1) and (2)}\}}$$

$\Rightarrow$ Existence of a UE,

$\Rightarrow$ Uniqueness if c_a is strictly increasing around the equilibrium.

- In comparison

$$\boxed{x \text{ is a SO} \Leftrightarrow x \in \arg\min\{\textstyle\sum_{a\in A} x_a c_a(x_a) \text{, subject to (1) and (2)}\}}$$

In order to find a good optimization problem that is equivalent to finding the user experience, we need to have two things first, that the cost function of each arc should depend only on the flow and that arc, and second, that this function should be positive and monotonically increasing. If we have this, then finding the user equilibrium is equivalent to solving a complex optimism problem. That is to minimize the sum of all the arcs, this integral from 0 to the flow on that arc of the cost function, subject to the constraints 1 and 2 that we saw in the previous slide. That's called Becksman transformation. And why this is convex, simply because if this function is increasing, then this integral is a convex function. And the sum of convex functions is again convex. It is well known that for any complex function that we want to minimize that exist the minimum, so we have existence of a user equilibrium. And in order to obtain uniqueness, we must ensure that this cause function CA is not only positive and monitor increasing, but it's also strictly increasing around the equilibrium. To prove that minimizing this expression is equivalent to solving the user equilibrium is a bit technical.

Notes

Summary

8m 26s

Static route choice in general networks

- Congestion games:
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- In comparison

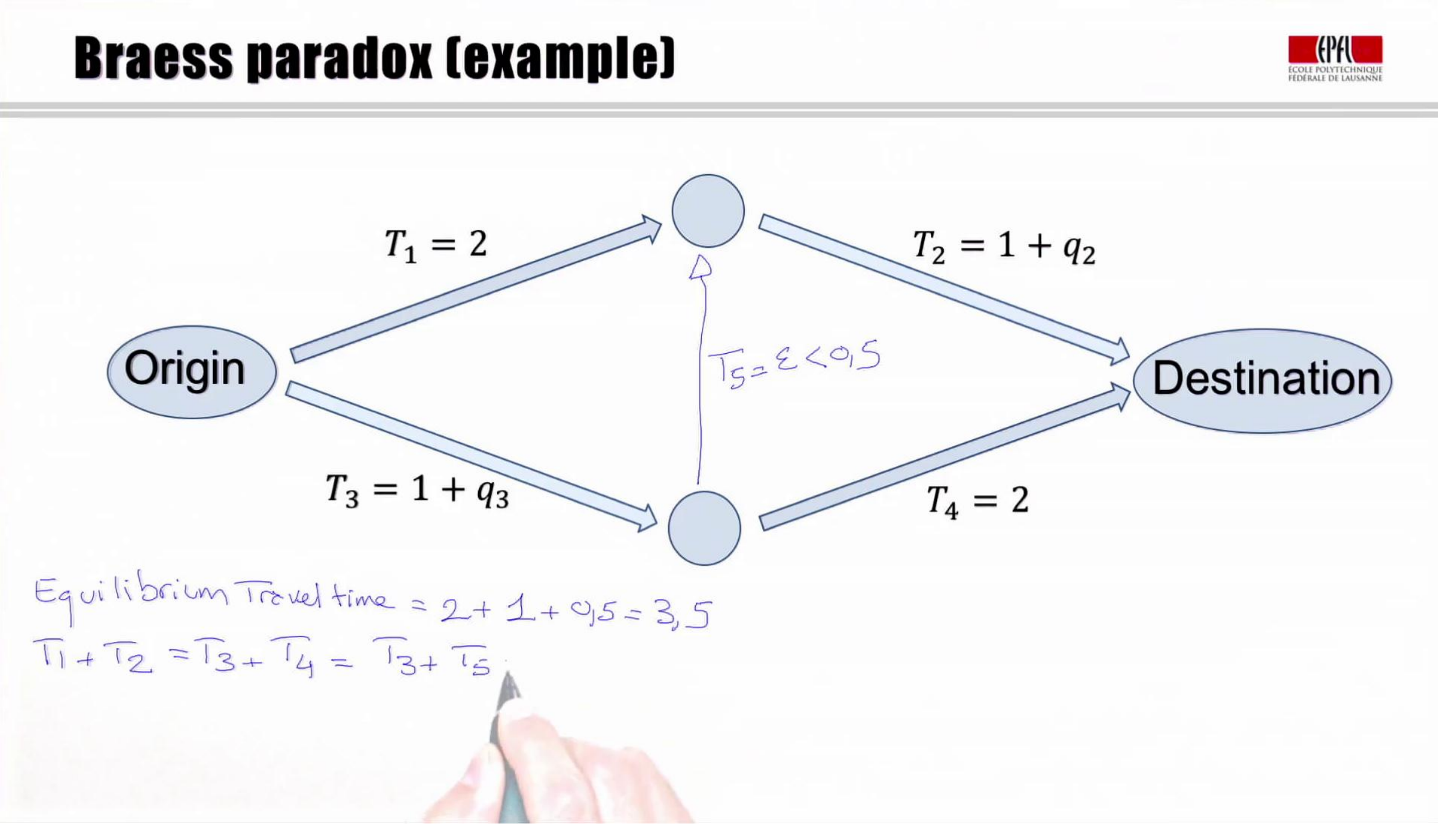
$$\boxed{x \text{ is a SO} \Leftrightarrow x \in \arg\min\left\{\sum_{a\in A} x_a c_a(x_a) \text{, subject to (1) and (2)}\right\}}$$

