

Course material

Course:

## Digital Signal Processing II

Video:

### 2.1.5.b The ideal lowpass filter

Concepts (extracted from automatically generated subtitles):

**Low-pass filter. Magnitude response. Impulse response. Ideal version of the low-pass filter. Sinc function. Low frequencies. Low-pass filter lets.  $\omega$  periodicity of any frequency response. Frequency response transitions. Frequency response.  $2\omega_c$ . High frequencies. Real function. Special functions.**



[to video sequence search](#)  
(within Digital Signal Processing II.)

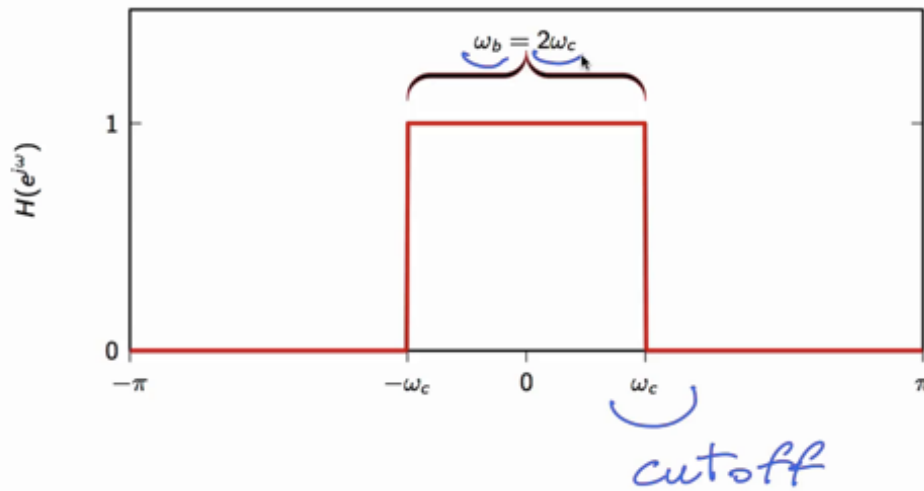


[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

## What is the best lowpass we can think of?



So, what is the best low-pass filter that we can think of? Well, as we said, the low-pass filter lets low frequencies go through and attenuates or kills high frequencies. So, an ideal version of the low-pass filter would leave frequencies in the low band untouched and would completely attenuate, completely kill frequencies in the higher band. So, the magnitude would be 1 over the pass band and identically 0 over what we call the stop band. To make matters even better, we would require the magnitude response to be a real function so that the filter has zero phase and therefore introduces no delay. So, if we call  $\omega_c$  the cut-off frequency, where the frequency response transitions from 1 to 0, we can say that the ideal low-pass has a bandwidth of  $\omega_b$ , which is equal to twice  $\omega_c$ . Formally, we can write the frequency response of the ideal low-pass,

### notes

### summary

0m 0s



## Ideal lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \quad (2\pi\text{-periodicity implicit})$$

- ▶ perfectly flat passband
- ▶ infinite attenuation in stopband
- ▶ zero-phase (no delay)

$H(e^{j\omega})$  is equal to 1 for  $\omega$  between  $-\omega_c$  and  $\omega_c$  and 0 otherwise.

notes

summary

1m 7s



## Ideal lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \quad \underbrace{(2\pi\text{-periodicity implicit})}$$

- ▶ perfectly flat passband
- ▶ infinite attenuation in stopband
- ▶ zero-phase (no delay)

And remember, whenever we write a specification like this, we tacitly imply the 2- $\pi$  periodicity of any frequency response, of any Fourier transform.

notes

summary

1m 13s



## Ideal lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \quad (2\pi\text{-periodicity implicit})$$

- ▶ perfectly flat passband
- ▶ infinite attenuation in stopband
- ▶ zero-phase (no delay)

So, this means that the ideal low-pass has a perfectly flat pass band, which is identical to 1,

notes

summary

1m 25s



## Ideal lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \quad (2\pi\text{-periodicity implicit})$$

- ▶ perfectly flat passband
- ▶ infinite attenuation in stopband
- ▶ zero-phase (no delay)

has infinite attenuation in the stop band, and again has zero phase and no delay. Let's try and compute the impulse response from the frequency response

notes

summary

1m 37s



## Ideal lowpass filter: impulse response

$$\begin{aligned}h[n] &= \text{IDFT} \{H(e^{j\omega})\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\&= \frac{1}{\pi n} \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \\&= \frac{\sin \omega_c n}{\pi n}\end{aligned}$$

by taking the inverse Fourier transform of big H of e to the j omega. So, we can write the inverse DTFT formula, which we know by heart.

