

9-2. Rotating frame revisited

Equation of motion for \vec{M} (always valid in any reference frame) in presence of B_0

$$\frac{d\vec{M}}{dt} = -\gamma \Delta \vec{B}^{\text{eff}} \times \vec{M}$$

magnetization **precesses** in xy plane with frequency $\gamma \Delta B^{\text{eff}}/2\pi$

Rotating frame: reference frame rotating about z at frequency ω_{RF}

Case I: non-rotating reference frame ($\omega_{\text{RF}}=0$)

\Rightarrow magnetization **precesses** in xy plane with frequency $\gamma B_0/2\pi$



So now I want to review the rotating frame description of the magnetization because this is really important that we have a good grasp of this description. So, the equation of motion for the magnetization has to be valid in any reference frame in the presence of a magnetic field is given by the Larmor equation which states that the magnetization precesses around ΔB effective with the frequency $\gamma \Delta B$ effective. If the B effective is along z, then this means that the magnetization will precess in the xy plane with a frequency that's given by $\gamma \Delta B$ effective over 2π . Now, let's re-look at the rotating reference frame. So we have a reference frame that rotates around z, about z, with a frame frequency ω_{RF} . Let's take the first case: The rotating reference frame does not rotate. That's the lab frame; that's what-- our reference frame. So, ω_{RF} is zero. In this case, the magnetization precesses in the xy plane with a frequency γB_0 over 2π . So, if you take the reference frame here, we have the $B=0$ here; we have our magnetization here.

Notes

Summary



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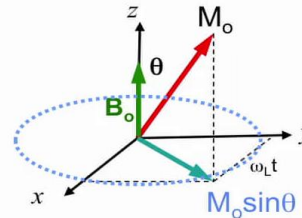
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Case II: rotating frame with $\omega_{\text{RF}} = \omega_L$

\Rightarrow magnetization is **stationary** ("precesses" in xy with **zero** frequency)

Equation of motion is still valid, i.e. precession frequency $\gamma \Delta B^{\text{eff}}/2\pi$

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We're just putting the magnetization at a certain angle, so we're just looking at the component that's transverse. That's effect -- this component here, the component that's perpendicular to B_0 does not precess. So this magnetization here, the transverse magnetization will precess with an angular velocity ω_L . So it will -- this angle will increase with $\omega_L t$, it goes around B_0 , encircles the transverse component. So if we take the transverse component, then the transverse component will simply do this motion. That's the Larmor equation in the non-rotating reference frame. Now we'll take the second case, and that is the rotating reference frame. And now we'll place ourselves into the rotating reference frame that rotates exactly with the Larmor frequency. In this rotating frame, as we are rotating exactly with the magnetization, this magnetization is, by definition, stationary. It does not move, or in other words, it precesses in xy with zero frequency. Now, the equation of motion that we have up here still has to be valid in this reference frame. So, the precession frequency is equal to $\gamma \Delta B^{\text{eff}}$ over 2π , or in this case, the ΔB^{eff} will be zero.

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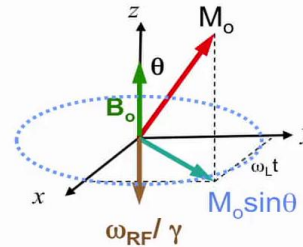
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Equation of motion is still valid, i.e. precession frequency $\gamma \Delta B^{\text{eff}}/2\pi$

$\Rightarrow \Delta B^{\text{eff}} = 0$



Larmor frequency Ω in the rotating frame:

$$\Omega = \gamma \Delta B^{\text{eff}}$$

$$\Delta B^{\text{eff}} = B_0 - \omega_{\text{RF}}/\gamma$$

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And our magnetization will do this motion, namely, none, because we are in the reference frame where it doesn't move. Now, the Larmor frequency in [inaudible], in the rotating reference frame is given by *gamma-delta-B-effective*, and this *gamma-delta-B* -- this *delta-B-effective* is given by *B-zero minus omega-R-F over gamma*. So what this means is we are to get from here to the lab frame, to the rotating reference frame that rotates with a Larmor frequency, what we have had to do to still have the equation work, we have to subtract a fictitious magnetic field which in this case equals to *omega-L over gamma* and *omega-L* is *gamma-B-zero*, so the *delta-B-effective* was *B-zero minus B-zero* equals *zero*. But this holds true for any *omega-R-F* for any angular velocity of a rotating reference frame, and we can calculate a *delta-B-effective*.

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Supplement: Rotating frame

What are the quantum-mechanical equivalencies ?