So I do not explain it here. But if you want, you can verify and the previous word choice examples, in this case is at least minimizing this expression, let's indeed to the user equilibrium. In comparison, the social optimum is much more straightforward, because it is already formulated as an optimization problem, we just have to minimize the total condition cost for society, which is this, the sum of the other arc A of the cost under arc A, time the flow under Arc A. And again, the state X, should satisfy the constraints 1 and 2 of the previous slide so that it is consistent with the demand.

Notes

Summary

9m 54s

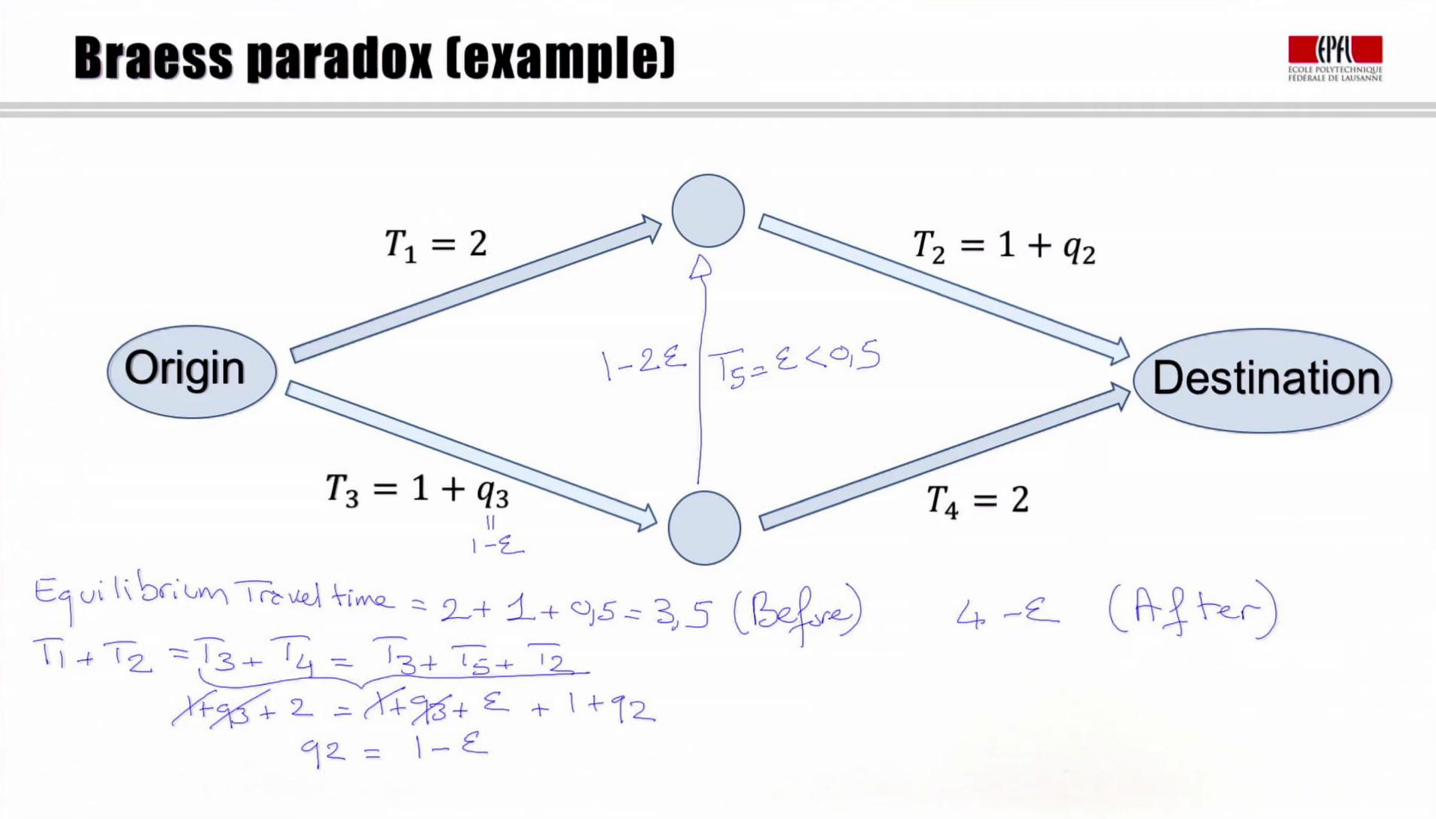


For now that we have the tools to find equilibrium, let's see an interesting example that shows all the importance of externalities, we have two routes, and they are symmetric, they both have one arc with the constant travel time and one arc with the travel time that depends on the demand. So since this case is clearly symmetric, the equilibrium is quite simple, we should 50% of users on each route. So it gives us an equilibrium travel time of 3.5. Let us now consider that we add another route that goes like this, and that has a very small travel time smaller than 0.5. Clearly, if we keep the previous school choice, that we have 0.5 and a route and 0.5 on the lower route, then the optimal route is to do this. Because the travel time on that route is 1.5 + epsilon + 1.5, which is less than 3.5. However, we cannot have all the demand on that route, because then the travel time would be 2+2+epsilon, which is larger than 2+2. A similar reasoning shows that we cannot have only of the three rules that are used, so all the rules should be used. As a consequence, the travel time should be equal, so that we have T1+ T2, the travel time and that road should be equal to T3 +T4 the travel time on that route should be equal to T3+T5+T2.