notes

summary

1m 46s



## Ideal lowpass filter: impulse response

$$\begin{aligned}h[n] &= \text{IDFT} \{H(e^{j\omega})\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\&= \frac{1}{\pi n} \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \\&= \frac{\sin \omega_c n}{\pi n}\end{aligned}$$

And since big H is 0 from omega c to pi and from minus pi to minus omega c,

notes

summary

1m 56s



## Ideal lowpass filter: impulse response

$$\begin{aligned}h[n] &= \text{IDFT} \{H(e^{j\omega})\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\&= \frac{1}{\pi n} \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \\&= \frac{\sin \omega_c n}{\pi n}\end{aligned}$$

we can change the integration limits and just integrate from minus omega c to plus omega c. Incidentally, over that interval, big H is equal to 1, so this simplifies to just the integral of e to the j omega n between minus omega c and omega c. And if we solve that integral, which is rather easy to do,

notes

summary

2m 1s



## Ideal lowpass filter: impulse response

$$\begin{aligned}h[n] &= \text{IDFT} \{H(e^{j\omega})\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\&= \frac{1}{\pi n} \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \\&= \frac{\sin \omega_c n}{\pi n}\end{aligned}$$

we get  $\frac{1}{\pi n}$  that multiplies  $e^{j\omega_c n} - e^{-j\omega_c n}$  divided by  $2j$ .

notes

summary

2m 25s



## Ideal lowpass filter: impulse response

$$\begin{aligned}h[n] &= \text{IDFT} \{H(e^{j\omega})\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\&= \frac{1}{\pi n} \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \\&= \frac{\sin \omega_c n}{\pi n}\end{aligned}$$

We know from Euler's formula that this is equal to the sine of  $\omega_c n$ , and we are left also with  $\pi n$  at the denominator. So, we can plot this impulse response, and we find out that it has a nice oscillatory shape.

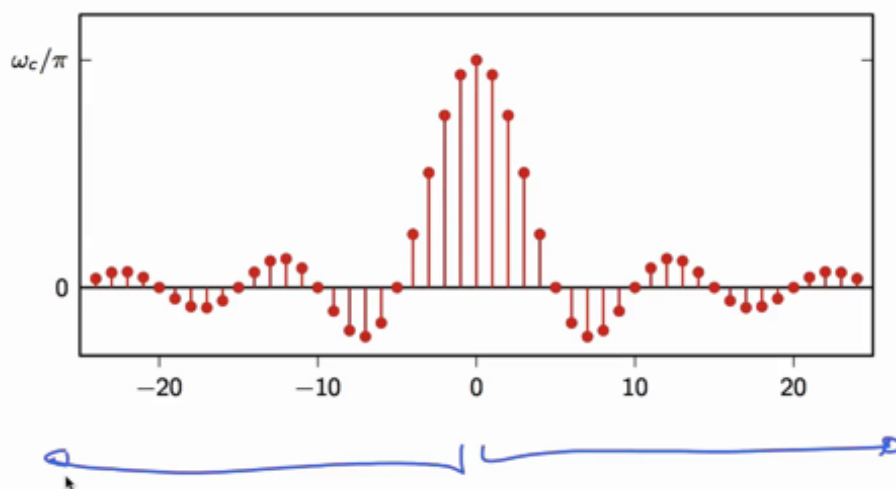
notes

summary

2m 37s



## Ideal lowpass filter: impulse response



We also notice that it is an infinite support impulse response, and as opposed to the other infinite support impulse responses we've seen so far, now the support is infinite both to the right and to the left. Now, this is a problem because it means that no matter how we compute the convolution,

notes

summary

2m 48s



## The bad news

- ▶ impulse response is infinite support, two-sided
  - ⇒ cannot compute the output in a finite amount of time
  - ⇒ that's why it's called "ideal"
- ▶ impulse response decays slowly in time
  - ⇒ we need a lot of samples for a good approximation

we will always have an infinite number of operations to compute. So, this low-pass filter cannot be exactly implemented in practice, and we will spend quite some time in the rest of the module to find ways to approximate this ideal behavior with computable filters. Another problem with the impulse response is that it decays very slowly in time as  $1/n$ , as a matter of fact,

