



Course material

Course:

Digital Signal Processing II

Video:

2.2.1.b The window method

Concepts (extracted from automatically generated subtitles):

Fourier transform of the filter. Product of the original impulse response times. Dual of the convolution theorem. Interval minus n . Fourier transform of the convolution. Indicator function. Dft of the product. Minus π . Main lobe start. Frequency domain of their fourier transforms. Inverse dft of the convolution. Approximated filter h . Fourier transform. Terms of the inner product. Convolution theorem.



[to video sequence search](#)
(within Digital Signal Processing II.)



[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n] w[n]$$

$$w[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{H}(e^{j\omega}) = ?$$

In order to understand intuitively what happens when we approximate a filter by

notes

summary

0m 0s



Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n] w[n]$$

$$w[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{H}(e^{j\omega}) = ?$$

truncation, we have to take a little bit of detour and look at the problem from a different perspective. So we could consider the approximated filter $\hat{h}$ of n as the product of the original impulse response

notes

summary

0m 1s



Understanding the Gibbs phenomenon

$$\hat{h}[n] = \underbrace{h[n]} \underbrace{w[n]}$$

$$w[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$



$$\hat{H}(e^{j\omega}) = ?$$

times a sequence w of n , which is just the indicator function for the interval minus n to plus n . So w of n is just a series of points of value 1 centered in 0 and going from minus n to n .

notes

summary

0m 18s



Understanding the Gibbs phenomenon

$$\hat{h}[n] = \underbrace{h[n]} \underbrace{w[n]}$$

$$w[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$



$$\hat{H}(e^{j\omega}) = ?$$

So with this notation the question now is how can we express the Fourier transform of the filter

notes

summary

0m 36s



Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n] w[n]$$

$$w[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{H}(e^{j\omega}) = ?$$

as the product of two sequences? And for that we have to study the modulation theorem. The modulation theorem is really the dual of the convolution theorem.

notes

summary

0m 42s



The modulation theorem

$$\text{DTFT} \{(x * y)[n]\} = X(e^{j\omega}) Y(e^{j\omega})$$



$$\text{DTFT} \{x[n] y[n]\} = (X * Y)(e^{j\omega})$$

You remember the convolution theorem states that the Fourier transform of the convolution of two sequences is the product in the frequency domain of their Fourier transforms. The modulation theorem

notes

summary

0m 52s



The modulation theorem

$$\text{DTFT} \{(x * y)[n]\} = X(e^{j\omega}) Y(e^{j\omega})$$

$$\text{DTFT} \{ \underbrace{x[n]} \underbrace{y[n]} \} = \underbrace{(X * Y)}(e^{j\omega})$$

gives us a result for the Fourier transform of the product of two sequences and tells us that their Fourier transform is the convolution of the Fourier transforms in the frequency domain. So what is a convolution in the frequency domain? Well in $\mathbb{C}^\infty$, i.e. in the space of infinite

notes

summary

1m 1s



Convolution of DTFTs

in $\mathbb{C}^\infty$:

$$\begin{aligned}(x * y)[n] &= \langle x^*[k], y[n - k] \rangle \\ &= \sum_{k=-\infty}^{\infty} x[k]y[n - k]\end{aligned}$$

in $L_2([-\pi, \pi])$:

$$\begin{aligned}(X * Y)(e^{j\omega}) &= \langle X^*(e^{j\sigma}), Y(e^{j(\omega-\sigma)}) \rangle \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) d\sigma\end{aligned}$$

support sequences, we can define the convolution in terms of the inner product between two sequences. So the convolution between x and y is the inner product between x conjugated with y time reversed

notes

summary

1m 19s



Convolution of DTFTs

in $\mathbb{C}^\infty$:

$$\begin{aligned} (x * y)[n] &= \langle x^*[k], y[n-k] \rangle \\ &= \sum_{k=-\infty}^{\infty} x[k]y[n-k] \end{aligned}$$

in $L_2([-\pi, \pi])$:

$$\begin{aligned} (X * Y)(e^{j\omega}) &= \langle X^*(e^{j\sigma}), Y(e^{j(\omega-\sigma)}) \rangle \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) d\sigma \end{aligned}$$

and delayed by n . And so if you apply the definition of the inner product in $\mathbb{C}^\infty$, that's exactly what you get. We can adopt the same strategy in L_2 of minus π to π , which is the space

notes

summary

1m 36s



Convolution of DTFTs

in $\mathbb{C}^\infty$:

