

Course material

Course:

Digital Signal Processing II

Video:

2.2.4.d Fractional delay and Hilbert filter

Concepts (extracted from automatically generated subtitles):

Ideal filters. Frequency domain. Frequency response of the fractional delay. Hilbert filter. Fractional delay. $J\omega$. Fourier transform of this signal. Fractional part. Impulse response. $J\omega n$. Representation of the input spectrum. Real number d . Transfer function. Magnitude response. J y of n look.



[to video sequence search](#)
(within Digital Signal Processing II.)



[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

Overview

- ▶ the fractional delay
- ▶ the Hilbert filter

1

Hi, in this module we want to talk about a couple more ideal filters, namely the fractional delay and the Hilbert filter. These ideal filters we will use them later in a variety of applications and when I say use them I of course mean I will use an approximation of these filters and we have already seen a variety of ways in which we can approximate ideal filters. So we have seen the ideal low pass and the various transformations that we can use to turn that into a whole different set of ideal filters. Let's start with something that is related to the low pass and it's called the fractional delay. To understand what the fractional delay does let's consider a simple delay what we have

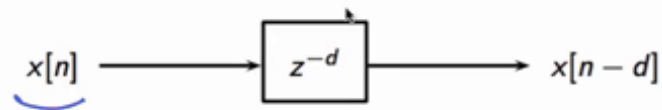
notes

summary

0m 0s



consider a simple delay...



$$H(e^{j\omega}) = e^{-j\omega d} \quad d \in \mathbb{Z}$$

2

seen I think in the first module of this class. We take an input signal X of n and we put it through a block that delays this input by a certain number of

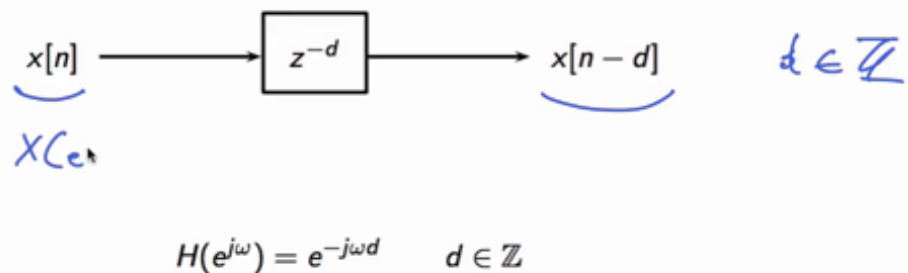
notes

summary

0m 41s



consider a simple delay...



2

samples d . So d now is an integer, right? So the output if we consider delay as a filter we have an input X of n and we have an output which is simply a delayed version of the input. If we consider this as a filter we can analyze the behavior of the delay in the frequency domain and we can derive the transfer function for the filter. This is very easy if the input is a generic sequence X of n and we indicate its DTFT as X of e to the j omega the output is the Fourier transform

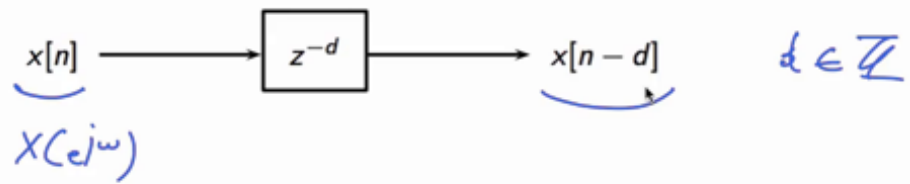
notes

summary

0m 50s



consider a simple delay...



$$H(e^{j\omega}) = e^{-j\omega d} \quad d \in \mathbb{Z}$$

2

of this signal here we apply the shift property of the Fourier transform and we

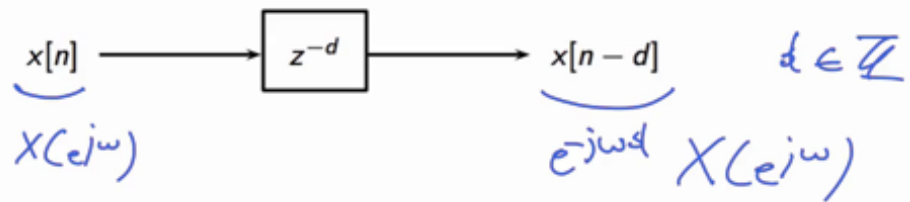
notes

summary

1m 22s



consider a simple delay...



$$H(e^{j\omega}) = e^{-j\omega d} \quad d \in \mathbb{Z}$$

2

find out that this is e to the minus j omega d time X of e to the j omega. And so from this relationship we can find the transfer function of the system as the output divided by the input and we get this formula here. So the transfer

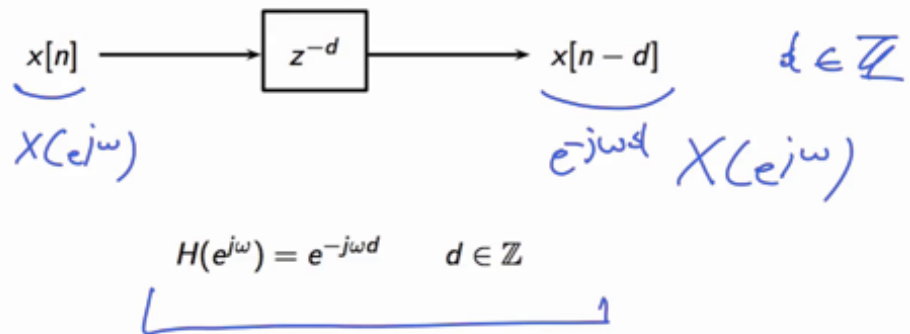
notes

summary

1m 27s



consider a simple delay...



