

Course material

Course:

Digital Signal Processing II

Video:

2.2.2.b Region of convergence and stability

Concepts (extracted from automatically generated subtitles):

Z-transform. Transfer function of the system. Convergence of the power series. Region of convergence. Z-transform of the impulse response. Partial fraction decomposition. Poles of the system. Phase term $\angle$. Infinite support sequence. Transfer functions. Complex plane. Impulse response. Rational transfer function. General convergence properties of the z-transform. Sum of the absolute values of the terms.



[to video sequence search](#)
(within Digital Signal Processing II.)



[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

ROC is defined by the absolute convergence of the power series:

$$z \in \text{ROC}\{X(z)\} \iff \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

How can we determine the ROC?

- ▶ ROC depends on the values of $x[n]$
- ▶ for rational transfer function we can use indirect methods
- ▶ we don't care about convergence in zero and infinity

20

In the previous lecture we have used the Z-transform liberally and we have defined

notes

summary

0m 0s



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20

the transfer function of the system as the Z-transform of the impulse response. But the impulse response we have seen can be an infinite support sequence and so the Z-transform of such a sequence is an infinite power series and we need to be concerned with the convergence of the power series. So we define the region of convergence for a Z-transform as the set of values for the variable z for which the Z-transform converges absolutely, namely for which the sum for n that goes

notes

summary

0m 1s



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20

to minus infinity to plus infinity of the absolute value of x of n times z to the minus n is less than infinity. The question is how do we determine the

notes

summary

0m 34s



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How can we determine the ROC?

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20

region of convergence? Clearly the region of convergence depends on the values of the sequence that we're taking the transform of. But remember that we're interested in transfer functions for systems and we're interested in not having to compute impulse response explicitly. So we will see that for rational transfer function we can use indirect methods to determine the region of convergence for the impulse response. We should also remark that we're not interested in convergence at zero or infinity. These are limit points that do not affect the general convergence properties of the Z-transform. With that in mind the first observation that we can make is that if we have a finite

notes

summary

0m 45s



observation #1:

for finite-support signals, the z-transform converges everywhere (except in 0 and/or ∞)

$$X(z) = \sum_{n=-M}^N x[n]z^{-n}$$


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support signal, which in the case of a transfer function means that the system is FIR, the Z-transform converges everywhere because the Z-transform is actually a finite degree polynomial in Z as we can see from the expression for the Z-transform where the summation goes from some integer big M to some other

notes

summary

1m 27s



observation #1:

for finite-support signals, the z-transform converges everywhere (except in 0 and/or ∞)

$$X(z) = \sum_{n=-M}^N x[n]z^{-n}$$


21

integers big N. So for finite support signals the region of convergence is

notes

summary

1m 47s



observation #2:

the region of convergence has circular symmetry: set $z = ae^{j\theta}$:

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty \iff \sum_{n=-\infty}^{\infty} |x[n]| |a^{-n}| < \infty$$

22

everywhere. The second observation is that the region of convergence will

notes

summary

1m 49s



observation #2:

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22

always have circular symmetry. Indeed set Z in polar coordinates like so. Z is equal to some magnitude a times a phase term e to the j theta. If you remember that the region of convergence is defined by the absolute convergence of the power series and replace the value for Z in polar coordinate in this summation here, you can see that the convergence depends only on the magnitude of Z and therefore the region of convergence will have circular

notes

summary

2m 1s



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22

symmetry, will not depend on the phase of its argument. The third observation is

notes

summary

2m 32s



observation #3:

$$x[n] = 0 \quad n < 0$$

for causal sequences, the ROC extends from a circle to infinity:
assume $z_0 \in \text{ROC}$ and $|z_1| > |z_0|$:

$$\sum_{n=0}^{\infty} |x[n] z_1^{-n}| = \sum_{n=0}^{\infty} \frac{|x[n]|}{|z_1^n|} \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z_0^n|} \leq \infty$$

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that for a causal sequence, namely a sequence such that $x[n] = 0$ for $n < 0$, for a causal sequence the region of convergence will

notes

summary

2m 40s



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23

extend from a circle to infinity. Why from a circle? Well because we have circular symmetry. And why to infinity? Well assume that we have a point z_0 on the complex plane that already belongs to the region of convergence. Now let's pick any other point on the plane that is larger than z_0 in magnitude.

