

Course material

Course:

Digital Signal Processing II

Video:

2.1.4.a The convolution theorem

Concepts (extracted from automatically generated subtitles):

Dtft of the impulse response. Linear time. Frequency ω . $\int_{-\infty}^{\infty}$. Value of the dtft of the impulse response. Eigen sequence. $\int_{-\infty}^{\infty}$. K times. Invariant filter h . Frequency of a sinusoidal input. Convolution product. Little bit. Original input e . First fundamental property.



[to video sequence search](#)
(within Digital Signal Processing II.)



[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

A remarkable result

$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \\&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) e^{j\omega_0 n}\end{aligned}$$

What happens if we take a linear time invariant filter H and we use as the input a complex exponential of known frequency ω_0 ? Well, let's write out the convolution. Then we exploit the fact that the convolution is commutative and we

notes

summary

0m 0s



A remarkable result

$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)} \\&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) e^{j\omega_0 n}\end{aligned}$$

exchange the order of the terms in the convolution product. At which point we can start writing out the convolution sum explicitly and we have the sum for K that goes from minus infinity to plus infinity to infinity of H of K that multiplies e to the j omega 0 and minus K. Now we can take out the constant term that doesn't depend on K out of the sum and we have a leading term of e to the j

notes

summary

0m 15s



A remarkable result

$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \\&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) e^{j\omega_0 n}\end{aligned}$$

$e^{j\omega_0 n}$ that multiplies the sum for K that goes from minus infinity to plus infinity of H of K

notes

summary

0m 36s



A remarkable result

$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)} \\&= \underbrace{e^{j\omega_0 n}}_{\text{times } e \text{ to the minus } j \omega_0 K.} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) e^{j\omega_0 n}\end{aligned}$$

times e to the minus j omega 0 K.

notes

summary

0m 37s



A remarkable result

$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)} \\&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) e^{j\omega_0 n}\end{aligned}$$

But now this guy here, we know very well what it is, is the DTFT of the impulse response computed in ω_0 . So it's big H of $e^{j\omega_0}$ which multiplies our

notes

summary

0m 49s



A remarkable result

$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)} \\&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) e^{j\omega_0 n}\end{aligned}$$

original input $e^{j\omega_0 n}$ to the system.

notes

summary

0m 59s



A remarkable result

$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \\&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) \underbrace{e^{j\omega_0 n}}\end{aligned}$$

Hence the name eigen sequences just like an eigenvector is a vector that when multiplied by a matrix gives a scaled version of itself an eigen sequence is a sequence that when input to a linear time invariant filter returns the sequence itself times

notes

summary

1m 5s



A remarkable result

$$\begin{aligned}y[n] &= e^{j\omega_0 n} * h[n] \\&= h[n] * e^{j\omega_0 n} \\&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)} \\&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\&= H(e^{j\omega_0}) \underbrace{e^{j\omega_0 n}}\end{aligned}$$

a scaling factor which happens to be the value of the DTFT of the impulse response at the frequency of the input.

notes

summary

1m 24s



A remarkable result



- ▶ complex exponentials are *eigensequences* of LTI systems, i.e., linear filters cannot change the frequency of sinusoids
- ▶ DTFT of impulse response determines the frequency characteristic of a filter

So the first fundamental property that we can glean from this derivation is that a linear time invariant system cannot change the frequency of a sinusoidal input. And so since the sinusoidal input is a pure frequency component it's clear that the DTFT of the impulse response fully determines the

notes

summary

1m 25s



A remarkable result



- ▶ complex exponentials are *eigensequences* of LTI systems, i.e., linear filters cannot change the frequency of sinusoids
- ▶ DTFT of impulse response determines the frequency characteristic of a filter

frequency characteristic of a filter at a given frequency. Let's examine a little bit more what happens when we process a complex exponential with a linear time invariant filter.

notes

summary

1m 52s



Magnitude and phase

$$A \in \mathbb{R}$$

If $H(e^{j\omega_0}) = Ae^{j\theta}$, then

$$\mathcal{H}\{e^{j\omega_0 n}\} = Ae^{j(\omega_0 n + \theta)}$$

amplitude:
amplification ($A > 1$)
or attenuation ($0 \leq A < 1$)

phase shift:
delay ($\theta < 0$)
or advancement ($\theta > 0$)

If we write the value of the DTFT of the impulse response at ω_0 as $Ae^{j\theta}$ where A is a real number and θ of course is an angle between

notes

summary

2m 2s



Magnitude and phase

If $H(e^{j\omega_0}) = Ae^{j\theta}$, then

$$A \in \mathbb{R}$$
$$\theta \in [-\pi, \pi]$$

$$\mathcal{H}\{e^{j\omega_0 n}\} = Ae^{j(\omega_0 n + \theta)}$$

amplitude:
amplification ($A > 1$)
or attenuation ($0 \leq A < 1$)

phase shift:
delay ($\theta < 0$)
or advancement ($\theta > 0$)

minus pi and pi if you will. Then we have that the

notes

summary

2m 25s



Magnitude and phase

If $H(e^{j\omega_0}) = Ae^{j\theta}$, then

$$A \in \mathbb{R}$$
$$\theta \in [-\pi, \pi]$$

$$\mathcal{H}\{e^{j\omega_0 n}\} = Ae^{j(\omega_0 n + \theta)}$$

amplitude:
amplification ($A > 1$)
or attenuation ($0 \leq A < 1$)

phase shift:
delay ($\theta < 0$)
or advancement ($\theta > 0$)

processed complex exponential is the original complex exponential which has been scaled by the amplitude a and has been delayed or advanced by a phase term θ . So if a is larger than 1 then we have an amplification if a is less than 1 then we have an attenuation of the input sinusoid and the phase shift again if it's bigger than 0 then we have an advancement of the sinusoid and if it's less than 0 then we have a delay.

notes

summary

2m 30s



The convolution theorem

In general:

$$\text{DTFT} \{x[n] * h[n]\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions. By linearity...