2

function of a simple delay is $e^{-j\omega d}$. So we have said that

notes

summary

1m 43s



question

what happens if, in $H(e^{j\omega})$ we use a non-integer $d \in \mathbb{R}$?

for a standard delay d is an integer but in this transfer function formula here there is no requirement for d to be an integer. So the question is what happens if we replace d which up to now has been an integer number by a real number d ? Surprising as it may seem using this real quantity for the delay will result in what's called this fractional delay namely the filter with a known integer d will compute an output which is the input delayed by an integer number of samples plus a fractional part. Let's start by looking in more detail at the

notes

summary

1m 49s



impulse response

$$\begin{aligned}h[n] &= \text{IDTFT} \left\{ e^{-j\omega d} \right\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega d} e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-d)} d\omega \\&= \frac{1}{\pi(n-d)} \frac{e^{j\pi(n-d)} - e^{-j\pi(n-d)}}{2j} \\&= \frac{\sin \pi(n-d)}{\pi(n-d)} \\&= \text{sinc}(n-d)\end{aligned}$$

6

frequency response of the fractional delay. The magnitude response is identically one. This is of course the magnitude of a complex exponential which is one independent of frequency. So the filter can be classified as an all-pass filter. It doesn't alter the frequency distribution of the input which is consistent with what we would expect from a simple delay. The phase response is linear which once again is consistent with the response of the delay and the slope of this line will be proportional to the delay. The impulse response can be obtained by taking the inverse DTFT of the frequency response. So we take the integral from minus pi to pi of $e^{-j\omega d}$ times $e^{j\omega n}$ to the

notes

summary

2m 26s



impulse response

$$\begin{aligned}h[n] &= \text{IDTFT} \left\{ e^{-j\omega d} \right\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega d} e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-d)} d\omega \\&= \frac{1}{\pi(n-d)} \frac{e^{j\pi(n-d)} - e^{-j\pi(n-d)}}{2j} \\&= \frac{\sin \pi(n-d)}{\pi(n-d)} \\&= \text{sinc}(n-d)\end{aligned}$$

6

$j\omega n$ and if we work through the integral which is elementary we end up

notes

summary

3m 7s



impulse response

$$\begin{aligned}h[n] &= \text{IDTFT} \left\{ e^{-j\omega d} \right\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega d} e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-d)} d\omega \\&= \frac{1}{\pi(n-d)} \frac{e^{j\pi(n-d)} - e^{-j\pi(n-d)}}{2j} \\&= \frac{\sin \pi(n-d)}{\pi(n-d)} \\&= \text{sinc}(n-d)\end{aligned}$$

6

with an impulse response which is the ratio of sine of pi that multiplies n minus D divided by pi that multiplies n minus D. Now this function here you should recognize by now as a sinc function namely it's sinc of n minus D. Now remember that the sinc function is equal to zero for all integer values of its argument except when the argument is zero at which point the sinc is equal to one. Shape of the sinc if you want is like this so here it's equal to one and

notes

summary

3m 16s



impulse response

$$\begin{aligned}h[n] &= \text{IDTFT} \{e^{-j\omega d}\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega d} e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-d)} d\omega \\&= \frac{1}{\pi(n-d)} \frac{e^{j\pi(n-d)} - e^{-j\pi(n-d)}}{2j} \\&= \frac{\sin \pi(n-d)}{\pi(n-d)} \\&= \text{sinc}(n-d)\end{aligned}$$



6

we'll cross the x-axis for all integer values of the argument. So here if D is an integer as in the case of the classic delay that we have seen so far the sinc

notes

summary

3m 49s



impulse response

$$\begin{aligned}h[n] &= \text{IDTFT} \{e^{-j\omega d}\} \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega d} e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-d)} d\omega \\&= \frac{1}{\pi(n-d)} \frac{e^{j\pi(n-d)} - e^{-j\pi(n-d)}}{2j} \\&= \frac{\sin \pi(n-d)}{\pi(n-d)} \\&= \text{sinc}(n-d)\end{aligned}$$



6

collapses to a simple delta function. So for instance if D is equal to three you

notes

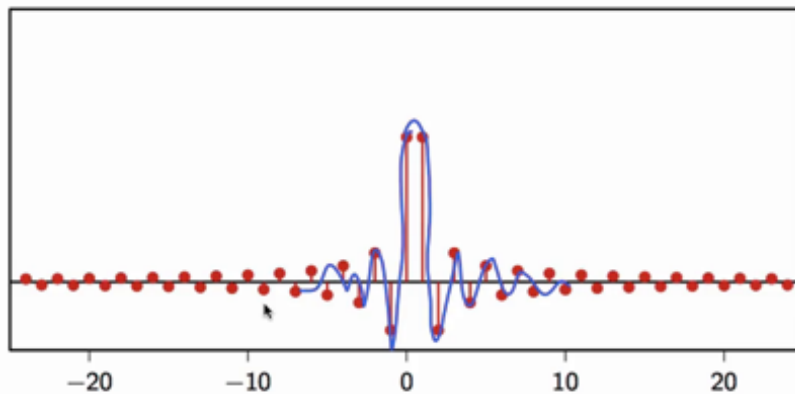
summary

4m 1s



Fractional delay: impulse response

$$d = 0.5$$



8

will have the sinc of n minus three will have a single nonzero value for n equal to three in which case it will be equal to one here and will be zero everywhere else. But if D is not an integer then the impulse response will have an infinite number of nonzero values and it will look like a sinc function. So here is for instance the case for D equal to 0.5 so a fractional delay of a half a sample and if you were to visualize the continuous version of the sinc function it would probably look like this. So it's something like this symmetric and so here we have the samples of this