Notes

Summary

10m 35s



And let's focus only on one of this to the equation. So maybe there's just this one. What do we have, it's simply 1+Q3+2= 1+Q3+epsilon+1+ Q2 So this term, simplify. And we have that Q2 =1- epsilon. With the same reason, and we can find that Q3=1-epsilon. And then, since we must have conservation of flow at every link, we find that link five, we have a flow of 1-2epsilon. So what is the new equilibrium travel time? Now, it's 2+1+1- epsilon, that's 4-epsilon. So that was before. And that's after. So we see that the new equilibrium travel time is larger than the previous one. This is surprising, we have added more capacity to the network, and the outcome is not as good. Why? It's simply because the new arc that we have added allow users to use shorter links with larger externalities. They chose this so that they shorten their own travel time, but by doing so, they'll actually increase the travel time of others. That's an example of a road here that should not be constructed. If it exists, it should be modified in such a way that it can be used to take this problematic route.

Notes

Summary

12m 23s

Summary

To summarize, we have seen that the classic route choice problem belongs to the general class of contextual games, and we have introduced a methodology to solve such games. We have highlighted the difference between the user equilibrium and the social optimum and what it means in terms of user behavior and externalities. We have shown that some network configurations are better than others, in the sense that it the equilibrium is naturally closer to the optimum. In the next video, we will see a more difficult type of equilibrium where people can choose the departure time. Thank you.

Notes

Summary

14m 11s