notes

summary

2m 50s



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23

Let's now see if the Z transform is absolutely convergence in Z1. We write out the sum of the absolute values of the terms in the Z transform summation

notes

summary

3m 7s



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23

and we notice that the powers of Z, because of the causality of the sequence,

notes

summary

3m 20s



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assume $z_0 \in \text{ROC}$ and $|z_1| > |z_0|$:

$$\sum_{n=0}^{\infty} |x[n] z_1^{-n}| = \sum_{n=0}^{\infty} \frac{|x[n]|}{|z_1^n|} \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z_0^n|} < \infty$$

23

are only negative. So we can write a sum that goes from zero to plus infinity of the values of the sequence in absolute value divided by the absolute value of z_1 to the power of N . Now each term in the sum is less than or equal to

notes

summary

3m 29s



observation #3:

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for causal sequences, the ROC extends from a circle to infinity:
assume $z_0 \in \text{ROC}$ and $|z_1| > |z_0|$:

$$\sum_{n=0}^{\infty} |x[n] z_1^{-n}| = \sum_{n=0}^{\infty} \frac{|x[n]|}{|z_1^n|} \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z_0^n|} \leq \infty$$

23

this term here, which is the corresponding term in the Z transform

notes

summary

3m 41s





notes

3m 49s



so where are the convergence problems?

in general, difficult question; but we're only interested in rational transfer functions!

25

somewhere on the complex plane which could be shrunk down just to the origin if the convergence is everywhere and the convergence will be everywhere outside of this circular boundary. Where are the convergence problems? In general this is a difficult question but we're only interested in rational transfer

notes

summary

4m 1s



so where are the convergence problems?

in general, difficult question; but we're only interested in rational transfer functions!

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functions and for this case the problem is simpler. Remember the transfer

notes

summary

4m 22s



Consider the transfer function for an LTI system:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

It can always be factored as:

$$H(z) = b_0 \frac{\prod_{n=1}^M (1 - z_n z^{-1})}{\prod_{n=1}^N (1 - p_n z^{-1})}$$

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function for an LTI system is a ratio of finite degree polynomials like so and a rational transfer function like this can always be factored as such. We find all the roots of the numerator and all the roots of the denominator and then we

notes

summary

4m 27s



Consider the transfer function for an LTI system:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

It can always be factored as:

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↑
poles

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transform this as a product of monomial terms where each term captures one root per term. The roots of the denominator are called the poles of the system whereas the roots of the numerator are called the zeros of the system. Clearly

notes

summary

4m 44s



Consider the transfer function for an LTI system:

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It can always be factored as:

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} zero

↑
poles

26

the only trouble spots for the region of convergence are the poles of the system

notes

summary

5m 0s



- ▶ z_n 's: *zeros* of the transfer function
- ▶ p_n 's: *poles* of the transfer function
- ▶ only trouble spots for ROC are the poles

because that's where the rational transfer function explodes because the denominator goes to zero. So we know that the region of convergence will extend

notes

summary

5m 4s



We know:

- ▶ ROC extends outwards
- ▶ ROC cannot include poles

ROC extends outwards from a circle touching the largest-magnitude pole

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outwards for a causal sequence and we assume to have a causal system so the impulse response will be causal. We also know that the region of convergence cannot include the poles so it's reasonable to assume that for a causal

notes

summary

5m 14s



We know:

- ▶ ROC extends outwards
- ▶ ROC cannot include poles

ROC extends outwards from a circle touching the largest-magnitude pole

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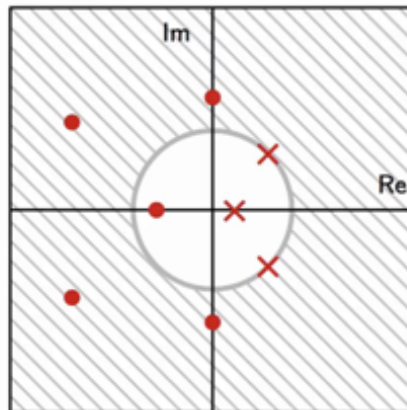
system the region of convergence will extend outwards from circle that is just touching the largest pole in magnitude. Let's have an example. Suppose