$$\begin{aligned}(x * y)[n] &= \langle x^*[k], y[n - k] \rangle \\ &= \sum_{k=-\infty}^{\infty} x[k]y[n - k]\end{aligned}$$

in $L_2([-\pi, \pi])$:

$$\begin{aligned}(X * Y)(e^{j\omega}) &= \langle X^*(e^{j\sigma}), Y(e^{j(\omega-\sigma)}) \rangle \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) d\sigma\end{aligned}$$

where DTFTs live, and define the convolution between two Fourier transforms as the inner product between the first Fourier transform conjugated and the second Fourier transform frequency reversed

notes

summary

1m 48s



Convolution of DTFTs

in $\mathbb{C}^\infty$:

$$\begin{aligned}(x * y)[n] &= \langle x^*[k], y[n - k] \rangle \\ &= \sum_{k=-\infty}^{\infty} x[k]y[n - k]\end{aligned}$$

in $L_2([-\pi, \pi])$:

$$\begin{aligned}(X * Y)(e^{j\omega}) &= \langle X^*(e^{j\sigma}), Y(e^{j(\omega-\sigma)}) \rangle \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) d\sigma\end{aligned}$$

and delayed by omega. And if we apply the definition of the inner product for L2 of minus pi pi, we get that the convolution between two Fourier transforms is one over two pi times the integral from minus pi to pi of big X of e to the j sigma times big Y of e to the j omega minus sigma in

notes

summary

2m 0s



Modulation theorem: proof

$$\begin{aligned}\text{IDTFT}\{(X * Y)(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X * Y)(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\omega n} d\sigma d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\sigma n} e^{j(\omega-\sigma)n} d\sigma d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) e^{j\sigma n} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j(\omega-\sigma)}) e^{j(\omega-\sigma)n} d\omega \\&= x[n] y[n]\end{aligned}$$

this sigma. With this notation in place, we can prove that the DTFT of the product of two sequences is the convolution of their Fourier transforms by working backwards. So we start with the inverse

notes

summary

2m 25s



Modulation theorem: proof

$$\begin{aligned}\text{IDTFT}\{(X * Y)(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X * Y)(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\omega n} d\sigma d\omega \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\sigma n} e^{j(\omega-\sigma)n} d\sigma d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) e^{j\sigma n} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j(\omega-\sigma)}) e^{j(\omega-\sigma)n} d\omega \\ &= x[n] y[n]\end{aligned}$$

DTFT of the convolution of two Fourier transforms. We can write it out explicitly like so.
And if we

notes

summary

2m 37s



Modulation theorem: proof

$$\begin{aligned}\text{IDTFT}\{(X * Y)(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X * Y)(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\omega n} d\sigma d\omega \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\sigma n} e^{j(\omega-\sigma)n} d\sigma d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) e^{j\sigma n} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j(\omega-\sigma)}) e^{j(\omega-\sigma)n} d\omega \\ &= x[n] y[n]\end{aligned}$$

expand the definition of the convolution inside the integral that defines the inverse DTFT, we have the following expression where here you have the convolution and outside here you have the inverse DTFT. Now we use the same trick that we used when we proved the convolution theorem, namely we replace omega with omega minus sigma plus sigma in the exponential here. And we manage to split the

notes

summary

2m 40s



Modulation theorem: proof

$$\begin{aligned}\text{IDTFT} \{(X * Y)(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X * Y)(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\omega n} d\sigma d\omega \quad \omega = \omega - \sigma + \sigma \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\sigma n} e^{j(\omega-\sigma)n} d\sigma d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) e^{j\sigma n} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j(\omega-\sigma)}) e^{j(\omega-\sigma)n} d\omega \\&= x[n] y[n]\end{aligned}$$

complex exponential in a way that will allow us to separate the contribution due to X and to Y.

notes

summary

3m 8s



Modulation theorem: proof

$$\begin{aligned}\text{IDTFT}\{(X * Y)(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X * Y)(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\omega n} d\sigma d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\sigma n} e^{j(\omega-\sigma)n} d\sigma d\omega \\&= \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) e^{j\sigma n} d\sigma}_{x[n]} \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j(\omega-\sigma)}) e^{j(\omega-\sigma)n} d\omega \\&= x[n] y[n]\end{aligned}$$