notes

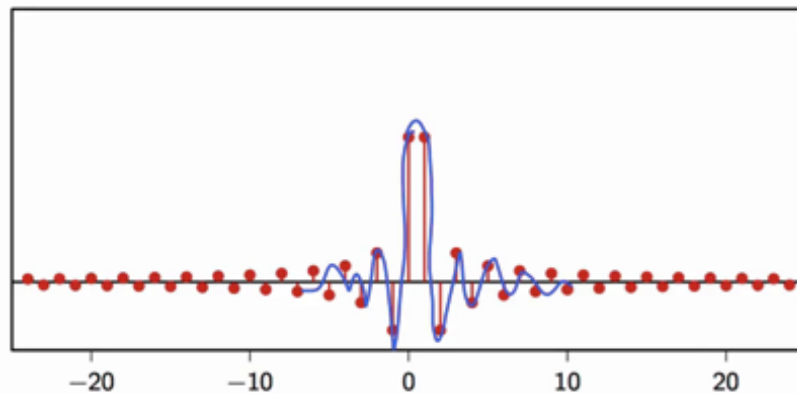
summary

4m 10s



Fractional delay: impulse response

$$d = 0.5$$



8

impulse response and you can see it's an ideal filter because once again you have an impulse response that is infinite and two-sided. We can say change the

notes

summary

4m 54s



the Hilbert filter

fractional delay value to 0.3 and you have a different shape for the impulse response. 0.1 would look like this. Now if you look at the shape of this impulse response you will see that the peak of the impulse response is in the vicinity of the integer part of the delay. So for instance here where the delay is 0.1 you have that the peak of the impulse response is in 0 and the rest of the impulse response will use all the remaining samples in the signal to build an intermediate value between the original samples. This is a little bit complicated to explain now. It will be very clear once we study the sampling theorem and the relationship between continuous time models of signals and discrete time models. For now suffice it to say that we can actually interpolate in discrete time and find intermediate values of a discrete time sequence using just discrete time filters like the fractional delay. However the price we pay is that we in theory need to use an ideal filter so something that we cannot really compute in practice. We can therefore approximate the fractional delay and obtain arbitrarily good approximations of inter sample values for sequences. The Hilbert filter is another ideal filter whose approximated versions are used in practice and especially in communication systems. So to understand the Hilbert filter let's consider this weird problem if you will.

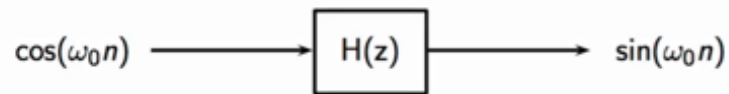
notes

summary

5m 5s



a quirky machine



can we build such a thing?

12

We have a sinusoid, cosine of $\omega_0 n$ for a given frequency ω_0 and we ask ourselves whether we can build a machine that turns a cosine into a sine. In other words introduces a phase shift of π over 2 or 90 degrees, call it as you want,

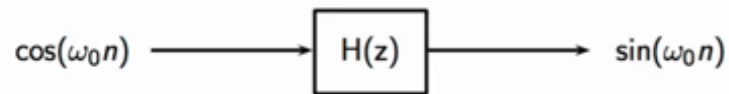
notes

summary

6m 39s



a quirky machine



can we build such a thing?

12

to turn a cosine into a sine. So if we are given a precise frequency ω_0 we

notes

summary

7m 0s



in the frequency domain

$$H(e^{j\omega})[\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)] = -j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]$$

we can derive two values:

$$\begin{cases} H(e^{j\omega_0}) &= -j \\ H(e^{-j\omega_0}) &= +j \end{cases}$$

13

can use the convolution theorem and try to find the frequency response of a filter that produces such an effect. We remember that the Fourier transform or the formal Fourier transform of a cosine is the sum of two deltas, delta of omega

notes

summary

7m 10s



in the frequency domain

$$\underbrace{H(e^{j\omega})[\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)]}_{\text{input}} = \underbrace{-j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]}_{\text{output}}$$

we can derive two values:

$$\begin{cases} H(e^{j\omega_0}) &= -j \\ H(e^{-j\omega_0}) &= +j \end{cases}$$

13

minus omega 0 plus delta of omega plus omega 0. These are direct deltas in the frequency domain. So we multiply this input by the frequency response of this quirky machine that we're trying to design and we want the output to be the Fourier transform of a sine at the same frequency omega 0 whose formal Fourier transform we know. So it's minus j that multiplies the periodic Dirac delta at omega minus omega 0 minus the periodic Dirac delta of omega plus omega 0. So once we have this formula into place we can derive the value of the transfer function of the filter in two specific points in the frequency axis, namely omega 0 and minus omega 0. And for this equation to hold we have that the frequency response of

notes

summary

7m 26s



in the frequency domain

$$H(e^{j\omega})[\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)] = -j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]$$

we can derive two values:

$$\begin{cases} H(e^{j\omega_0}) &= -j \\ H(e^{-j\omega_0}) &= +j \end{cases}$$

13

this filter will have to be minus j for a frequency equal to ω_0 and plus j for a frequency equal to minus ω_0 . Now if we want this relationship to be good for all frequencies then we figure out that the frequency response of the filter will have to have this pattern for all frequencies between 0 and π and 0 and minus π . So the final frequency response that produces this

notes

summary

8m 14s



in the frequency domain

$$H(e^{j\omega})[\tilde{\delta}(\omega - \omega_0) + \tilde{\delta}(\omega + \omega_0)] = -j[\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0)]$$

we can derive two values:

$$\begin{cases} H(e^{j\omega_0}) &= -j \\ H(e^{-j\omega_0}) &= +j \end{cases}$$

13

transformation from cosine into sine is a filter whose frequency response is

notes

summary

8m 36s



in the frequency domain

for the machine to work at all frequencies:

$$H(e^{j\omega_0}) = \begin{cases} -j & \text{for } 0 \leq \omega < \pi \\ +j & \text{for } -\pi \leq \omega < 0 \end{cases} \quad (2\pi\text{-periodic})$$