## notes

## summary

3m 13s



## The bad news

- ▶ impulse response is infinite support, two-sided
  - ⇒ cannot compute the output in a finite amount of time
  - ⇒ that's why it's called "ideal"
- ▶ impulse response decays slowly in time
  - ⇒ we need a lot of samples for a good approximation

$$1/n$$

and so we need a lot of samples to get a good approximation of the impulse response if we were to implement the filter directly. In spite of the shortcomings, the ideal low-pass filter is so important

### notes

### summary

3m 35s



## Nevertheless...

The sinc-rect pair:

$$\text{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

(note that  $\text{sinc}(x) = 0$  when  $x$  is a nonzero integer)

that we use two special functions to describe its impulse response and its frequency response. So, we start with a rect function that, as the name suggests, looks like a rectangle,

notes

summary

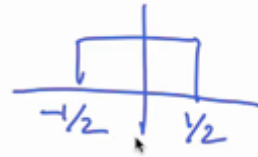
3m 46s



## Nevertheless...

The sinc-rect pair:

$$\text{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}$$



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(note that  $\text{sinc}(x) = 0$  when  $x$  is a nonzero integer)

and it is indeed the indicator function on the real line from 1 half to minus 1 half. The sinc function, in this particular case it's a normalized sinc function,

notes

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summary

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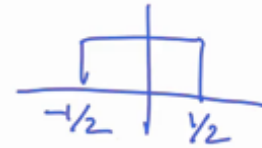
4m 1s



## Nevertheless...

The sinc-rect pair:

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(note that  $\text{sinc}(x) = 0$  when  $x$  is a nonzero integer)

is simply the ratio between sin of pi of x divided by pi of x for all real numbers other than 0,

notes

summary

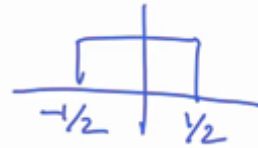
4m 13s



## Nevertheless...

The sinc-rect pair:

$$\text{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}$$



$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

(note that  $\text{sinc}(x) = 0$  when  $x$  is a nonzero integer)

and it is prolonged by continuity in 0 to the value of 1. Note that the sinc function is equal to 0 whenever its argument is a non-zero integer, because in that case the sin of  $\pi x$  becomes 0.

notes

summary

4m 18s



## The ideal lowpass in canonical form

$$\text{rect}\left(\frac{\omega}{2\omega_c}\right) \xleftrightarrow{\text{DTFT}} \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c}{\pi}n\right)$$

!

With this notation in place, we can express the ideal low-pass filter in canonical form. The frequency response,  $h$  of  $e$  to the  $j\omega$ , will be simply the rect of  $\omega$  divided by  $2\omega_c$ ,

notes

summary

4m 36s



## The ideal lowpass in canonical form

$$\boxed{\text{rect}\left(\frac{\omega}{2\omega_c}\right)} \xleftrightarrow{\text{DTFT}} \boxed{\frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c}{\pi}n\right)}$$

$H(e^{j\omega})$

where  $\omega_c$  is the cut-off frequency of the filter. So, it will look like that.

notes

summary

4m 48s



## The ideal lowpass in canonical form

$$\text{rect}\left(\frac{\omega}{2\omega_c}\right) \xleftrightarrow{\text{DTFT}} \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c}{\pi}n\right)$$

$H(e^{j\omega})$



The impulse response will be the sinc function normalized by  $\omega_c$  over  $\pi$ , both in terms of its argument and in terms of magnitude. So, here's an example with  $\omega_c$  equal to  $\pi/3$ .

