



Course material

Course:

Digital Signal Processing II

Video:

2.2.4.c FIR design

Concepts (extracted from automatically generated subtitles):

Fir filters. Case of fir filters. Fir design methods. Center tap. Impulse response. Types of filters. Optimal fir filters. Minus n. Auxiliary filter h. Equi-ripple error. Parks-mclellan design algorithm. Linear phase. Good way. Fourier transform. Odd number of taps.



[to video sequence search](#)
(within Digital Signal Processing II.)



[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

FIR: optimal minimax design

FIR filters are a digital signal processing "exclusivity".

In the 70s Parks and McClellan developed an algorithm to design optimal FIR filters:

- ▶ linear phase
- ▶ equiripple error in passband and stopband

algorithm proceeds by minimizing the maximum error in passband and stopband

Let's now look at FIR design methods. FIR filters, as we said, are a digital signal processing exclusivity, and so are the algorithms to design them. In the case of FIR filters, the rational transfer function is no longer a ratio of polynomials, but just one polynomial in Z . So optimization procedures do exist that can achieve optimal results for a given set of specifications. In the 70s, Parks and McClellan developed an algorithm to design optimal FIR filters that yields filters with linear phase and that have equi-ripple error both in passband and stopband.

notes

summary

0m 0s



FIR: optimal minimax design

FIR filters are a digital signal processing "exclusivity".

In the 70s Parks and McClellan developed an algorithm to design optimal FIR filters:

- ▶ linear phase
- ▶ equiripple error in passband and stopband

algorithm proceeds by **minimizing** the **maximum** error in passband and stopband

The algorithm proceeds by minimizing the maximum error in passband and stopband of the transfer function. Linearity of the phase is achieved by designing an impulse response which is either symmetric or antisymmetric.

notes

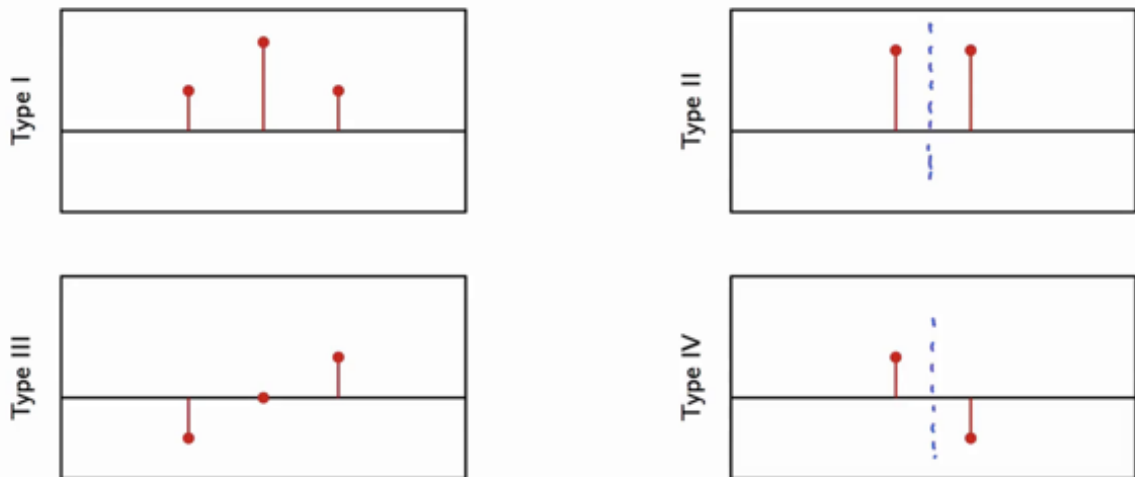
summary

0m 36s



Linear phase

Linear phase derives from a symmetric or antisymmetric impulse response



We end up with four types of filters according to whether the impulse response is even length or odd length, and symmetric or antisymmetric. Type 1 filters, probably the most common, have an odd length impulse response, and they are symmetric around the center tap. Type 3 filters have an odd number of taps, and they are antisymmetric around the center tap, which imposes, of course, that the center tap is zero. Type 2 and type 4 filters are symmetric and antisymmetric filters respectively, both of which have an even number of taps. That means that the center of symmetry of these filters falls in between samples, and so they both introduce a non-integer linear phase factor of one half sample. Let's look in more detail at the phase properties of a type 1 FIR filter.

