

Course material

Course:

Digital Signal Processing II

Video:

2.1.3.b Filter stability

Concepts (extracted from automatically generated subtitles):

Magnitude of the convolution sum. Times x of n minus k . Impulse response. Magnitude of the output of the filter. Bounded input. Minus infinity. Linear filters. Output stability. Case h . Good news. Sufficient condition. Bounded output. Well-behaved input. Minus n . Magnitude of the sum.



[to video sequence search](#)
(within Digital Signal Processing II.)



[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

Stability

- ▶ key concept: avoid “explosions” if the input is nice
- ▶ a nice signal is a bounded signal: $|x[n]| < M$ for all n
- ▶ Bounded-Input Bounded-Output (BIBO) stability: if the input is nice the output should be nice

Let's talk now about stability. Stability is important because it guarantees that the system will not behave unexpectedly if the input to the system is well behaved. In signal processing a well-behaved input is a bounded input,

notes

summary

0m 0s



Stability

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- ▶ a nice signal is a bounded signal: $|x[n]| < M$ for all n
- ▶ Bounded-Input Bounded-Output (BIBO) stability: if the input is nice the output should be nice

namely a signal for which we know the maximum excursion. We want a system to

notes

summary

0m 12s



Stability

- ▶ key concept: avoid “explosions” if the input is nice
- ▶ a nice signal is a bounded signal: $|x[n]| < M$ for all n
- ▶ Bounded-Input Bounded-Output (BIBO) stability: if the input is nice the output should be nice

behave well when the input is bounded. And the concept of bounded input bounded output stability or BIBO stability for short formalizes this by requiring that a system produces a bounded output when the input is bounded. So a well-behaved output for a well-behaved input. The fundamental stability theorem for linear

notes

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Fundamental Stability Theorem

A filter is BIBO stable if and only if its impulse response is absolutely summable

filters states that a filter is BIBO stable if and only if its impulse response is absolutely summable. So we're going to prove this and we're going to prove both the necessary and sufficient condition. So we start with the sufficient

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Proof ($\Rightarrow$)

Proof:

Hypotheses:

- ▶ $|x[n]| < M$ ✓
- ▶ $\sum_n |h[n]| = L < \infty$

Thesis:

- ▶ $|y[n]|$ bounded

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\ &< ML \end{aligned}$$

condition and we say that our hypothesis are that our input is bounded and that the impulse response of the filter is absolutely summable. And what we want to

notes

summary

1m 4s



Proof ($\Rightarrow$)

Proof:

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $\sum_n |h[n]| = L < \infty$

Thesis:

- ▶ $|y[n]|$ bounded

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\ &< ML \end{aligned}$$

prove is that in this case the output will be bounded. Now the proof is very simple. We start by considering the magnitude of the output of the filter

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1m 13s



Proof ($\Rightarrow$)

Proof:

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- ▶ $|x[n]| < M$
- ▶ $\sum_n |h[n]| = L < \infty$

Thesis:

- ▶ $|y[n]|$ bounded

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\ &< ML \end{aligned}$$

which is just the magnitude of the convolution sum written here as the sum for k that goes to minus infinity to plus infinity of h of k times x of n minus k . Notice here that we use the commutative property of the convolution and we actually time-reverse and delay the input rather than the impulse response. Now the magnitude of the sum is maximized by the sum of the magnitude of

notes

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1m 25s



Proof ($\Rightarrow$)

Proof:

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $\sum_n |h[n]| = L < \infty$

Thesis:

- ▶ $|y[n]|$ bounded

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\ &< ML \end{aligned}$$

its terms and this in turn is maximized by the following expression where we

notes

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Proof ($\Rightarrow$)

Proof:

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $\sum_n |h[n]| = L < \infty$

Thesis:

- ▶ $|y[n]|$ bounded

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k] \underbrace{x[n-k]}| \\ &\leq \underbrace{M}_{\text{blue}} \sum_{k=-\infty}^{\infty} |h[k]| \\ &< ML \end{aligned}$$

have replaced the magnitude of the input by its upper bound m and we're left therefore with m that multiplies the sum from minus infinity to plus infinity of

notes

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2m 1s



Proof ($\Rightarrow$)