14

identically minus j for ω between 0 and π and plus j for ω between minus π and 0. And this of course like all filters like all discrete time Fourier transformed is 2π periodic so you repeat this pattern every 2π . If we want to visualize the frequency response of the filter it will look like so. Now

notes

summary

8m 37s



impulse response

$$\begin{aligned}h[n] &= \frac{1}{2\pi} \int_{-\pi}^0 j e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^{\pi} -j e^{j\omega n} d\omega \\&= \frac{1}{2\pi n} [1 - e^{-j\pi n} - e^{-j\pi n} + 1] \\&= \begin{cases} \frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}\end{aligned}$$

17

please remark here that we're not showing the magnitude we're showing the actual value and this is an imaginary axis for a change so we have j for negative frequencies and minus j for positive frequencies. If we were to look at the magnitude of this filter then we will have that the magnitude is identically 1 and so again we have an all-pass filter whereas the phase, if we go back to this, will be equal to pi over 2 for negative frequencies and minus pi over 2 for positive frequencies. Although the frequency response of the filter is purely imaginary surprisingly enough the impulse response is actually a real

notes

summary

9m 8s



impulse response

$$\begin{aligned}h[n] &= \frac{1}{2\pi} \int_{-\pi}^0 j e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^{\pi} -j e^{j\omega n} d\omega \\&= \frac{1}{2\pi n} [1 - e^{-j\pi n} - e^{-j\pi n} + 1] \\&= \begin{cases} \frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}\end{aligned}$$

17

valued sequence. We can obtain this by taking the inverse Fourier transform. This is even simpler than in the fractional delay because we have an integral that we can split into two intervals so between minus pi and 0 we will integrate j times e to the j omega n and on the positive frequency axis we will integrate minus j that multiplies e to the j omega n. If we work out this

notes

summary

9m 45s



impulse response

$$\begin{aligned}h[n] &= \frac{1}{2\pi} \int_{-\pi}^0 j e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^{\pi} -j e^{j\omega n} d\omega \\&= \frac{1}{2\pi n} [1 - e^{-j\pi n} - e^{-j\pi n} + 1] \\&= \begin{cases} \frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}\end{aligned}$$

17

these integrals we obtain 2 divided by pi over n for n odd and 0 for n even. So if we plot the impulse response it will look like this. It will go down with

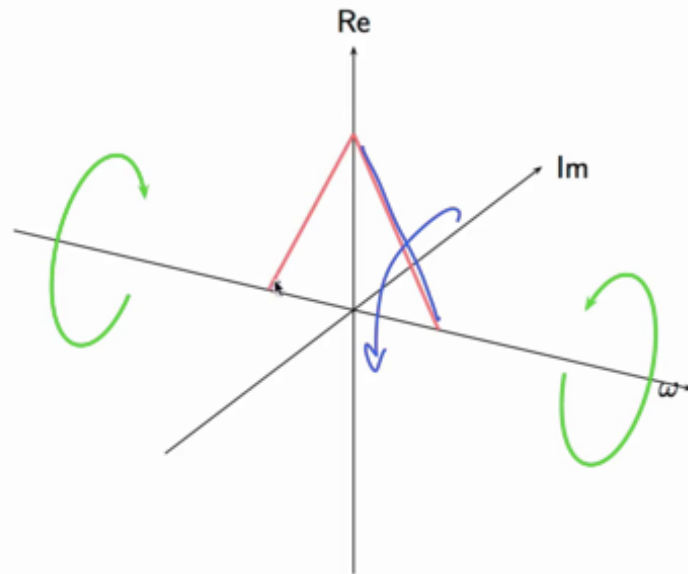
notes

summary

10m 13s



effect of the Hilbert filter (real part)



20

hyperbolic decay so $1/n$ and every other sample is equal to 0. Again this is an ideal filter because the impulse response is infinite and two-sided. Again the decay is inversely proportional to the index which means we can get reasonably good approximations with a finite number of samples if we want to implement the Hilbert filter. Now why would we want to implement the Hilbert filter? Well this is a useful building block in a demodulator. In order to understand how to use the Hilbert filter for demodulation let's look at the effect of the Hilbert filter on an arbitrary input signal. The filter will introduce a phase shift in the signal and a different phase shift for the positive and negative frequencies. To understand the behavior of the filter let's look at a representation of the input spectrum by displaying both the real and the imaginary part on a three-dimensional plot. So here suppose that the input is real valued so we have a classic pattern where the real part of the spectrum is symmetric and the imaginary part is antisymmetric. We will plot the real part here on the vertical plane and the imaginary part on the horizontal plane and this is the frequency axis. The Hilbert filter will introduce a 90 degree clockwise rotation of the spectrum for the positive frequencies and a 90 degree counterclockwise rotation for the negative frequencies. Let's look at the real part first so imagine that the real part of the spectrum has this triangular shape. When we apply the Hilbert filter this part will be rotated by 90 degrees in this direction and will become imaginary

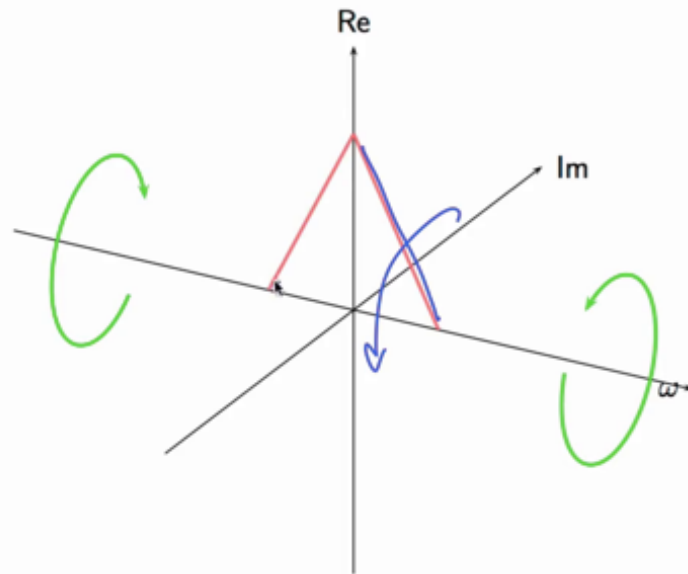
notes

summary

10m 24s



effect of the Hilbert filter (real part)