notes

summary

5m 28s





that you factor your transfer function and you find that it has five zeros and three poles and graphically the convention is to represent the zeros as dots on the complex plane and the poles as crosses. So these poles that are complex conjugate and we will see why in a second are the largest poles in magnitude. So the region of convergence for the system we posit is this shaded area here that extends outwards from the poles that have the largest magnitude. Now we have to prove this and the proof is a bit tedious but we will sketch it

notes

summary

5m 39s



- ▶ $H(z) = B(z)(1/A(z))$
- ▶ with no loss of generality, assume $B(z) = 1$
- ▶ assume all poles distinct
- ▶ use partial fraction decomposition:

$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

and give the intuition. So we start with the generic transfer function we call b of z the numerator and a of z the denominator. Now with no loss of

notes

summary

6m 19s



- ▶ $H(z) = \mathcal{B}(z)(1/A(z))$
- ▶ with no loss of generality, assume $B(z) = 1$
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$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

generality we can assume that the numerator is equal to one so we assume we have a pure feedback system. Why is that? Because we can always decompose the feed forward from the feedback part. This is the cascade of two systems and the

notes

summary

6m 28s





$$H(z) = B(z) \left(1/A(z) \right)$$

▶ with no loss of generality, assume $B(z) = 1$

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$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

first system is a purely feed forward system. We have seen that there are no convergence problems for this kind of systems. The second system is where the

notes

summary

6m 45s



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convergence problem can happen. So we will study the convergence for a transfer function of the form 1 over a of z and we will assume further that all

notes

summary

6m 54s



- ▶ $H(z) = B(z)(1/A(z))$
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the poles in this transfer function are distinct. We can use then partial fraction decomposition to go from a factored expression for the transfer function which is the product of monomial terms each one of which

notes

summary

7m 3s



- ▶ $H(z) = B(z)(1/A(z))$
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- ▶ use partial fraction decomposition:

$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$



captures a pole to a sum of monomial terms again where each term captures one

notes

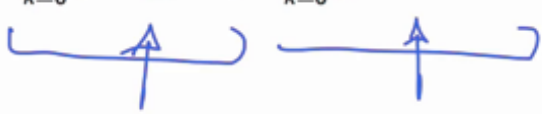
summary

7m 22s



- ▶ $H(z) = B(z)(1/A(z))$
- ▶ with no loss of generality, assume $B(z) = 1$
- ▶ assume all poles distinct
- ▶ use partial fraction decomposition:

$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$



pole but we go from a product to a sum. Partial fraction decomposition is a standard algebraic procedure and let's have a simple example just to refresh your high school memories. Suppose we start with this fraction here 1 over 1

notes

summary

7m 28s



ROC for causal systems - Proof (sketch)



Example:

$$\frac{1}{1 - 5z^{-1} + 6z^{-2}} = \frac{1}{(1 - 2z^{-1})(1 - 3z^{-1})}$$
$$= \frac{c_0}{1 - 2z^{-1}} + \frac{c_1}{1 - 3z^{-1}}$$

$$c_0 + c_1 = 1$$

$$2c_0 + 3c_1 = 0$$

$$\frac{1}{1 - 5z^{-1} + 6z^{-2}} = \frac{-2}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}$$

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minus 5 z to the minus 1 plus 6 z to the minus 2. If I find the roots of this

notes

summary

7m 44s



ROC for causal systems - Proof (sketch)



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$$\frac{1}{1 - 5z^{-1} + 6z^{-2}} = \frac{-2}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}$$

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polynomial they are 2 and 3 and so I can factor this as the product of 1 over 1

notes

summary

7m 49s



Example:

$$\frac{1}{1 - 5z^{-1} + 6z^{-2}} = \frac{1}{(1 - 2z^{-1})(1 - 3z^{-1})}$$

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$$\frac{1}{1 - 5z^{-1} + 6z^{-2}} = \frac{-2}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}$$