notes

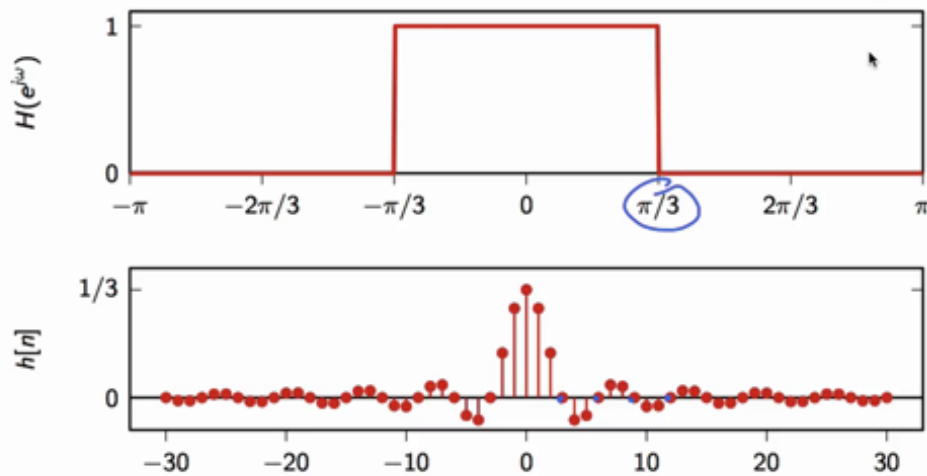
summary

4m 58s



## Example

$$\omega_c = \pi/3: H(e^{j\omega}) = \text{rect}(3\omega/2\pi), h[n] = (1/3)\text{sinc}(n/3)$$



We have a low-pass characteristic with cut-off frequency  $\pi/3$ , and in the time domain we have  $\text{sinc}(n/3)$  multiplied by  $1/3$ , which means that every third sample, as we said, is going to be 0, because every time  $n$  is a multiple of 3, the argument of the sinc will be an integer. So, here's a little known fact about the sinc function.

notes

summary

5m 13s



## Little-known fact

- ▶ the sinc is not absolutely summable
- ▶ the ideal lowpass is not BIBO stable!
- ▶ example for  $\omega_c = \pi/3$ :  $h[n] = (1/3) \text{sinc}(n/3)$
- ▶ take  $x[n] = \text{sign}\{\text{sinc}(-n/3)\}$  and

$$y[0] = (x * h)[0] = \frac{1}{3} \sum_{k=-\infty}^{\infty} |\text{sinc}(k/3)| = \infty$$

The sinc function is not absolutely summable. The proof is not too complicated, it only involves basic calculus,

notes

summary

5m 41s



## Little-known fact

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and you can find it in the references. As a consequence, the ideal low-pass is not bi-bostable. We can use the technique that we developed in the proof of the stability theorem, and so, for instance, if we take  $\omega_c$  equal to  $\pi/3$ , the impulse response of the ideal filter will be  $1/3 \text{sinc}(n/3)$ ,

## notes

## summary

5m 49s



## Little-known fact

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$$y[0] = (x * h)[0] = \frac{1}{3} \sum_{k=-\infty}^{\infty} |\text{sinc}(k/3)| = \infty$$

we build an artificial bounded input signal, which is simply the sine of sinc of minus n over 3, and so if we compute the convolution in n equal to 0, so we have the convolution product here, and it is 1 third, sum from minus infinity to plus infinity of the absolute value of sinc of k over 3,

notes

summary

6m 11s



## Little-known fact

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- ▶ take  $x[n] = \text{sign}\{\text{sinc}(-n/3)\}$  and

$$y[0] = (x * h)[0] = \frac{1}{3} \sum_{k=-\infty}^{\infty} |\text{sinc}(k/3)| = \infty$$

which, because of the non-absolute summability of the sinc, diverges to infinity. The divergence, however, is very, very slow.

notes

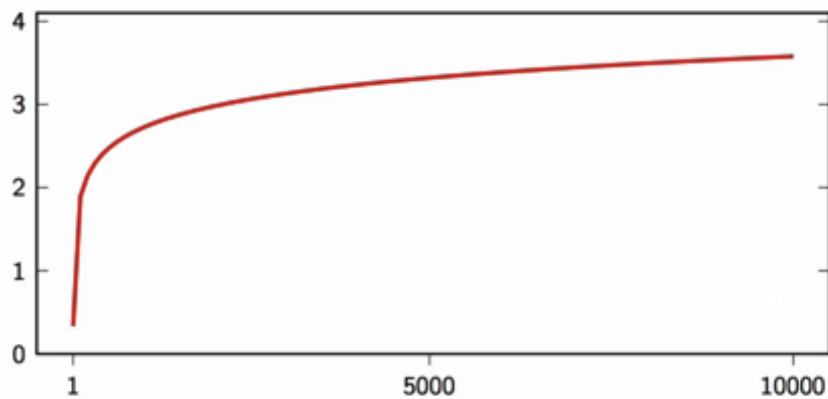
summary

6m 32s



Divergence is however very slow...

$$s(n) = (1/3) \sum_{k=-n}^n |\text{sinc}(k/3)|$$



You can see here a plot that shows that for 10,000 samples, we're still below 4 in terms of the value of the sum.

notes

summary

6m 39s