Proof:

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $\sum_n |h[n]| = L < \infty$

Thesis:

- ▶ $|y[n]|$ bounded

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\ &< ML \end{aligned}$$

the magnitude of the impulse response. But we know that the impulse response is absolutely summable and therefore the output will be bounded by the product mL .

notes

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2m 13s



Proof ($\Rightarrow$)

Proof:

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $\sum_n |h[n]| = L < \infty$

Thesis:

- ▶ $|y[n]|$ bounded

$$\begin{aligned}\underline{|y[n]|} &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\ &< \underline{ML}\end{aligned}$$

which is finite. As for the necessary part we'll proceed as follows. The

notes

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Proof ($\Leftarrow$)

Proof (by contradiction):

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $|y[n]| < P$

Thesis:

- ▶ $h[n]$ absolutely summable

- ▶ assume $\sum_n |h[n]| = \infty$

- ▶ build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$

- ▶ clearly, $x[n]$ is bounded

- ▶ however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

hypothesis now are that both input and output are bounded and we want to prove

notes

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Proof ($\Leftarrow$)

Proof (by contradiction):

Hypotheses:

▶ $|x[n]| < M$ ✓

▶ $|y[n]| < P$ ✓

Thesis:

▶ $h[n]$ absolutely summable

▶ assume $\sum_n |h[n]| = \infty$

▶ build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$

▶ clearly, $x[n]$ is bounded

▶ however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

that in this case h of n must be absolutely summable. The proof will proceed by contradiction and it is a little bit tricky but not difficult at

notes

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Proof ($\Leftarrow$)

Proof (by contradiction):

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $|y[n]| < P$

Thesis:

- ▶ $h[n]$ absolutely summable

- ▶ assume $\sum_n |h[n]| = \infty$

- ▶ build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$

- ▶ clearly, $x[n]$ is bounded

- ▶ however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

all. So assume by contradiction that the impulse response is not absolutely summable therefore the sum of its magnitude is plus infinity. At which point we can build an artificial input as follows. x of n will be equal to plus

notes

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Proof ($\Leftarrow$)

Proof (by contradiction):

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $|y[n]| < P$

Thesis:

- ▶ $h[n]$ absolutely summable

- ▶ assume $\sum_n |h[n]| = \infty$

- ▶ build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$

- ▶ clearly, $x[n]$ is bounded

- ▶ however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

1 if h of minus n is positive and minus 1 if h of minus n is negative. Well clearly x of n built as such is a bounded signal because it's between plus

notes

summary

3m 9s



Proof ($\Leftarrow$)

Proof (by contradiction):

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $|y[n]| < P$

Thesis:

- ▶ $h[n]$ absolutely summable

- ▶ assume $\sum_n |h[n]| = \infty$

- ▶ build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$

- ▶ clearly, $x[n]$ is bounded

- ▶ however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

$n=0$ ↑

1 and minus 1. However if we try to compute the response of the system defined by h of n in 0 we have that the output y of 0 is the convolution between our artificial input x of n and the impulse response computed in 0. So we write out the convolution sum explicitly. So it's a sum for k that goes from minus infinity to plus infinity of h of k times x of minus k because remember n is equal to 0 here in this convolution and of course by definition this will be the

notes

summary

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Proof ($\Leftarrow$)

Proof (by contradiction):

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $|y[n]| < P$

Thesis:

- ▶ $h[n]$ absolutely summable

- ▶ assume $\sum_n |h[n]| = \infty$

- ▶ build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$

- ▶ clearly, $x[n]$ is bounded

- ▶ however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

$n \rightarrow \infty$

sum of the magnitude of h of k because we built a signal like so and because of

notes

summary

4m 4s



Proof ($\Leftarrow$)

Proof (by contradiction):

Hypotheses:

- ▶ $|x[n]| < M$
- ▶ $|y[n]| < P$

Thesis:

- ▶ $h[n]$ absolutely summable

- ▶ assume $\sum_n |h[n]| = \infty$

- ▶ build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$

- ▶ clearly, $x[n]$ is bounded

- ▶ however

$$y[0] = (x * h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

$n \rightarrow \infty$

our initial assumption this will be plus infinity. So we have proven that h of n must be absolutely summable because if it were not I could find a bounded input that would make the system explode. So the good news is that FIR filters are

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The good news

FIR filters are always stable

always stable because their impulse response only contains a finite number of non-zero values and therefore the sum of their absolute values will always be finite. When it comes to IIR filters on the other hand we have to explicitly

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Checking the stability of IIRs

Let's check the Leaky Integrator:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h[n]| &= |1 - \lambda| \sum_{n=0}^{\infty} |\lambda|^n \\ &= \lim_{n \rightarrow \infty} |1 - \lambda| \frac{1 - |\lambda|^{n+1}}{1 - |\lambda|} \\ &< \infty \quad \text{for } |\lambda| < 1\end{aligned}$$

stability is guaranteed for $|\lambda| < 1$

check for stability and for the time being the only way we know how to do so is by going through the sum of the absolute values of the impulse response and if we do the exercise with our friend the leaky integrator we get the sum for n that goes from 0 to infinity of magnitude of lambda to the power of

notes

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Checking the stability of IIRs

Let's check the Leaky Integrator:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h[n]| &= |1 - \lambda| \sum_{n=0}^{\infty} |\lambda|^n \\ &= \lim_{n \rightarrow \infty} |1 - \lambda| \frac{1 - |\lambda|^{n+1}}{1 - |\lambda|} \\ &< \infty \quad \text{for } |\lambda| < 1\end{aligned}$$

stability is guaranteed for $|\lambda| < 1$

n multiplied by a leading factor of 1 minus lambda in magnitude. This is just

notes

summary

5m 0s



Checking the stability of IIRs

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$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h[n]| &= |1 - \lambda| \sum_{n=0}^{\infty} |\lambda|^n \\ &= \lim_{n \rightarrow \infty} |1 - \lambda| \frac{1 - |\lambda|^{n+1}}{1 - |\lambda|} \\ &< \infty \quad \text{for } |\lambda| < 1\end{aligned}$$

stability is guaranteed for $|\lambda| < 1$

a geometric sum and we know that it's equal to the limit for n that goes to infinity of 1 minus magnitude of lambda to the n plus 1 divided by 1 minus lambda. Always multiplied by this factor that doesn't really bother us. This limit

notes

summary

5m 1s



Checking the stability of IIRs

Let's check the Leaky Integrator:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h[n]| &= |1 - \lambda| \sum_{n=0}^{\infty} |\lambda|^n \\ &= \lim_{n \rightarrow \infty} \underbrace{|1 - \lambda|}_{< \infty} \frac{1 - |\lambda|^{n+1}}{1 - |\lambda|} \\ &< \infty \quad \text{for } |\lambda| < 1\end{aligned}$$

stability is guaranteed for $|\lambda| < 1$

converges only if lambda is less than 1 in magnitude and therefore stability of

notes

summary

5m 21s



Checking the stability of IIRs

Let's check the Leaky Integrator:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h[n]| &= |1 - \lambda| \sum_{n=0}^{\infty} |\lambda|^n \\ &= \lim_{n \rightarrow \infty} |1 - \lambda| \frac{1 - |\lambda|^{n+1}}{1 - |\lambda|} \\ &< \infty \quad \text{for } |\lambda| < 1\end{aligned}$$

stability is guaranteed for $|\lambda| < 1$

the leaky integrator is guaranteed only for lambda less than 1 in magnitude. Later on in this module we will study indirect methods for filter stability

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summary

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Checking the stability of IIRs

We will study indirect methods for filter stability later in this Module.

that will allow us to determine whether a filter is stable or not without going through the impulse response explicitly.

notes

summary

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