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minus 2 times z to the minus 1 and multiplies 1 over 1 minus 3 z to the minus 1 and I want to express this as the sum of two terms with the same roots as before. So these are two placeholder coefficients and I need to find c0 and c1.

notes

summary

8m 1s



ROC for causal systems - Proof (sketch)



Example:

$$\frac{1}{1 - 5z^{-1} + 6z^{-2}} = \frac{1}{(1 - 2z^{-1})(1 - 3z^{-1})}$$
$$= \frac{c_0}{1 - 2z^{-1}} + \frac{c_1}{1 - 3z^{-1}}$$

$$c_0 + c_1 = 1$$

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$$\frac{1}{1 - 5z^{-1} + 6z^{-2}} = \frac{-2}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}$$

32

How do I do that? Well I compute the sum and I get a system of equations that relates c_0 and c_1 and if I solve the system of equation I find that c_0 is

notes

summary

8m 17s



$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

- ▶ remember the leaky integrator...
- ▶ each term corresponds to an exponential sequence:

$$\mathcal{Z}\{p_k^n u[n]\} = \frac{1}{(1 - p_k z^{-1})}$$

- ▶ the ROC for each term is $|z| > |p_k|$
- ▶ intersection of all ROCs is $|z| > |p_{\max}|$

equal to minus 2 and c1 is equal to 3. So the original fraction has been decomposed into the sum of two first-order terms and each term captures one of the poles in the original fraction. So back to our transfer function we said our transfer function is of the form 1 over a of z. We

notes

summary

8m 30s



$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

$\frac{1}{A(z)}$

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33

product to a sum of terms. We will need to find the c_k coefficients but this is really immaterial at this point. Now each term here corresponds to an exponential

notes

summary

8m 50s



$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

- ▶ remember the leaky integrator...
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- ▶ the ROC for each term is $|z| > |p_k|$
- ▶ intersection of all ROCs is $|z| > |p_{\max}|$

$$\frac{1-\lambda}{1-\lambda z^{-1}} \quad \lambda^n u[n]$$

sequence. If you remember the leaky integrator, the leaky integrator had a transfer function of the form $1 - \lambda$ divided by $1 - \lambda z^{-1}$ and the corresponding impulse response was of the form $\lambda^n u[n]$ times $1 - \lambda$. That's again not important. So a

notes

summary

9m 2s



$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

- ▶ remember the leaky integrator...
- ▶ each term corresponds to an exponential sequence:

$$\frac{1-\lambda}{1-\lambda z^{-1}} \quad \lambda^n u[n]$$

$$\mathcal{Z}\{p_k^n u[n]\} = \frac{1}{(1 - p_k z^{-1})}$$

- ▶ the ROC for each term is $|z| > |p_k|$
- ▶ intersection of all ROCs is $|z| > |p_{\max}|$

33

first-order term with a single pole corresponds in the time domain to an

notes

summary

9m 23s



$$\underline{H(z)} = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

- ▶ remember the leaky integrator...
- ▶ each term corresponds to an exponential sequence:

$$\frac{1-\lambda}{1-\lambda z^{-1}} \quad \lambda^n u[n]$$

$$\mathcal{Z}\{p_k^n u[n]\} = \frac{1}{(1 - p_k z^{-1})}$$

- ▶ the ROC for each term is $|z| > |p_k|$
- ▶ intersection of all ROCs is $|z| > |p_{\max}|$

33

exponential sequence and we now have decomposed the entire transfer function into a sum of first-order terms and so each one of these terms will correspond to a distinct exponential sequence where the ratio of the exponential sequence, the lambda in the leaky integrator if you will, corresponds to the value of

notes

summary

9m 30s



$$\underline{H(z)} = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

- ▶ remember the leaky integrator...
- ▶ each term corresponds to an exponential sequence:

$$\frac{1-\lambda}{1-\lambda z^{-1}} \quad \underline{\lambda^n u[n]}$$

$$\mathcal{Z}\{p_k^n u[n]\} = \frac{1}{(1 - p_k z^{-1})}$$

- ▶ the ROC for each term is $|z| > |p_k|$
- ▶ intersection of all ROCs is $|z| > |p_{\max}|$