And indeed when we do so, we have a first part here which is just the inverse DTFT of big X. And here we have the inverse DTFT of big Y. The fact that the argument to the complex exponential

notes

summary

3m 14s



Modulation theorem: proof

$$\begin{aligned}\text{IDTFT}\{(X * Y)(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X * Y)(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\omega n} d\sigma d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\sigma n} e^{j(\omega-\sigma)n} d\sigma d\omega \\&= \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) e^{j\sigma n} d\sigma}_{x[n]} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j(\omega-\sigma)}) e^{j(\omega-\sigma)n} d\omega}_{y[n]} \\&= x[n] y[n]\end{aligned}$$

has a minus sigma term doesn't really bother us because the integral is between minus pi and pi. And of course the DTFT is 2π periodic, so an offset does not change the result of the integral. And so finally we have indeed what we were looking for, the product of the two time domain sequences.

notes

summary

3m 28s



Modulation theorem: proof

$$\begin{aligned}\text{IDTFT} \{(X * Y)(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X * Y)(e^{j\omega}) e^{j\omega n} d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\omega n} d\sigma d\omega \\&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) Y(e^{j(\omega-\sigma)}) e^{j\sigma n} e^{j(\omega-\sigma)n} d\sigma d\omega \\&= \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) e^{j\sigma n} d\sigma}_{x[n]} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j(\omega-\sigma)}) e^{j(\omega-\sigma)n} d\omega}_{y[n]} \\&= x[n] y[n]\end{aligned}$$

As an interesting aside, let's look back at the sinusoidal modulation result with the help of the

notes

summary

3m 49s



Aside: sinusoidal modulation revisited

$$\begin{aligned}\text{DTFT}\{x[n] \cos \omega_c n\} &= X(e^{j\omega}) * \frac{1}{2}[\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)] \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma + \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma \\ &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]\end{aligned}$$

modulation theorem we just studied. So the DTFT of a sequence $x[n]$ multiplied by cosine of $\omega_c n$ turns out to be the convolution of the DTFT of $x[n]$ with the DTFT of the cosine of $\omega_c n$,

notes

summary

3m 53s



Aside: sinusoidal modulation revisited

$$\begin{aligned}\text{DTFT}\{x[n] \cos \omega_c n\} &= \underbrace{X(e^{j\omega})}_{\text{DTFT of } x[n]} * \underbrace{\frac{1}{2}[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]}_{\text{DTFT of } \cos \omega_c n} \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \delta(\sigma - \omega + \omega_c) d\sigma + \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \delta(\sigma - \omega - \omega_c) d\sigma \\ &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]\end{aligned}$$

which is one half delta of omega minus omega c plus delta of omega plus omega c. This delta here

notes

summary

4m 9s



Aside: sinusoidal modulation revisited

$$\begin{aligned}\text{DTFT}\{x[n] \cos \omega_c n\} &= \underbrace{X(e^{j\omega})}_{\text{DTFT of } x[n]} * \underbrace{\frac{1}{2}[\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)]}_{\text{DTFT of } \cos \omega_c n} \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma + \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma \\ &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]\end{aligned}$$

is the delta defined over the real line, so it's that functional that isolates the value of a

notes

summary

4m 16s



Aside: sinusoidal modulation revisited

$$\begin{aligned}
 \text{DTFT} \{x[n] \cos \omega_c n\} &= X(e^{j\omega}) * \frac{1}{2} [\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] \\
 &= \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \delta(\sigma - \omega + \omega_c) d\sigma + \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \delta(\sigma - \omega - \omega_c) d\sigma \\
 &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]
 \end{aligned}$$

$\delta(t) \quad t \in \mathbb{R}$

function when it is used under the integral sign. So because of the distributive property of the convolution we can split the above convolution product into two terms, which we write out

notes

summary

4m 25s



Aside: sinusoidal modulation revisited

$$\begin{aligned}
 \text{DTFT} \{x[n] \cos \omega_c n\} &= \underbrace{X(e^{j\omega})}_{\substack{\delta(t) + \in \mathbb{R} \\ \swarrow}} * \underbrace{\frac{1}{2}[\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)]}_{\substack{\delta(t) + \in \mathbb{R} \\ \swarrow}} \\
 &= \underbrace{\frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma}_{\substack{\delta(t) + \in \mathbb{R} \\ \swarrow}} + \underbrace{\frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma}_{\substack{\delta(t) + \in \mathbb{R} \\ \swarrow}} \\
 &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]
 \end{aligned}$$

explicitly here as convolution integrals. And we have for instance in the first case the integral of the product between the DTFT of X and delta of sigma minus omega plus omega c in the sigma.