notes

summary

0m 46s



Linear phase (Type I)

$$h[C + n] = h[C - n]$$



$$h'[n] = h[n + C]$$

$$h'[n] = h'[-n]$$

$$H(z) = z^{-C} H'(z)$$

A type 1 filter is an odd length symmetric FIR, so it will look maybe like this, where the taps are symmetric around an index capital C. The symmetry relationship can be simplified if we shift the filter so that the center tap falls in zero. So we define an auxiliary filter H' of n , which is simply H of n plus capital C , and this filter now will be centered in zero and symmetric around zero.

notes

summary

1m 38s



Linear phase (Type I)

$$h[C + n] = h[C - n]$$

$$h'[n] = h[n + C]$$

$$h'[n] = h'[-n]$$

$$H(z) = z^{-C} H'(z)$$



We can express that by saying that H' of n is equal to H' of minus n .

notes

summary

2m 3s



Linear phase (Type I)

$$H'(z) = \sum_{n=-M}^M h'[n]z^{-n}$$
$$= h'[0] + \sum_{n=1}^M h'[n](z^n + z^{-n})$$

$$H'(e^{j\omega}) = h'[0] + \sum_{n=1}^M h'[n](e^{j\omega n} + e^{-j\omega n})$$
$$= h'[0] + 2 \sum_{n=1}^M h'[n] \cos \omega n \in \mathbb{R}$$

The relationship between H' and H in the Z domain is simply multiplication by Z to the minus capital C . If we now compute the Z transform of H' of n , we have the sum for n that goes from minus capital M to M of H' of n times Z to the minus n .

notes

summary

2m 18s



Linear phase (Type I)

$$\begin{aligned} H'(z) &= \sum_{n=-M}^M h'[n]z^{-n} \\ &= h'[0] + \sum_{n=1}^M h'[n](z^n + z^{-n}) \end{aligned}$$

$$\begin{aligned} H'(e^{j\omega}) &= h'[0] + \sum_{n=1}^M h'[n](e^{j\omega n} + e^{-j\omega n}) \\ &= h'[0] + 2 \sum_{n=1}^M h'[n] \cos \omega n \in \mathbb{R} \end{aligned}$$

Because of the symmetry, we can isolate the center tap, H' of zero, and then write the rest of the summation by collecting equal terms.

notes

summary

2m 36s



Linear phase (Type I)

$$\begin{aligned} H'(z) &= \sum_{n=-M}^M h'[n]z^{-n} \\ &= \underbrace{h'[0]} + \underbrace{\sum_{n=1}^M h'[n](z^n + z^{-n})} \\ H'(e^{j\omega}) &= h'[0] + \sum_{n=1}^M h'[n](e^{j\omega n} + e^{-j\omega n}) \\ &= h'[0] + 2 \sum_{n=1}^M h'[n] \cos \omega n \in \mathbb{R} \end{aligned}$$

And so we have the sum for n that goes from 1 to capital M of H' of n that multiplies Z to the n plus Z to the minus n.

notes

summary

2m 38s



Linear phase (Type I)

$$H'(z) = \sum_{n=-M}^M h'[n]z^{-n}$$

$$= \underbrace{h'[0]} + \underbrace{\sum_{n=1}^M h'[n](z^n + z^{-n})} \quad ?$$

$$H'(e^{j\omega}) = h'[0] + \sum_{n=1}^M h'[n](e^{j\omega n} + e^{-j\omega n})$$

$$= h'[0] + 2 \sum_{n=1}^M h'[n] \cos \omega n \in \mathbb{R}$$

If we now replace z with $e^{j\omega}$, we get the Fourier transform of H' , which is simply H' of zero plus this sum for n that goes from 1 to capital M of H' of n that multiplies $e^{j\omega n}$ plus $e^{-j\omega n}$.

notes

summary

2m 49s



Linear phase (Type I)

$$H'(z) = \sum_{n=-M}^M h'[n]z^{-n}$$
$$= \underbrace{h'[0]} + \underbrace{\sum_{n=1}^M h'[n](z^n + z^{-n})}$$

$z = e^{j\omega}$

$$H'(e^{j\omega}) = h'[0] + \sum_{n=1}^M h'[n](e^{j\omega n} + e^{-j\omega n})$$
$$= h'[0] + 2 \sum_{n=1}^M h'[n] \cos \omega n \in \mathbb{R}$$

But we know that this is simply twice the cosine of omega n, and so we obtain that the Fourier transform of H' of n is purely real.