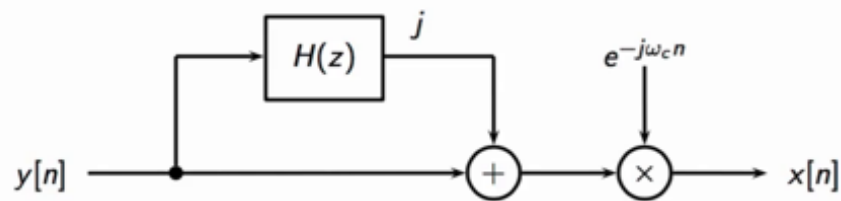
20

and this part here corresponding to the negative frequencies will be rotated 90

notes

summary

Hilbert demodulation



$x[n] \cos \omega_c n$

23

degrees in this direction and will become imaginary. Graphically if we were to show this rotation as it unfolds we start with a triangular shape and then we rotate it until it becomes like so. So the real part of the spectrum has now become the imaginary part of the spectrum and from symmetric it will become antisymmetric. Similarly the imaginary part of the spectrum will be rotated in the same way and from antisymmetric here will become real and symmetric like so. So if we look at the effect on the combined spectrum we start with this real and imaginary part and after applying the Hilbert filter to this input we end up with this spectrum here where the imaginary part and the real part have been exchanged and modified so that they preserve their symmetry and antisymmetry. So let's see how we can use the Hilbert filter to effectively perform demodulation. This here is a Hilbert demodulator. The input signal is supposed to be a modulated signal so an original signal x of n multiplied by a cosine at $\omega_c n$ where this is the carrier at frequency ω_c .

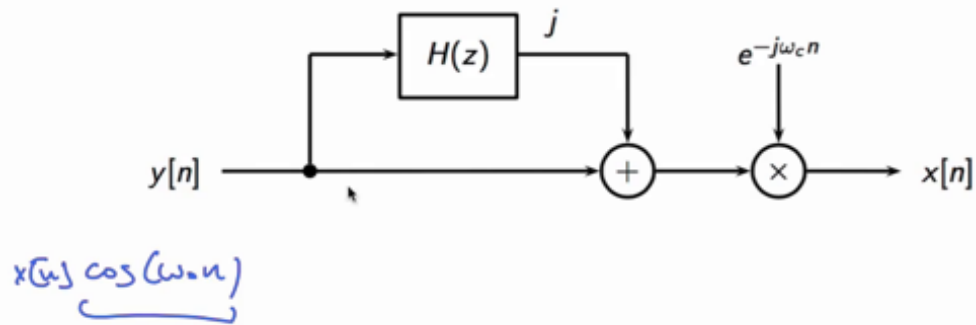
notes

summary

12m 13s



Hilbert demodulation



23

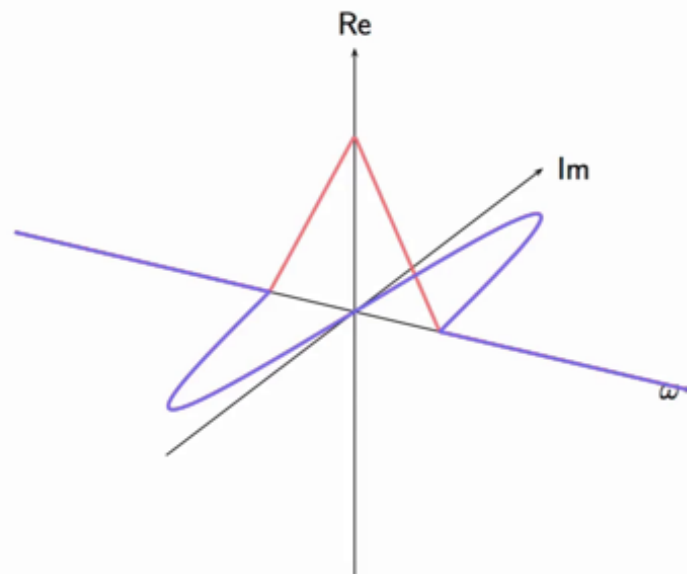
So this signal is split into two identical parts. One is passed through as

notes

summary

13m 22s



$x[n]$ 

24

notes

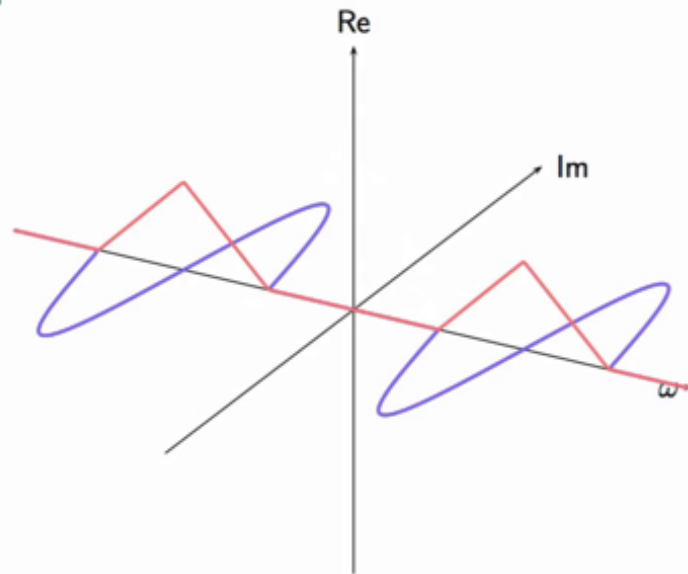
summary

13m 28s



Hilbert demodulation

$$x[n] \cos(\omega_0 n) = y[n]$$



25

looks like so. Two copies of the original signal here again we show the real part on the vertical plane and the imaginary part on the horizontal plane. The top branch of the demodulator remember here is the signal and the top branch will