notes

summary

4m 40s



Aside: sinusoidal modulation revisited

$$\begin{aligned}
 \text{DTFT} \{x[n] \cos \omega_c n\} &= \underbrace{X(e^{j\omega})}_{\delta(t) + \epsilon \mathbb{R}} * \underbrace{\frac{1}{2} [\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)]}_{\delta(t) + \epsilon \mathbb{R}} \\
 &= \underbrace{\frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma}_{\delta(t) + \epsilon \mathbb{R}} + \underbrace{\frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma}_{\delta(t) + \epsilon \mathbb{R}} \\
 &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]
 \end{aligned}$$

Similar here we have the complementary term on the negative axis, the integral of the product

notes

summary

4m 55s



Aside: sinusoidal modulation revisited

$$\begin{aligned}
 \text{DTFT} \{x[n] \cos \omega_c n\} &= \underbrace{X(e^{j\omega})}_{\substack{\delta(t) + \in \mathbb{R} \\ \swarrow}} * \underbrace{\frac{1}{2} [\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)]}_{\substack{\delta(t) + \in \mathbb{R} \\ \swarrow}} \\
 &= \underbrace{\frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma}_{\substack{\delta(t) + \in \mathbb{R} \\ \swarrow}} + \underbrace{\frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma}_{\substack{\delta(t) + \in \mathbb{R} \\ \swarrow}} \\
 &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]
 \end{aligned}$$

between the Fourier transform of X and delta of sigma minus omega minus omega c . Because of the sifting property of the delta, these integrals will isolate the value for the Fourier transform

notes

summary

5m 1s



Aside: sinusoidal modulation revisited

$$\begin{aligned}
 \text{DTFT} \{x[n] \cos \omega_c n\} &= \underbrace{X(e^{j\omega})}_{\delta(t) + \in \mathbb{R}} * \underbrace{\frac{1}{2}[\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)]}_{\delta(t) + \in \mathbb{R}} \\
 &= \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma + \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma \\
 &= \frac{1}{2} [X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)})]
 \end{aligned}$$

for the value of sigma where the argument to the delta is equal to zero. So in the first case

notes

summary

5m 13s



Aside: sinusoidal modulation revisited

$$\begin{aligned}\text{DTFT}\{x[n] \cos \omega_c n\} &= X(e^{j\omega}) * \frac{1}{2}[\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)] \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma + \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma \\ &= \frac{1}{2} \left[X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)}) \right]\end{aligned}$$

we will have that the delta will kick in for sigma equal to omega minus omega c, and in the second term it will kick in for sigma equal to omega plus omega c. And so in the end the final result is the well-known modulated signal, one half the Fourier transform of the signal centered in omega c plus

notes

summary

5m 15s



Aside: sinusoidal modulation revisited

$$\begin{aligned}\text{DTFT}\{x[n] \cos \omega_c n\} &= X(e^{j\omega}) * \frac{1}{2}[\tilde{\delta}(\omega - \omega_c) + \tilde{\delta}(\omega + \omega_c)] \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega + \omega_c) d\sigma + \frac{1}{4\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \tilde{\delta}(\sigma - \omega - \omega_c) d\sigma \\ &= \frac{1}{2} \left[X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)}) \right]\end{aligned}$$

one half the Fourier transform of the signal centered in minus omega c. So this is another way to arrive at the sinusoidal modulation result. But now we should go back to the reason why we