notes

summary

3m 1s



Linear phase (Type I)

$$\begin{aligned} H'(z) &= \sum_{n=-M}^M h'[n]z^{-n} \\ &= h'[0] + \sum_{n=1}^M h'[n](z^n + z^{-n}) \end{aligned}$$

$$\begin{aligned} H'(e^{j\omega}) &= h'[0] + \sum_{n=1}^M h'[n](e^{j\omega n} + e^{-j\omega n}) \\ &= h'[0] + 2 \sum_{n=1}^M h'[n] \cos \omega n \in \mathbb{R} \end{aligned}$$

And this means that the Fourier transform of H' of n is zero-phase. We can get the Fourier transform of H of n as simply the zero-phase Fourier transform times a linear phase factor $e^{-j\omega n}$.

notes

summary

3m 4s



Linear phase (Type I)

$$H(e^{j\omega}) = \underbrace{\left[h[C] + 2 \sum_{n=1}^M h[n] \cos(n - C)\omega \right]}_{\in \mathbb{R}} \underbrace{e^{-j\omega C}}$$

So we have proven that type 1 filters do indeed possess linear phase.

notes

summary

3m 34s



Linear phase (Type I)

$$H(e^{j\omega}) = \underbrace{\left[h[C] + 2 \sum_{n=1}^M h[n] \cos(n - C)\omega \right]}_{\in \mathbb{R}} \underbrace{e^{-j\omega C}}$$

A similar proof can be carried out for the other three types of linear phase FIRs.

notes

summary

3m 44s



Minimax lowpass

Magnitude response:

- ▶ equiripple in passband and stopband

Design parameters:

- ▶ order N (number of taps)
- ▶ passband edge ω_p
- ▶ stopband edge ω_s
- ▶ ratio of passband to stopband error δ_p/δ_s

Test value:

- ▶ passband max error
- ▶ stopband max error

Let's now look at the Parks-McLellan design algorithm, also known as the MINIMAX filter design algorithm. The magnitude response of the filters designed by this algorithm is equi-ripple both in the passband and the stopband.

notes

summary

3m 49s



Minimax lowpass

Magnitude response:

- ▶ equiripple in passband and stopband

Design parameters:

- ▶ order N (number of taps) ✓
- ▶ passband edge ω_p ✓
- ▶ stopband edge ω_s
- ▶ ratio of passband to stopband error δ_p/δ_s

Test value:

- ▶ passband max error
- ▶ stopband max error

The design parameters are the order, which in this case corresponds to the number of taps. We also specify the passband edge, ω_p , and the stopband edge, ω_s .

notes

summary

4m 1s



Minimax lowpass

Magnitude response:

- ▶ equiripple in passband and stopband

Design parameters:

- ▶ order N (number of taps) ✓
- ▶ passband edge ω_p ✓
- ▶ stopband edge ω_s ✓
- ▶ ratio of passband to stopband error δ_p/δ_s ✓

Test value:

- ▶ passband max error
- ▶ stopband max error

So we specify the width of the transition band, and we specify the ratio of the passband to stopband error, which we call δ_p/δ_s . Note that we're not specifying the absolute error, which means we can weigh the importance of the error by privileging the passband or the stopband. We run the algorithm and we obtain a filter that we have to check against the maximum passband and stopband error specified in the requirements.

notes

summary

4m 18s



Minimax lowpass

Magnitude response:

- ▶ equiripple in passband and stopband

Design parameters:

- ▶ order N (number of taps)
- ▶ passband edge ω_p
- ▶ stopband edge ω_s
- ▶ ratio of passband to stopband error δ_p/δ_s

Test value:

- ▶ passband max error
- ▶ stopband max error

If the error in either band exceeds the requirements of the specs, then we have to increase the order and run the algorithm again.