notes

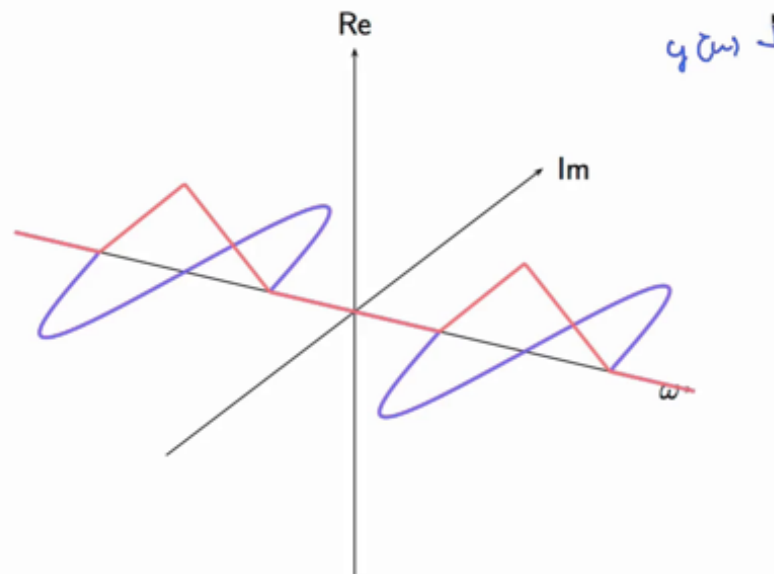
summary

14m 1s



Hilbert demodulation: $jy[n] * h[n]$

$y[n]$



26

have a Hilbert filter and then multiplication by j . We can interchange

notes

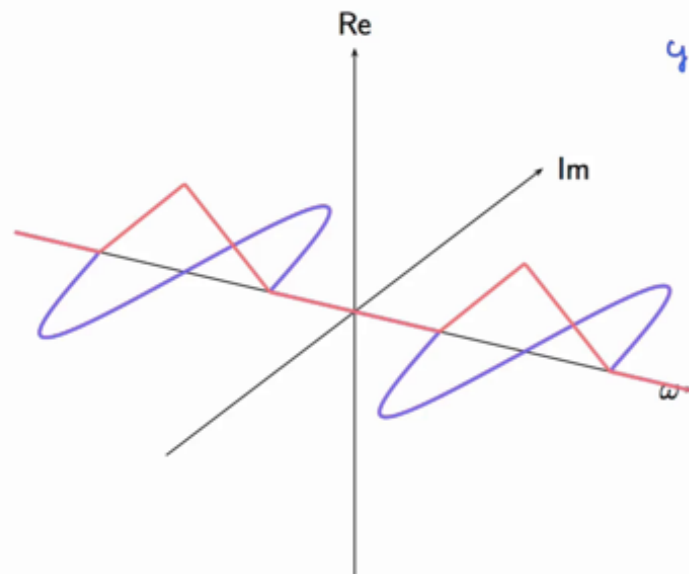
summary

14m 13s



Hilbert demodulation: $jy[n] * h[n]$

$y[n]$



26

Hilbert filter and multiplication. So how does $jy[n]$ look in the frequency domain? Well multiplication by j is just counterclockwise rotation by 90 degrees

notes

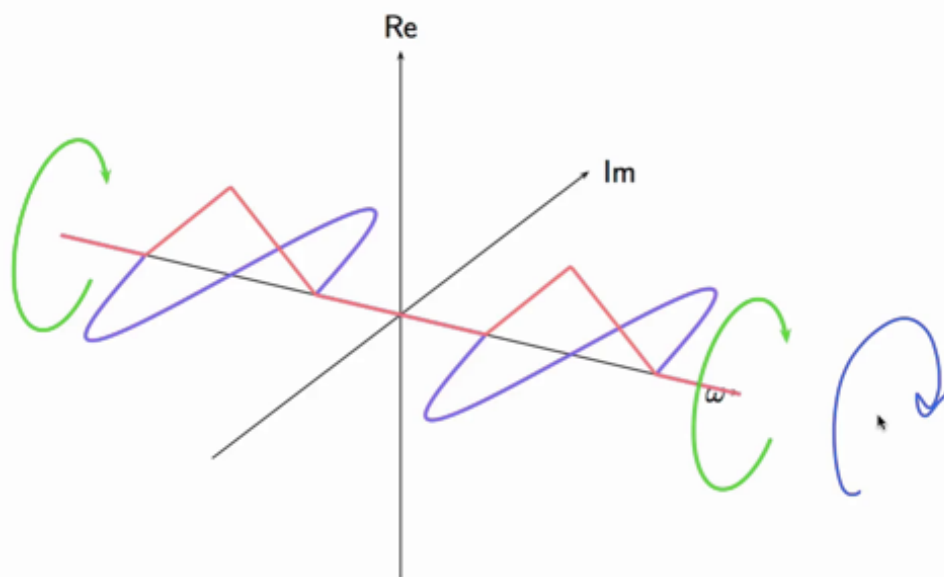
summary

14m 18s



Hilbert demodulation: $jy[n] * h[n]$

$jy[n]$



and it's the same for positive and negative frequencies. So we take the spectrum here and we just rotate this by 90 degrees so that the imaginary part becomes real and the real part becomes imaginary but there is no change in