notes

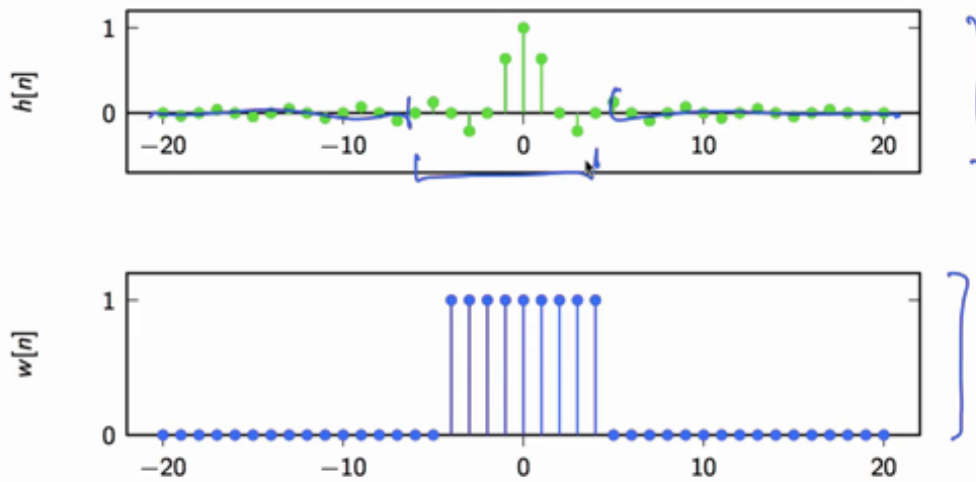
summary

5m 40s



Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n] w[n]$$



started this detour into the modulation theorem and try to understand what the Gibbs phenomenon is all about. So remember in the beginning of our detour we were at a point where we had expressed the approximate filter $\hat{h}$ of n as the product between the ideal impulse response h of n times an indicator function that serves the purpose of isolating the points of the ideal impulse response that we want to keep. So if this is the ideal impulse response h of n and this is w of n you can see that this window basically kills all these guys here and kills all these guys here and leaves us with an FIR approximation to the impulse response. In the frequency domain this

notes

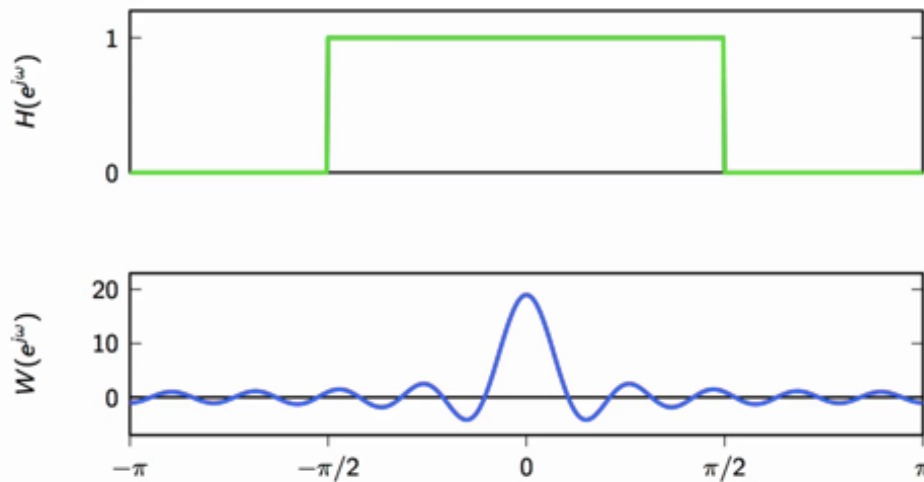
summary

5m 56s



Understanding the Gibbs phenomenon

$$\hat{H}(e^{j\omega}) = (H * W)(e^{j\omega}), \quad W(e^{j\omega}) = \frac{\sin(\omega(2N+1)/2)}{\sin(\omega/2)}$$



corresponds to the convolution of the Fourier transforms of the two actors. The Fourier transform of the ideal impulse response is the rect function that we know very well. The Fourier transform of the indicator function we have seen many times before and it is actually the Fourier transform of a zero-centered moving average filter. So its formula is here and it's sine of omega 2 n plus 1 over 2 divided by sine of omega over 2. So here we're going to try and compute the convolution