notes

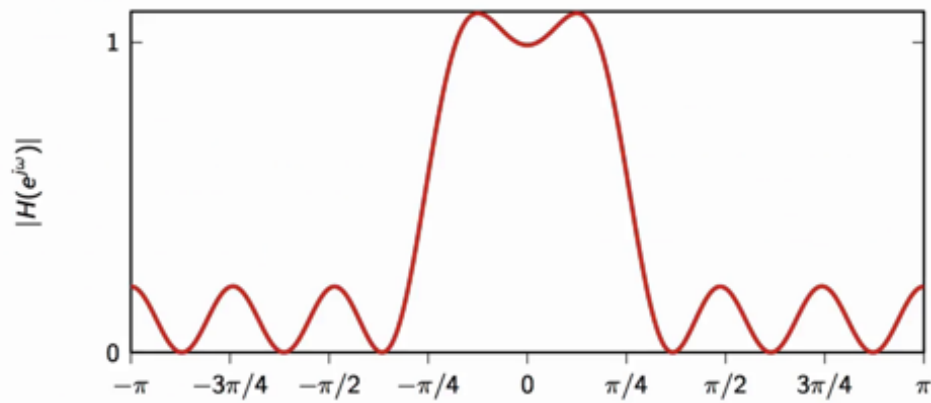
summary

4m 43s



Minimax lowpass example

$$N = 9, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 10$$



The frequency response of a MINIMAX filter looks like this.

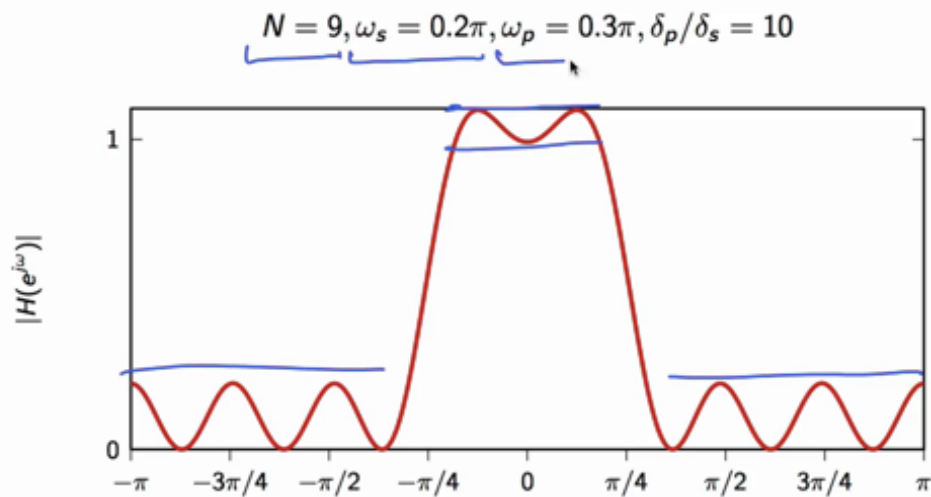
notes

summary

4m 49s



Minimax lowpass example



This is a typical equi-ripple characteristic, and you can see that the extrema of the frequency response all touch the edge of the tolerance regions, both in passband and stopband. This is designed for a 9-tap filter with a transition band between 0.2π and 0.3π . We can increase the number of taps to 19 for the same transition band, and we can see that we get steeper transitions and lower errors in passband and stopband.

notes

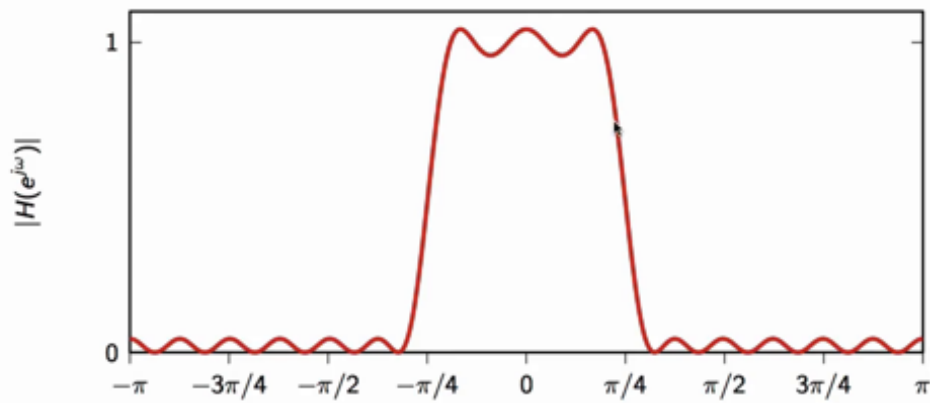
summary

5m 1s



Minimax lowpass example

$$N = 19, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 10$$



And we can go up to, say, 51 taps.