The convolution theorem tries to generalize this result by asking the question what is the DTFT of the convolution of two sequences or in other words

notes

summary

3m 4s



The convolution theorem

In general:

$$\text{DTFT} \{x[n] * h[n]\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions. By linearity...

what is the DTFT of the output of a filter? The intuition here is that the DTFT reconstruction formula tells us that any signal is made up of infinitely many

notes

summary

3m 13s



The convolution theorem

In general:

$$\text{DTFT} \{x[n] * h[n]\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions. By linearity...

$$x(e^{j\omega}) e^{j\omega n}$$

sinusoidal components of the form x to the e to the j omega times e to the j omega n and if I were to process a single sinusoidal component

notes

summary

3m 25s



The convolution theorem

In general:

$$\text{DTFT} \{x[n] * h[n]\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions. By linearity...

$$\underline{x(e^{j\omega}) e^{j\omega n}}$$

independently via a filter H , I would get a value

notes

summary

3m 35s



The convolution theorem

In general:

$$\text{DTFT} \{x[n] * h[n]\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions. By linearity...

$$\underbrace{x(e^{j\omega}) e^{j\omega n}} \rightarrow x(e^{j\omega}) H(e^{j\omega}) e^{j\omega n}$$

like so $H e^{j\omega n}$. So I wouldn't be surprised if the DTFT of the convolution of two sequences was just a product of the Fourier transforms.

notes

summary

3m 44s



The convolution theorem

In general:

$$\text{DTFT} \{x[n] * h[n]\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions. By linearity...

$$\underbrace{x(e^{j\omega}) e^{j\omega n}} \rightarrow \underbrace{x(e^{j\omega}) H(e^{j\omega}) e^{j\omega n}}$$

So let's go see how we can derive this more formally.

notes

summary

4m 0s



The convolution theorem

$$\begin{aligned}\text{DTFT}\{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x * h)[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega(n-k)} e^{-j\omega k} \\ &= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega(n-k)} \\ &= H(e^{j\omega}) X(e^{j\omega})\end{aligned}$$

If we write out the DTFT of the convolution of two sequences, well we can write out initially the formula for the DTFT

notes

summary

4m 1s



The convolution theorem

$$\begin{aligned}\text{DTFT}\{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x * h)[n] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega(n-k)} e^{-j\omega k} \\&= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega(n-k)} \\&= H(e^{j\omega}) X(e^{j\omega})\end{aligned}$$

which is the sum from n from minus infinity to infinity of the value of the convolution in n times e to the minus j omega n. Then we expand the formula for the convolution inside of the DTFT summation and we get a double summation

notes

summary

4m 13s



The convolution theorem

$$\begin{aligned}\text{DTFT}\{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x * h)[n] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \underbrace{x[k] h[n-k]} e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega(n-k)} e^{-j\omega k} \\&= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega(n-k)} \\&= H(e^{j\omega}) X(e^{j\omega})\end{aligned}$$

over the indices n and k of the convolution product here times this complex exponential here. In the next step we do a little trick whereby we add and subtract k

notes

summary

4m 31s



The convolution theorem

$$\begin{aligned}\text{DTFT}\{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x * h)[n] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega(n-k)} e^{-j\omega k} \\&= \underbrace{\sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}}_{= H(e^{j\omega})} \underbrace{\sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega(n-k)}}_{= X(e^{j\omega})} \\&= H(e^{j\omega}) X(e^{j\omega})\end{aligned}$$

from the argument n . So instead of putting n we write n minus k plus k and in so doing we manage to split the complex exponential in two components here that we will distribute across the summations. So now here we collect the terms that do not depend on n and here we put the rest. This first term here is the DTFT of x of n . That's really easy to see. The second term looks like a DTFT. We might be confused by the fact that the indices here and here are n minus k instead of n , but the fact that the index n ranges from minus infinity to plus infinity makes this term completely irrelevant. And so indeed we have the product of two DTFTs, the product of the DTFT of the impulse response

notes

summary

4m 43s



Frequency response

$$H(e^{j\omega}) = \text{DTFT} \{h[n]\}$$

Two effects:

- ▶ **magnitude:** amplification ($|H(e^{j\omega})| > 1$) or attenuation ($|H(e^{j\omega})| < 1$) of input frequencies
- ▶ **phase:** overall delay and shape changes

times the DTFT of the input. The Fourier transform of the impulse response is called the frequency response and just like in the case of the single complex exponential we can split the significance of the product in the frequency domain by separating the effects of the magnitude and those of the phase. So the magnitude of the frequency response will determine whether certain frequencies of the input signal are going to be amplified by the frequency response and certain frequencies of the input signal are going to be amplified or attenuated according to whether the frequency response is larger than one or smaller than one in magnitude in certain frequency bands. The effects of the phase are a little bit more difficult to qualify right now,

notes

summary

5m 37s



Frequency response

$$H(e^{j\omega}) = \text{DTFT} \{h[n]\}$$

Two effects:

- ▶ **magnitude:** amplification ($|H(e^{j\omega})| > 1$) or attenuation ($|H(e^{j\omega})| < 1$) of input frequencies
- ▶ **phase:** overall delay and shape changes

but it will be clearer later on that the phase will determine whether the signal will conserve its shape in the time domain or its shape will be altered.

notes

summary

6m 24s