notes

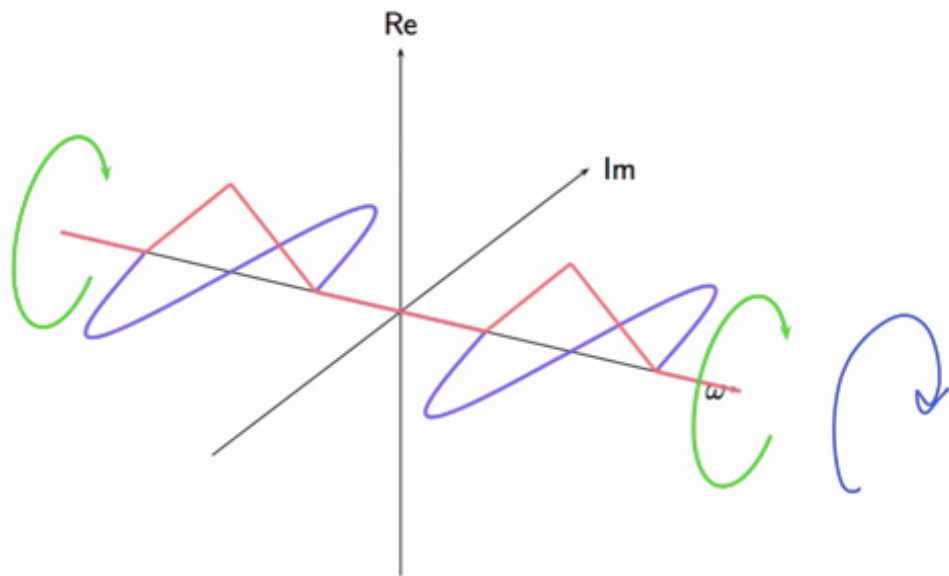
summary

14m 28s



Hilbert demodulation: $jy[n] * h[n]$

$jy[n]$



26

symmetry or anti-symmetry of the components. So when we do that we're

notes

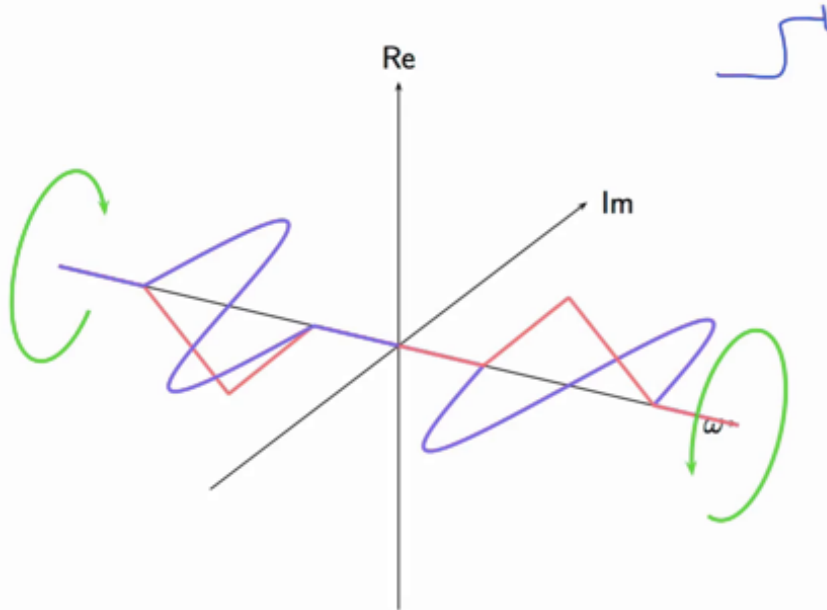
summary

14m 43s



Hilbert demodulation: $jy[n] * h[n]$

$$jy[n] * h[n]$$



just flipping the thing and now we look like so. Now we convolve this with the Hilbert filter. So now we will introduce this differential rotation between positive and negative frequencies. In particular we get clockwise in the positive frequencies and counterclockwise in the negative frequencies. Now remember that we have two branches in the demodulator. The first one which is the one we're showing here. Hilbert filter followed by multiplication by j and

notes

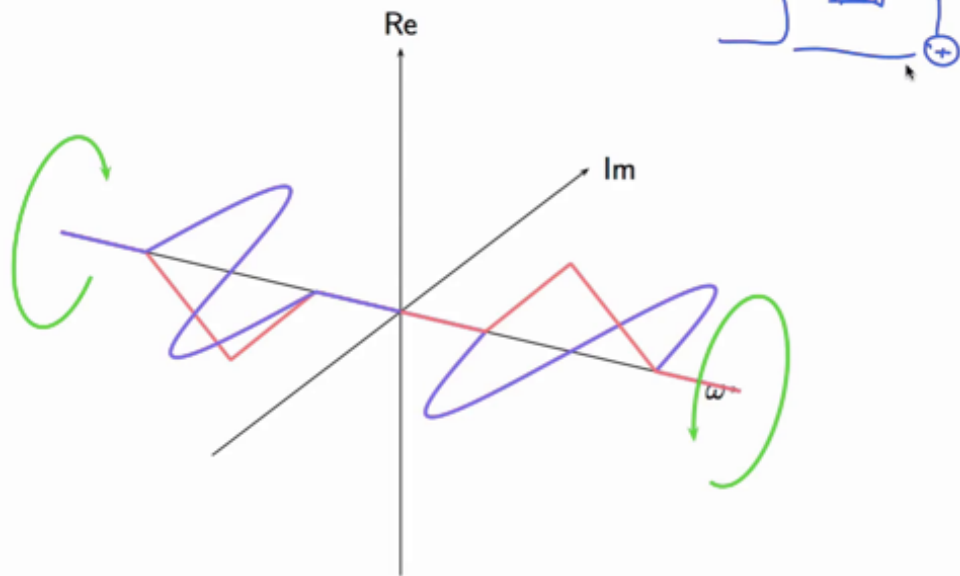
summary

14m 49s



Hilbert demodulation: $jy[n] * h[n]$

$$jy[n] * h[n]$$



26

here we sum this result back to the original input. If we look at the spectra in the two branches we see that for the negative frequencies the