notes

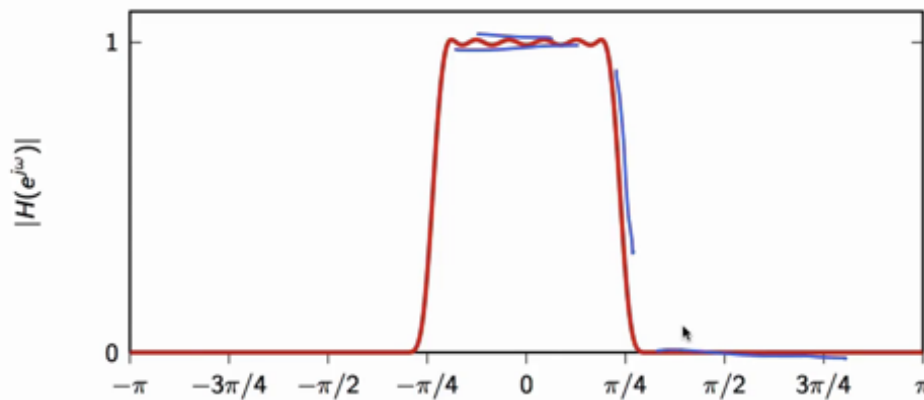
summary

5m 23s



Minimax lowpass example

$$N = 51, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 1$$



We get something that is actually quite good with very small error in passband, almost undetectable error in stopband, and steep transitions. A good way to compare the performance of different filters is to express their magnitude in decibels.

notes

summary

5m 30s



Magnitude response in decibels

- ▶ filter max passband magnitude G
- ▶ filter attenuation expressed in decibels as:

$$A_{\text{dB}} = 20 \log_{10}(|H(e^{j\omega})|/G)$$

- ▶ useful to compare attenuations between filters

So if we say that G is the maximum magnitude in the passband, the attenuation of the filter expressed in decibels is 20 times the log in base 10 of the magnitude of the frequency response divided by the maximum gain.

notes

summary

5m 47s



Magnitude response in decibels

- ▶ filter max passband magnitude G
- ▶ filter attenuation expressed in decibels as:

$$A_{\text{dB}} = 20 \log_{10}(|H(e^{j\omega})|/G)$$

- ▶ useful to compare attenuations between filters

With this notation, we can plot the frequency response of the fourth-order elliptic low pass, and we can see that it has an attenuation of about minus 30 dBs in the stopband.

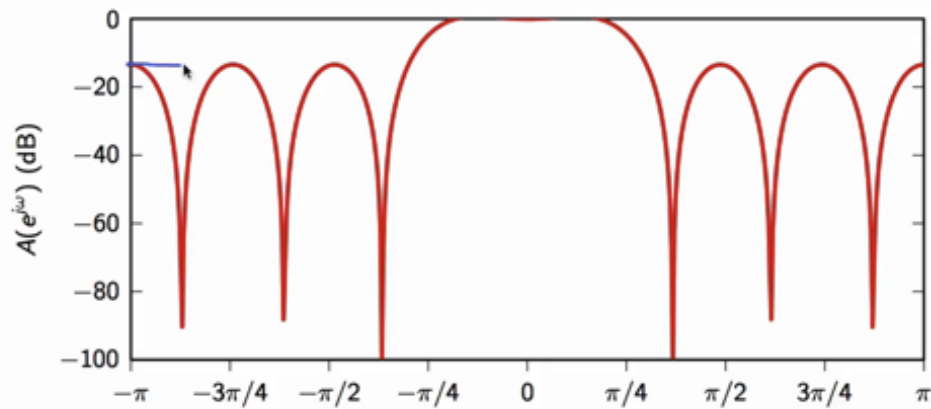
notes

summary

6m 1s



9-tap minimax lowpass, $\omega_c = \pi/4$, log scale



A comparable FIR filter, comparable in the sense of computational cost, is the 9-tap minimax, and we see that for that computational cost, we get an attenuation that is definitely less, because it's on the order of 10 dBs, approximately.