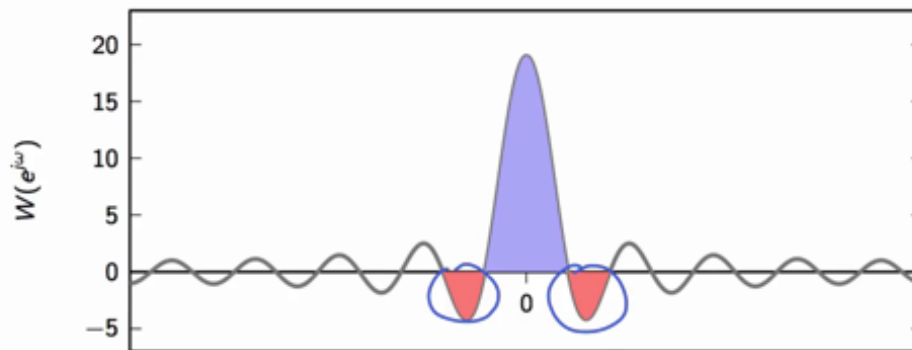
notes

summary

6m 41s



Mainlobe and sidelobes



between the Fourier transforms in a graphical way. Here are the actors involved in the play. We have h of e to the $j\omega$ which is the Fourier transform of the ideal filter. We have w of e to the $j\omega$ which is the Fourier transform of the indicator function. And here we have the result which is the integral of the product of these two guys. As ω moves along the axis here we will re-center the Fourier transform of the indicator function, take the product of the two and compute the integral. In the beginning we're just integrating the wiggles of the function over the non-zero part of the ideal impulse response. So what we have is an oscillatory behavior of small amplitude. Things start to become interesting when the main part of the Fourier transform of the indicator function starts to overlap with the support of the rect or the ideal filter. As we approach the transition band of the filter we see that the value of the convolution starts to grow. And as a matter of fact it grows even more with the entire main lobe of the Fourier transform of the window is under the rect function. As we move along the ripples that trail the main lobe start to get integrated and so the behavior in the passband is again oscillatory as these little ripples enter and exit the the main integration interval. As we reach the other transition band we have exactly the symmetric behavior at the other edge of the band. The shape of the Fourier transform of the approximate filter will depend then on the shape of the Fourier transform of the

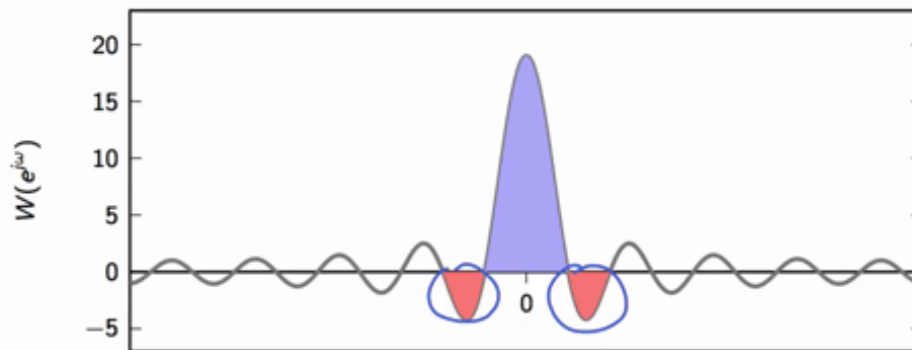
notes

summary

7m 8s



Mainlobe and sidelobes



indicator function. So if we look at it here we see that we have what is called a main lobe here in blue that will determine how steeply the approximate filter will transition from the stop band to the passband. The width of the main lobe is what determines the steepness whereas the amplitude of the so-called side lobes will determine the amplitude of the error on either side of the transition band and this is what determines the Gibbs phenomenon. So in terms of

notes

summary

What if we change the window?

We want:

- ▶ narrow mainlobe so that transition is sharp
- ▶ small sidelobe so Gibbs error is small
- ▶ short window so FIR is efficient

in terms of our requirements what we would like to have is a very narrow main lobe so that the transition is very sharp and at the same time we would like to have very small side lobes so that the Gibbs error is kept low and we would also like to have a short window so that the FIR will be

notes

summary

9m 30s



What if we change the window?

We want:

- ▶ narrow mainlobe so that transition is sharp
- ▶ small sidelobe so Gibbs error is small
- ▶ short window so FIR is efficient

very conflicting requirements!

efficient and these are very very conflicting requirements. There is a large body of literature that is concerned with developing the best possible window in order to approximate an LDO filter. For instance we have used a rectangular window to truncate the impulse response but if we use a

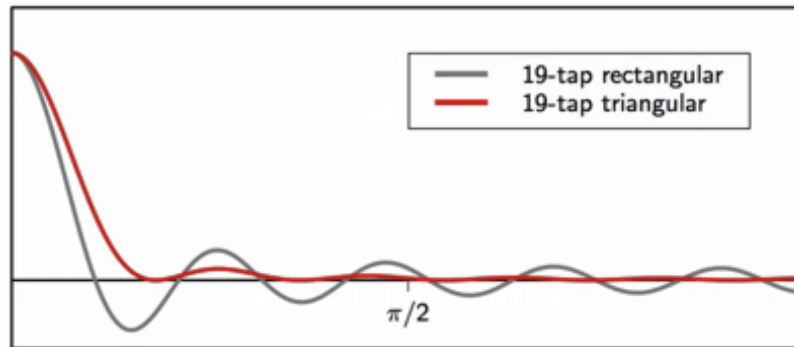
notes

summary

9m 43s



Rectangular vs Triangular Window



triangular window that therefore weights the impulse response and tapers it to zero in a more gradual manner what we have is that we will be able to attenuate the Gibbs error at the price of a wider main lobe. So here you have the comparison between a 19-tap rectangular window in gray and a 19-tap triangular window in red and you see that the side lobes are much smaller for the triangular window but the width of the main lobe has increased.

notes

summary

9m 59s