Schrödinger representation:

$$i\hbar \frac{d}{dt} |\psi_S(t)\rangle = H_S |\psi_S(t)\rangle$$

If $H_S = \text{const}$ in t:

$$|\psi_S(t)\rangle = e^{-iH_S t/\hbar}$$

NB.

$$\langle I_z \rangle \equiv \langle \psi_S(t) | I_z | \psi_S(t) \rangle$$



Now here, I want to make a short digression into quantum mechanics. Not what we strictly need quantum mechanics for understanding most of the MRI principles. We can actually work very much with a classical principle. But we want to remind ourselves that magnetic resonance is a quantum mechanical manifestation. It is a property, comes out of quantum mechanics. Quantum mechanics is the proper way of describing things, and what I want to do here is build a bridge between our description, classical description, and the quantum mechanical theory that's behind this. So let's look at the Schrödinger representation. We have the derivative of the cat is equals to the Hamiltonians times the cat, and if the Hamiltonian is constant in time, then we can write the solution very simply. The expectation value of a operator I_z is given by this term here in the Schrödinger representation. So this is all standard theory from quantum mechanics. Now, we have some equivalencies here, and that is the magnetization along z is the expectation value of the operator I_z , magnetization along x , that of I_x , and the magnetization along y , the expectation value of I_y . So in other words, our macroscopic quantities that we observe the magnetization are the expectation values of the quantum mechanical operators associated.

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Quantum mechanical equivalencies:

$$M_z \propto \langle I_z \rangle, M_x \propto \langle I_x \rangle, M_y \propto \langle I_y \rangle$$

For one spin-1/2 (^1H), i.e. two energy levels

$$I_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad I_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

How to determine $\langle I_x(t) \rangle$ etc ?

⇒ Split H_S into time-invariant and -dependent terms:

$$i\hbar \frac{d}{dt} |\psi_S(t)\rangle = [H_S^0 + V(t)] |\psi_S(t)\rangle$$

Interaction representation

(Higher order perturbation theory)

$$|\psi_I(t)\rangle \equiv e^{iH_S^0 t/\hbar} |\psi_S(t)\rangle$$

$$i\hbar \frac{d}{dt} |\psi_I(t)\rangle = V_I(t) |\psi_I(t)\rangle$$

$$V_I(t) = e^{iH_S^0 t/\hbar} V_S(t) e^{-iH_S^0 t/\hbar}$$

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So, this is very nice. So, for a spin one-half that's a hydrogen nucleus at the proton, we have two energy levels, so we have -- this is the operator for I_z , I_x , and I_y . These are the Pauli spin matrices that describe the system. Now, how do we determine the time dependence of I_x , that is of the m_x magnetization? And the way this is done here is we split the Hamiltonian into time-invariant and time-dependent terms. So we can write our Schrödinger equation here. We'll split the Hamiltonian, this term here, into time-invariant term here and a time-dependent term. And then we impose that the time-invariant term is much bigger. Its eigenvalues are much bigger than those of the time-dependent term and we'll use here now the interaction representation, which comes from higher order perturbation theory. And if we do that, the transformation that is being done that comes out of this theory is that we have now the cat in the interaction representation and we will now enter this condition the representation equation of motion for the wave function becomes this term here. And we have now the V of t is transferred to the V in the interaction representation.

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For spin:

$$H_S^0 = \hbar \gamma B_0 I_z$$

$$V(t) = \hbar \gamma B_1 (\cos(\omega_{RF} t) I_x + \sin(\omega_{RF} t) I_y)$$

What is $V_I(t)$ [$\omega_{RF} = \gamma B_0$]?

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This is the transformation given here. Now for spin, for a nuclear spin, this is all general theory, now, for a nuclear spin, the static Hamiltonian is the one associated to the magnetic field. That's the one that generates the two energy levels. Hence, we have I_z here as the operator, and we can take the time-variant Hamiltonian given by this term here. Now what is this -- How does this transform if you do this transformation here if we use an *omega-R-F* here in this term here that is equal to *gamma-B-zero*? And actually, if we do this transformation we'll find that in the interaction representation, this time-variant term here, which we have in this term here, becomes time-independent. So, the beauty here is that we have used classically the description in the rotating reference frame to make things stationary. If we do this in the interaction representation, and we make our Hamiltonian in this representation stationary, we're essentially doing the same thing but in quantum mechanics, and we obtain our stationary Hamiltonian, this is equivalent to putting ourselves into the rotating reference frame.

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Quantum mechanical equivalencies:

$$B_0 \propto \langle I_z \rangle, B_{1x,y} \propto \langle I_{x,y} \rangle$$

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So we have an additional quantum mechanical equivalent that is the B -zero is along I -z is proportional to the operator I -z at the B -one, and x or y is given by the operator I -x- I -y. That's what this term here is. This is the description of the R - F field. So, this is a short digression into quantum mechanics to remind ourselves that what is behind magnetic resonance is truly a quantum mechanical event.

Notes

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