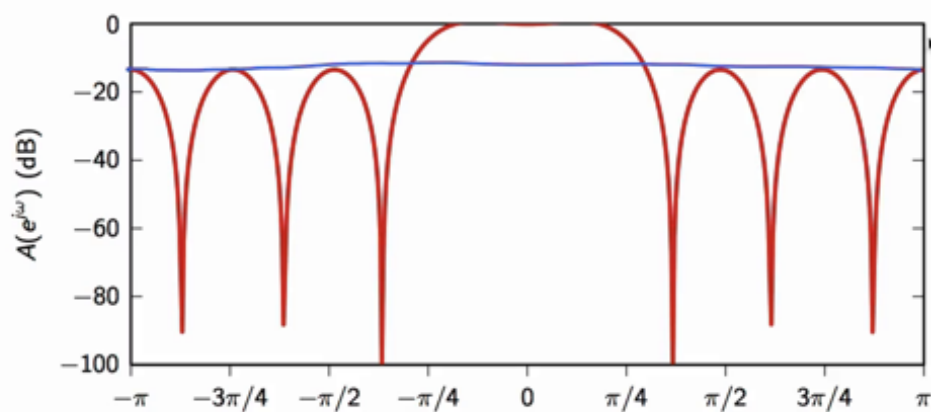
notes

summary

6m 11s



9-tap minimax lowpass, $\omega_c = \pi/4$, log scale



To get a performance comparable to the elliptic filter, we have to increase the length of the filter to 19 taps at least, and here we hit the approximate 30 dB mark.

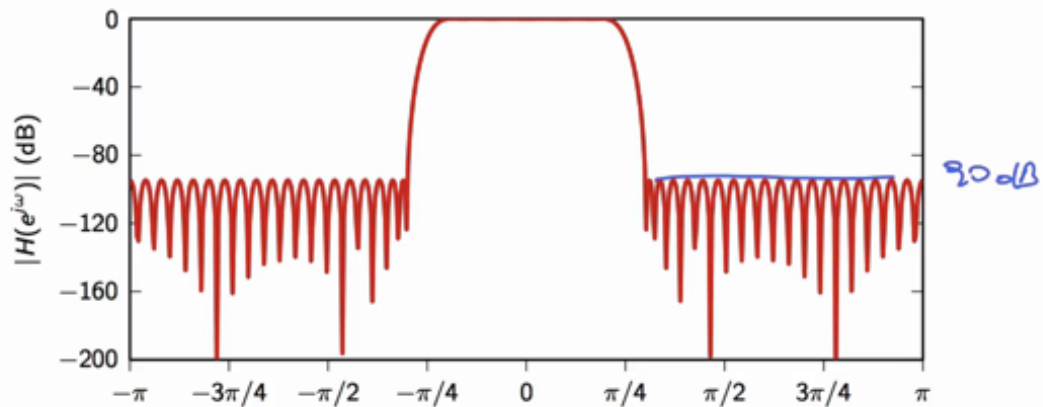
notes

summary

6m 34s



51-tap minimax lowpass, $\omega_c = \pi/4$, log scale



The 51-tap filter that we saw before has actually a very, very good attenuation. It's about 90 dBs. But of course, the price we pay is non-negligible in terms of the number of operations per output sample. Finally, as mentioned before, we just concentrated on low pass prototypes, but of course we can use any of these designs to obtain bandpass and high pass filters as well.

notes

summary

6m 37s



Life beyond lowpass

The IIR and FIR methods we just described can be used to design more general filter types than lowpass, with only minor modifications

- ▶ IIR bandpass and highpass can be obtained by modulating the lowpass response
- ▶ optimal FIR bandpass and highpass can be designed by the Parks-McClellan algorithm
- ▶ optimal FIR can also be designed with piecewise linear magnitude response
- ▶ the literature on filter design is vast: this is just the tip of the iceberg!

Optimal FIR bandpass and high pass filters can be designed directly with the Parks-McLellan algorithm.

notes

summary

7m 12s



Life beyond lowpass

The IIR and FIR methods we just described can be used to design more general filter types than lowpass, with only minor modifications

- ▶ IIR bandpass and highpass can be obtained by modulating the lowpass response
- ▶ optimal FIR bandpass and highpass can be designed by the Parks-McClellan algorithm
- ▶ optimal FIR can also be designed with piecewise linear magnitude response
- ▶ the literature on filter design is vast: this is just the tip of the iceberg!



You can also design optimal FIR with piecewise linear magnitude responses, so something that, for instance, goes like this. The literature on filter design is really, really vast, and we just scratched the surface.

notes

summary

7m 26s



Life beyond lowpass

The IIR and FIR methods we just described can be used to design more general filter types than lowpass, with only minor modifications

- ▶ IIR bandpass and highpass can be obtained by modulating the lowpass response
- ▶ optimal FIR bandpass and highpass can be designed by the Parks-McClellan algorithm
- ▶ optimal FIR can also be designed with piecewise linear magnitude response
- ▶ the literature on filter design is vast: this is just the tip of the iceberg!

It is a very fascinating topic that has occupied researchers for very many years and keeps them occupied to this day.

notes

summary

7m 42s