33

the pole. Now the region of convergence for each of these terms is z larger than

notes

summary

9m 48s



$$H(z) = \prod_{k=0}^{N-1} \frac{1}{(1 - p_k z^{-1})} = \sum_{k=0}^{N-1} \frac{c_k}{(1 - p_k z^{-1})}$$

- ▶ remember the leaky integrator...
- ▶ each term corresponds to an exponential sequence:

$$\mathcal{Z}\{p_k^n u[n]\} = \frac{1}{(1 - p_k z^{-1})}$$

- ▶ the ROC for each term is $|z| > |p_k|$
- ▶ intersection of all ROCs is $|z| > |p_{\max}|$

33

the value of the pole in magnitude. This is a standard convergence test for the geometric series and so the intersection of all the region of convergence will be z greater than the largest pole in magnitude. The general proof becomes a

notes

summary

9m 49s



- ▶ same for multiple poles, just more tedious

$$\mathcal{Z}\{np^n u[n]\} = \frac{pz^{-1}}{(1 - pz^{-1})^2}, \quad \text{ROC: } |z| > |p|$$

- ▶ all LTI impulse responses are linear combinations of weighed exponential sequences
- ▶ we could use an inverse z-transform to obtain $h[n]$

bit more tedious when we have to take into account poles that have a multiplicity larger than 1 but the gist remains the same. We can show that for instance a z-transform of the form $p z^{-1}$ to the minus 1 divided by $1 - p z^{-1}$ to

notes

summary

10m 14s



- ▶ same for multiple poles, just more tedious

$$\mathcal{Z}\{np^n u[n]\} = \frac{pz^{-1}}{(1 - \underbrace{pz^{-1}})^2}, \quad \text{ROC: } |z| > |p|$$

- ▶ all LTI impulse responses are linear combinations of weighed exponential sequences
- ▶ we could use an inverse z-transform to obtain $h[n]$

the minus 1 squared corresponds in the time domain to a weighed exponential sequence of the form n times p to the power of n times the step sequence and

notes

summary

10m 26s



- ▶ same for multiple poles, just more tedious

$$\mathcal{Z}\{np^n u[n]\} = \frac{pz^{-1}}{(1 - pz^{-1})^2}, \quad \text{ROC: } |z| > |p|$$

- ▶ all LTI impulse responses are linear combinations of weighed exponential sequences
- ▶ we could use an inverse z-transform to obtain $h[n]$

for this kind of weighed exponential sequences the region of convergence is once again z larger than p where p is the value of the pole. So the partial fraction expansion becomes a bit more cumbersome but the proof remains the same. What we have just shown is that all LTI impulse responses are linear

notes

summary

10m 40s



- ▶ same for multiple poles, just more tedious

$$\mathcal{Z}\{np^n u[n]\} = \frac{pz^{-1}}{(1 - pz^{-1})^2}, \quad \text{ROC: } |z| > |p|$$

- ▶ all LTI impulse responses are linear combinations of weighed exponential sequences
- ▶ we could use an inverse z-transform to obtain $h[n]$

combinations of weighed exponential sequences and the ratio of these exponential sequences is given by the poles of the system. We could use this to invert the z-transform and compute the impulse response explicitly but this is hardly ever useful in the sense that we don't need the impulse response to compute the output of a filter. We have the CCDE that describes the filter algorithmically. The fact that we can determine the region of convergence of a

notes

summary

11m 3s



Consider a filter with impulse response $h[n]$

► BIBO stability $\iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

► $1 \in \text{ROC} \iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$ because $H(z)$ converges absolutely in $z = 1$

a causal system is stable if and only if ROC includes the unit circle!

35

filter without having to explicitly compute the impulse response provides us also with a very useful and simple stability criterion for a filter. You remember from a few lectures ago that a necessary and sufficient condition for a

notes

summary

11m 32s



Consider a filter with impulse response $h[n]$

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a causal system is stable if and only if ROC includes the unit circle!