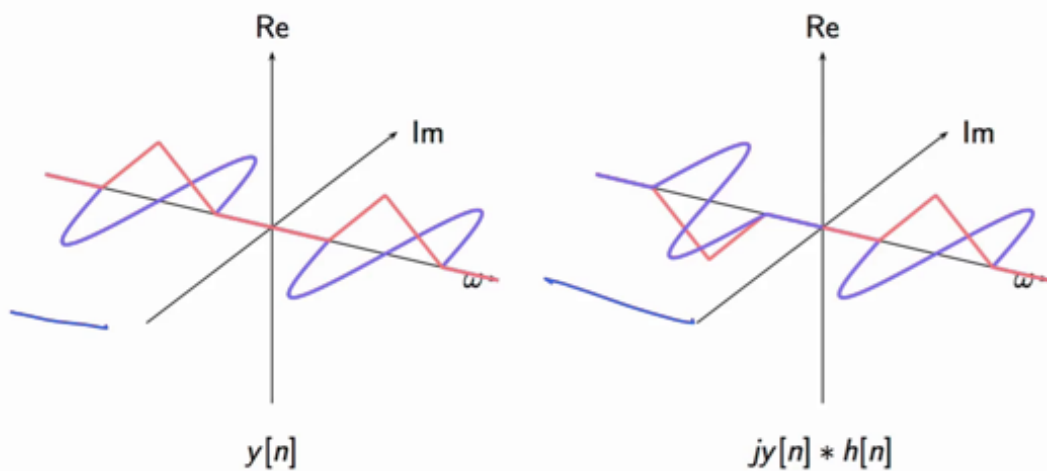
notes

summary

15m 12s



Hilbert demodulation: $jy[n] * h[n] + y[n]$



27

spectra between the top branch and the bottom branch are completely out of phase. So both the real parts and the imaginary parts are negative of each other. On the other hand for positive frequencies the spectra are in phase so they will sum up

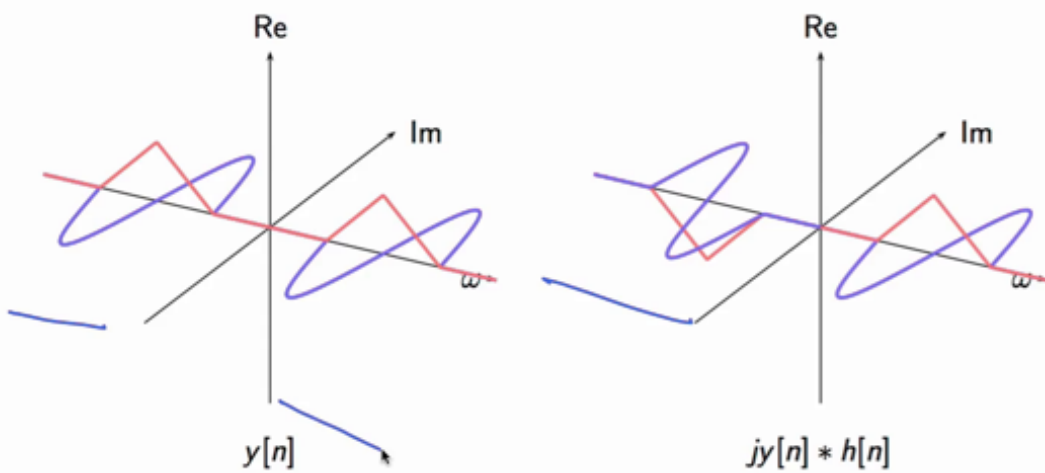
notes

summary

15m 21s



Hilbert demodulation: $jy[n] * h[n] + y[n]$



27

constructively rather than destructively. So when we sum these two signals

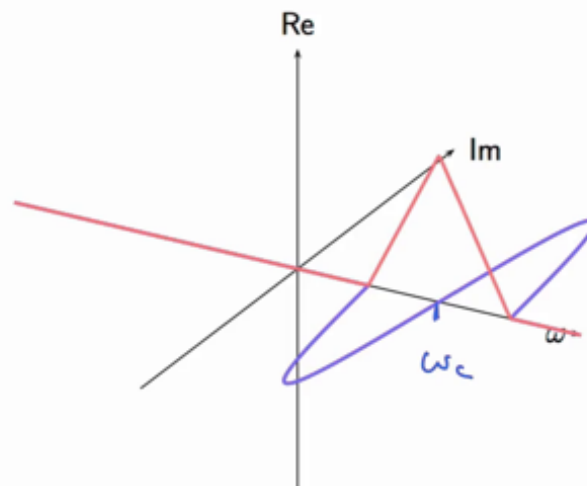
notes

summary

15m 36s



Hilbert demodulation: $jy[n] * h[n] + y[n] = x[n]e^{j\omega_0 n}$



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together the resulting spectrum is a one-sided spectrum where we have as you can see the same shape as the original signal X of n but simply translated in frequency and centered in ω_c . So now we can bring this back to baseband by shifting the spectrum just by ω_c and we know that we can just shift the spectrum by multiplying by a complex exponential at a proper frequency. So when we do that we bring back the spectrum here and we have completed the

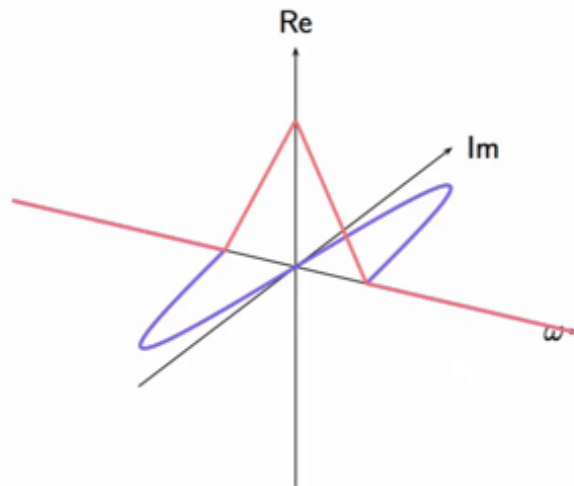
notes

summary

15m 37s



Hilbert demodulation: $(jy[n] * h[n] + y[n])e^{-j\omega_0 n}$



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demodulation process.

notes

summary

16m 10s