35

filter to be stable is that the impulse response is absolutely summable. Now if the point z equal to 1 belongs to the region of convergence of the z -transform of a causal impulse response h of n that means that the impulse response is absolutely summable. So a necessary and sufficient condition for the stability of a filter is at the point z equal to 1 belongs to the region of convergence. Now because the region of convergence is circularly symmetric this is equivalent

notes

summary

11m 49s



Consider a filter with impulse response $h[n]$

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35

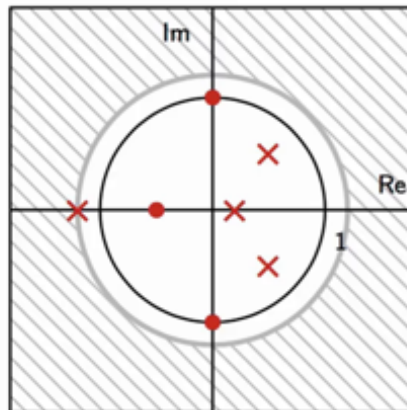
to saying that the whole unit circle belongs to the region of convergence. Another way to look at this is since the region of convergence for a causal filter extend outwards from a circle that touches the largest pole in

notes

summary

12m 18s





magnitude, a necessary and sufficient condition for a filter to be stable is that all the poles of the filter lay inside the unit circle. So here for instance you have an example where the poles are inside the unit circle, the region of convergence includes the unit circle and so the filter is stable. If one of the poles was outside of the unit circle then the unit circle will not be part of the region of convergence and the filter would be unstable. Of course for filters that don't have any poles the region of convergence always includes the unit circle because it corresponds to the entire complex plane and this confirms once again the stability of FIR filters. Before we conclude we should

notes

summary

12m 28s



$$y[n] = 2y[n-1] + x[n] \quad (\text{unstable!})$$

apply z-transform as a formal operator:

$$H(z) = Y(z)/X(z) = \frac{1}{1 - 2z^{-1}}$$

$$H(1) = -1 < \infty, \text{ so system is stable?}$$

38

clear a common confusion that mistakes the existence of the formal transfer function in z equal to 1 with the stability condition. And for that let's consider this simple constant coefficient different equation. This represents an unstable system. You can see that at each iteration the output gets multiplied by 2 so the impulse response is an exponentially growing sequence with ratio equal to 2. We can still apply the Z-transform as a purely

notes

summary

13m 14s



$$y[n] = 2y[n-1] + x[n] \quad (\text{unstable!})$$

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38

formal operator without worrying about the region of convergence to the above

notes

summary

13m 46s



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38

CCDE and we get a transfer function of the form 1 over 1 minus 2 times z to the minus 1. Now this function exists for z equal to 1 so is the system stable or

notes

summary

13m 49s



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38

unstable? The mistake was to derive the Z-transform of the system and identifying it with the Z-transform of the impulse response without taking the region of convergence into account. The region of convergence is independent of the actual value of the formal transfer function. If we go back to the example

notes

summary

14m 2s



ROC depends on $h[n]$, NOT on *formal* value of $H(z)$:

- ▶ $h[n] = 2^n u[n]$
- ▶ to apply the z-transform operator we *assume* to be in the ROC
- ▶ the region of convergence is $|z| > 2$ because

$$\sum_{n=0}^{\infty} a^n z^{-n} = \lim_{N \rightarrow \infty} \frac{1 - (a/z)^{N+1}}{1 - (a/z)} = \begin{cases} \frac{1}{1 - az^{-1}} & \text{if } |z| > |a| \\ \infty & \text{otherwise} \end{cases}$$

39

the impulse response for the previous unstable system is of the form 2 to the power of n times u of n. Now to apply the Z-transform operator we must assume to be in the region of convergence which for this kind of sequence here is z larger than 2 in magnitude. And so z equal to 1 does not belong to the regional convergence for H of z although the formal expression for H of z is defined in z equal to 1. This is easy to see if you write out explicitly the Z-transform of an exponential sequence. This is just a standard application of

notes

summary

14m 24s



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the sum of a geometric series and this is equal to the limit for the range of the sum that goes to infinity of this ratio here. Now this converges only if z

notes

summary

15m 0s



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39

is larger than a in magnitude otherwise this guy blows up. So in order to apply

notes

summary

15m 12s



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39

the Z-transform and obtain something that is actually the Z-transform of the impulse response we must be in the region of convergence. Now the function

notes

summary

15m 13s



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